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Finite-time synchronization and quasi-synchronization of fractional-order fuzzy BAM neural networks with time delays*

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Abstract. This paper concentrates on the finite-time synchronization (FTS) and the quasi-synchronization (QS) problems for a kind of fractional-order fuzzy BAM neural networks with time delays (FOFBAMNNs). In order to reach the goals of synchronization, two novel controllers are designed. Then, based on finite-time stability theorem, Lyapunov function theory, and several inequality techniques, through the application of two different designed controllers, several criteria for both FTS and QS are established. Moreover, more precise error level and settling times are given. The effectiveness of the derived criteria is ultimately validated through two simulations.

Keywords: finite-time synchronization, quasi-synchronization, BAM neural networks, Caputo derivative, fuzzy term.

1 Introduction

In 1988, Kosko proposed the bidirectional associative memory neural network (BAMNN) as the first instance of its kind [7, 8]. BAMNN typically comprises two layers of neurons, which are used to store and process two different types of data. BAMNN uses

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a bidirectional feedback mechanism, which enables the input and output to be mapped toeach other. Different from the traditional one-way associative memory model, the BAM neural network can realize the mutual mapping and association of information between two sets of neurons through the mechanism of bidirectional information propagation. This bidirectional associative mechanism makes the BAM neural network have broad application prospects in the fields of associative memory [19], pattern recognition, and image processing [32]. To date, considerable research has been conducted on the exponential stability, Mittag-Leffler-type stability, and asymptotic stability of delayed BAM neural networks, leading to numerous significant findings regarding global stability.

With the deepening of research, the theory and application of BAM neural network have been further developed, especially in the study of BAM neural network model under complex conditions such as time delay [14], stochastic perturbations [11], fusion of medical images and nonlinear dynamics [12], significant progress has been made. In most practical situations, time delay cannot be avoided. The neural network with time delay can simulate the dynamic behavior in the real system more realistically. Zhao et al. and Lin et al. developed two new predator–prey models with time delay in [34] and [15], respectively, and studied the relevant dynamical properties of the system. Cui et al. [4] studied the existence, uniqueness, and bifurcation behavior of a class of five-dimensional BAM neural network solutions that conform to objective reality. In addition, since fuzzy logic can handle uncertainty and ambiguity in the system, it is very meaningful to combine fuzzy terms with BAMNNs to form fuzzy BAMNNs (FBAMNNs). FBAMNNs can better cope with noise interference and enhance the model's antinoise ability, which find extensive application in areas such as automatic control, image processing, and pattern recognition. Therefore, the introduction of fuzzy terms enhances the robustness, adaptability, and accuracy of BAM neural networks. At present, there are many literatures on the synchronization problem of FBAMNNs, and good results have been obtained [26, 28].

Fractional-order calculus (FC) has been widely used in control system, biomedical engineering [27], materials science, and electrical engineering. Compared with traditional integer-order calculus, FC offers notable benefits in modeling memory effects and hereditary phenomena. This enables FC to serve as a robust tool for more accurately modeling real-world systems. Fractional-order neural networks (FONNs) offer a more precise characterization of dynamic behaviors. The nonlocality of fractional derivatives effectively captures historical dependencies, enhancing the system's generalization ability in handling complex nonlinear problems. Moreover, FONNs demonstrate greater adaptability, making them suitable for diverse dynamic environments such as time delay systems [1], chaotic systems [21], and uncertain systems, where traditional neural networks often struggle. Additionally, fractional-order learning rules accelerate network convergence, improve optimization efficiency, and enhance training robustness. Hence, integrating FC with BAM neural networks has become increasingly essential. In [20], a novel fractional-order integral inequality has been established, effectively accounting for the influence of both the delay factor and the order of the fractional derivative. Xu et al. [25] analyzed the stability and the emergence of Hopf bifurcation.

The synchronization of NNs, regarded as one of the most intriguing and essential research areas [2,23], has attracted extensive attention from scholars owing to its extensive use in areas including image encryption [24], circuit systems, cryptography, and so on. Synchronization can be classified based on different synchronization times into two main categories: infinite-time synchronization (IFTS) and finite-time synchronization (FTS). The former encompasses categories such as complete synchronization (CS) [16] and QS [22], Mittag-Leffler synchronization [17]. Among these, QS is notable for allowing small errors between systems, which makes it especially practical for systems involving fuzzy terms. However, in many real-world scenarios, it is usually preferable to implement synchronization within a finite time frame as it addresses the needs for fast response, stability, and accuracy. Consequently, studying the FTS of FOFBAMNNs holds significant value. Recently, several important findings on FTS in neural networks have been published.

Based on the above analysis, our study seeks to address both FTS and QS in delayed FOFBAMNNs. The primary contributions of this paper can be outlined as follows:

(i) Distinct from the model in [28, 30], Caputo derivative operator, time delay, and fuzzy terms are considered in the model, which makes the proposed model more general and less conservative.
(ii) By designing an innovative hybrid nonlinear controller, this paper derives a set of verifiable criteria for FTS. These criteria are expressed as algebraic inequalities and provide a clear estimation of the finite settling time.
(iii) In distinction from the adaptive controllers [16] and linear feedback controller [30], this paper introduces a novel and useful class of controller: adaptive nonlinear feedback controllers, which are being applied for the first time to achieve QS problem in FOFBAMNNs.

The structure of this paper is outlined as follows: Section 2 introduces key definitions and lemmas related to FC along with the considered FOFBAMNNs with fuzzy terms. In Section 3, several sufficient criteria for QS and FTS are derived based on newly designed controllers. Section 4 validates the theoretical outcomes through numerical simulations, while Section 5 concludes the paper and suggests future research directions.

Notations. For simplicity, recognizable symbols will be employed in the following sections. $\mathbb{R}$ and $\mathbb{R}^n$ represent the set of real numbers and n-dimensional real space, respectively. $\Gamma(\cdot)$ denotes the gamma function. The $\operatorname{sign}(\cdot)$ is the standard sign function. $\mathscr{C}^n([t_0, +\infty, \mathbb{R})$ represents a set of continuous nth-order differential function from $[t_0, +\infty)$ into $\mathbb{R}$. $\mathbb{N}_1^n = \{1, 2, \dots, n\}$, $\mathbb{M}_1^m = \{1, 2, \dots, m\}$, where $n, m \in \mathbb{R}$. In addition, for any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, the 1-norm and 2-norm of x is defined as $\|x\|_1 = \sum_{\zeta=1}^n |x_\zeta|$ and $\|x\|_2 = (\sum_{\zeta=1}^n x_\zeta^2)^{1/2}$.

2 Preliminaries and model description

Some of the necessary definitions and lemmas are listed in this part.

Definition 1. (See [18].) For a function $\Upsilon(\partial)$, the ε-order Caputo fractional derivative is defined as

$$ {}_{\partial_0}^{c}D_{\partial}^{\varepsilon}\Upsilon(\partial) = \frac{1}{\Gamma(m-\varepsilon)}\int\limits_{t_0}^{t}\frac{\Upsilon^{(m)}(\varsigma)}{(\partial-\varsigma)^{\varepsilon-m+1}}\,\mathrm{d}\varsigma, \quad \partial \geqslant \partial_0,\ m-1<\varepsilon\leqslant m. $$

Definition 2. (See [18].) For $0<s<1$ and $m\in\mathbb{R}$, the Mittag-Leffler function with one argument is defined as

$$ E_s(m) = \sum_{\rho=0}^{\infty}\frac{m^{\rho}}{\Gamma(\rho s+1)}. $$

Lemma 1. (See [33].) *Suppose function $\Upsilon(\partial)$ is continuous and differentiable on $\partial \in [\partial_0,\infty)$. Then for any constant $\breve{h}$ and $\partial\in[\partial_0,\infty)$,*

$$ {}_{\partial_0}^{c}D_{\partial}^{\varepsilon}\big(\Upsilon(\partial)-\breve{h}\big)^2 \leqslant 2\big(\Upsilon(\partial)-\breve{h}\big)\,{}_{\partial_0}^{c}D_{\partial}^{\varepsilon}\Upsilon(\partial). $$

Lemma 2. (*See* [3].) *For $\Upsilon(\partial)\in\mathscr{C}^1([\partial_0,+\infty),\mathbb{R})$ and $\varepsilon\in(0,1)$,*

$$ {}_{\partial_0}^{c}D_{\partial}^{\varepsilon}\big|\Upsilon(\partial)\big| \leqslant \operatorname{sign}\big(\Upsilon(\partial)\big)\,{}_{\partial_0}^{c}D_{\partial}^{\varepsilon}\Upsilon(\partial), \quad \partial\geqslant\partial_0. $$

Lemma 3. (*See* [18].) *Let $\partial\geqslant\partial_0$. Then $E_\lambda(\epsilon(\partial-\partial_0)^\lambda)$ is monotonically nonincreasing, and $0\leqslant E_\lambda(\epsilon(\partial-\partial_0)^\lambda)\leqslant 1$ for $\epsilon\leqslant 0$.*

Lemma 4. (*See* [5].) *If function $\Upsilon(\partial)\in\mathscr{C}^1([\partial_0,\ +\infty),\mathbb{R}^+)$ is positive defined and*

$$ {}_{\partial_0}^{c}D_{\partial}^{\varepsilon}\Upsilon(\partial) \leqslant -\sigma_1\Upsilon^{-\eta_1}(\partial)-\sigma_2\Upsilon^{-\eta_2}(\partial), \quad \Upsilon(\partial)\in\mathbb{R}^+\setminus\{0\}, $$

where $0<\varepsilon<1$, $\sigma_1>0$, $\sigma_2>0$, $\eta_1\geqslant 0$, and $\eta_1<\eta_2<1+2\eta_1$, then one has $\lim_{\partial\to\partial^\vartriangle}\Upsilon(\partial)=0$, and $\Upsilon(\partial)=0$, $\partial\geqslant\partial^\vartriangle$ with $\partial^\vartriangle\leqslant\widetilde{T}$, where

$$ \widetilde{T} = \Bigg[\frac{\Gamma(1+\varepsilon)}{\sigma_2 2^{(\eta_2-2\eta_1-1)/(1+\eta_1)}(1+\eta_2)}\Bigg(\bigg(\Upsilon^{1+\eta_1}(\partial_0)+\Big(\frac{\sigma_2}{\sigma_1}\Big)^{(1+\eta_1)/(\eta_2-\eta_1)}\bigg)^{(1+\eta_2)/(1+\eta_1)} - \Big(\frac{\sigma_2}{\sigma_1}\Big)^{(1+\eta_2)/(\eta_2-\eta_1)}\Bigg)\Bigg]^{1/\varepsilon} + \partial_0. $$

Remark 1. Compared with the fractional-order finite-time inequality (FO-FTI) in [6], the FO-FTI established by Lemma 4 contains two nonlinear terms. Therefore, the controller designed based on Lemma 4 is more flexible. In addition, when $\eta_2 = 0$, Lemma 4 is simplified to Lemma 9 in [5]. When $\sigma_2 = 0$, Lemma 4 is simplified to Lemma 8 in [10]. Further more, when $\eta_1 = 0$, Lemma 4 is simplified to Lemma 4 in [13]. Therefore, the result of Lemma 4 is less conservative, and the obtained settling time is more accurate.

Lemma 5. (*See* [9].) *Let $\Upsilon_1(\partial)$ and $\Upsilon_2(\partial)$ be two nonnegative, continuous functions satisfying*

$$ {}_{\partial_0}^{c}D_{\partial}^{\varepsilon}\big(\Upsilon_1(\partial)+\Upsilon_2(\partial)\big) \leqslant -\epsilon\Upsilon_1(\partial)+\varsigma, $$

where $0<\varepsilon<1$, $\epsilon>0$, and $\varsigma>0$. Then

$$ \Upsilon_1(\partial) \leqslant \left(\Upsilon_1(\partial_0)+\Upsilon_2(\partial_0)-\frac{\varsigma}{\epsilon}\right)E_{\varepsilon}\big(-\epsilon(\partial-t_0)^{\varepsilon}\big)+\frac{\varsigma}{\epsilon}, \quad \partial \geqslant \partial_0+\left(\frac{\Gamma(\varepsilon)}{\epsilon}\right)^{1/(1-\varepsilon)}. $$

Next, we consider the following FOFBAMNN:

$$ \begin{aligned} {}_{\partial_0}^{c}D_{\partial}^{\varepsilon}\chi_{\vartheta}(\partial) &= -p_{\vartheta}\chi_{\vartheta}(\partial)+\sum_{\nu=1}^{m}\breve{a}_{\vartheta\nu}\Theta_{\nu}\big(\Lambda_{\nu}(\partial)\big)+\bigwedge_{\nu=1}^{m}\breve{\alpha}_{\vartheta\nu}\Theta_{\nu}\big(\Lambda_{\nu}(\partial-\breve{\tau})\big) \\ &\quad+\bigvee_{\nu=1}^{m}\breve{\beta}_{\vartheta\nu}\Theta_{\nu}\big(\Lambda_{\nu}(\partial-\breve{\tau})\big)+\breve{I}_{\vartheta}(\partial), \\ {}_{\partial_0}^{c}D_{\partial}^{\varepsilon}\Lambda_{\nu}(\partial) &= -q_{\nu}\Lambda_{\nu}(\partial)+\sum_{\vartheta=1}^{n}\hat{b}_{\nu\vartheta}\Xi_{\vartheta}\big(\chi_{\vartheta}(\partial)\big)+\bigwedge_{\vartheta=1}^{n}\hat{m}_{\nu\vartheta}\Xi_{\vartheta}\big(\chi_{\vartheta}(\partial-\breve{\tau})\big) \\ &\quad+\bigvee_{\vartheta=1}^{n}\hat{w}_{\nu\vartheta}\Xi_{i}\big(\chi_{\vartheta}(\partial-\breve{\tau})\big)+\breve{\mathcal{I}}_{\nu}(\partial), \end{aligned} \tag{1} $$

where $0<\varepsilon<1$, $\vartheta\in\mathbb{N}_1^n$, $\nu\in\mathbb{M}_1^m$. n and m represent the number of neurons in the first and second layers, respectively. The neuron states are denoted by $\chi_{\vartheta}(\partial)$ and $\Lambda_{\nu}(\partial)$, while p_{ϑ} and q_{ν} signify positive decay rates. Elements of the feedback templates are represented by $\breve{a}_{\vartheta\nu}$ and $\hat{b}_{\nu\vartheta}$, whereas $\breve{\alpha}_{\vartheta\nu}$ and $\hat{m}_{\nu\vartheta}$ are part of the fuzzy feedback MAX templates, and $\breve{\beta}_{\vartheta\nu}$ and $\hat{w}_{\nu\vartheta}$ are elements of the fuzzy feedback MIN templates. The activation functions are denoted as Θ_{ν} and Ξ_{ϑ} with internal inputs given by $\breve{I}_{\vartheta}(\partial)$ and $\breve{\mathcal{I}}_{\nu}(\partial)$. The constant delay is represented by $\breve{\tau}$, and $\bigwedge$ and $\bigvee$ refer to the *fuzzy AND* and *fuzzy OR* operations, respectively. The initial conditions of (1) can be described as

$$ \chi_{\vartheta}(s)=\varphi_{\vartheta}^{1}(s), \quad \Lambda_{\nu}(s)=\psi_{\nu}^{1}(s), \quad s\in[\partial_0-\breve{\tau},\partial_0],\ \vartheta\in\mathbb{N}_1^n,\ \nu\in\mathbb{M}_1^m. $$

Assumption 1. For any $\vartheta,\iota\in\mathbb{N}^+$ and $\iota,\tilde{\iota}\in\mathbb{R}$, there exist $\xi_{\iota}>0$ and $\rho_{\vartheta}>0$ such that $|\Theta_{\iota}(\iota)-\Theta_{\iota}(\tilde{\iota})|\leqslant\xi_{\iota}|\iota-\tilde{\iota}|$ and $|\Xi_{\vartheta}(\iota)-\Xi_{\vartheta}(\tilde{\iota})|\leqslant\rho_{\vartheta}|\iota-\tilde{\iota}|$.

Then the corresponding system for the drive system (1) can be written as follows:

$$ \begin{aligned} {}_{\partial_0}^{c}D_{\partial}^{\varepsilon}\tilde{\chi}_{\vartheta}(\partial) &= -p_{\vartheta}\tilde{\chi}_{\vartheta}(\partial)+\sum_{\nu=1}^{m}\breve{a}_{\vartheta\nu}\Theta_{\nu}\big(\tilde{\Lambda}_{\nu}(\partial)\big)+\bigwedge_{\nu=1}^{m}\breve{\alpha}_{\vartheta\nu}\Theta_{\nu}\big(\tilde{\Lambda}_{\nu}(\partial-\breve{\tau})\big) \\ &\quad+\bigvee_{\nu=1}^{m}\breve{\beta}_{\vartheta\nu}\Theta_{\nu}\big(\tilde{\Lambda}_{\nu}(\partial-\breve{\tau})\big)+\breve{I}_{\vartheta}(\partial)+u_{\vartheta}(\partial), \\ {}_{\partial_0}^{c}D_{\partial}^{\varepsilon}\tilde{\Lambda}_{\nu}(\partial) &= -q_{\nu}\tilde{\Lambda}_{\nu}(\partial)+\sum_{\vartheta=1}^{n}\hat{b}_{\nu\vartheta}\Xi_{\vartheta}\big(\tilde{\chi}_{\vartheta}(\partial)\big)+\bigwedge_{\vartheta=1}^{n}\hat{m}_{\nu\vartheta}\Xi_{\vartheta}\big(\tilde{\chi}_{\vartheta}(\partial-\breve{\tau})\big) \\ &\quad+\bigvee_{\vartheta=1}^{n}\hat{w}_{\nu\vartheta}\Xi_{i}\big(\tilde{\chi}_{\vartheta}(\partial-\breve{\tau})\big)+\breve{\mathcal{I}}_{\nu}(\partial)+v_{\nu}(\partial). \end{aligned} \tag{2} $$

The initial conditions of the corresponding system (2) are

$$\chi_\vartheta(s)=\varphi_\vartheta^2(s),\quad \Lambda_\nu(s)=\psi_\nu^2(s),\quad s\in[\partial_0-\breve{\tau},\partial_0],\ \vartheta\in\mathbb{N}_1^n,\ \nu\in\mathbb{M}_1^m.$$

The synchronization error is given by $\tilde{e}_\vartheta(\partial)=\tilde{\chi}_\vartheta(\partial)-\chi_\vartheta(\partial)$ and $\tilde{z}_\nu(\partial)=\tilde{\Lambda}_\nu(\partial)-\Lambda_\nu(\partial)$. From (1) and (2) the error system is formulated as follows:

$$\begin{aligned}
{}_{\partial_0}^{c}D_\partial^{\varepsilon}\tilde{e}_\vartheta(\partial)&=-p_\vartheta\tilde{e}_\vartheta(\partial)+\sum_{\nu=1}^{m}\breve{a}_{\vartheta\nu}\Theta_\nu\big(\tilde{\Lambda}_\nu(\partial)\big)-\sum_{\nu=1}^{m}\breve{a}_{\vartheta\nu}\Theta_\nu\big(\Lambda_\nu(\partial)\big)\\
&\quad+\bigwedge_{\nu=1}^{m}\breve{\alpha}_{\vartheta\nu}\Theta_\nu\big(\tilde{\Lambda}_\nu(\partial-\breve{\tau})\big)-\bigwedge_{\nu=1}^{m}\breve{\alpha}_{\vartheta\nu}\Theta_\nu\big(\Lambda_\nu(\partial-\breve{\tau})\big)\\
&\quad+\bigvee_{\nu=1}^{m}\breve{\beta}_{\vartheta\nu}\Theta_\nu\big(\tilde{\Lambda}_\nu(\partial-\breve{\tau})\big)-\bigvee_{\nu=1}^{m}\breve{\beta}_{\vartheta\nu}\Theta_\nu\big(\Lambda_\nu(\partial-\breve{\tau})\big)+u_\vartheta(\partial),\\
{}_{\partial_0}^{c}D_\partial^{\varepsilon}\tilde{z}_\nu(\partial)&=-q_\nu\tilde{z}_\nu(\partial)+\sum_{\vartheta=1}^{n}\hat{b}_{\nu\vartheta}\Xi_\vartheta\big(\tilde{\chi}_\vartheta(\partial)\big)-\sum_{\vartheta=1}^{n}\hat{b}_{\nu\vartheta}\Xi_\vartheta\big(\chi_\vartheta(\partial)\big)\\
&\quad+\bigwedge_{\vartheta=1}^{n}\hat{m}_{\nu\vartheta}\Xi_\vartheta\big(\tilde{\chi}_\vartheta(\partial-\breve{\tau})\big)-\bigwedge_{\vartheta=1}^{n}\hat{m}_{\nu\vartheta}\Xi_\vartheta\big(\chi_\vartheta(\partial-\breve{\tau})\big)\\
&\quad+\bigvee_{\vartheta=1}^{n}\hat{w}_{\nu\vartheta}\Xi_\vartheta\big(\tilde{\chi}_\vartheta(\partial-\breve{\tau})\big)-\bigvee_{\vartheta=1}^{n}\hat{w}_{\vartheta\nu}\Xi_\vartheta\big(\chi_\vartheta(\partial-\breve{\tau})\big)+v_\nu(\partial).
\end{aligned}\tag{3}$$

Lemma 6. (*See* [31].) *Let* χ_ν,Λ_ν *be two states of FOFBAMNNs* (3)*. We can get*

$$\begin{aligned}
\left|\bigwedge_{\nu=1}^{m}\breve{\alpha}_{\vartheta\nu}\Theta_\nu(\chi_\nu)-\bigwedge_{\nu=1}^{m}\breve{\alpha}_{\vartheta\nu}\Theta_\nu(\Lambda_\nu)\right|&\leqslant\sum_{\nu=1}^{m}|\breve{\alpha}_{\vartheta\nu}|\big|\Theta_\nu(\chi_\nu)-\Theta_\nu(\Lambda_\nu)\big|,\\
\left|\bigvee_{\nu=1}^{m}\breve{\alpha}_{\vartheta\nu}\Theta_\nu(\chi_\nu)-\bigvee_{\nu=1}^{m}\breve{\alpha}_{\vartheta\nu}\Theta_\nu(\Lambda_\nu)\right|&\leqslant\sum_{\nu=1}^{m}|\breve{\alpha}_{\vartheta\nu}|\big|\Theta_\nu(\chi_\nu)-\Theta_\nu(\Lambda_\nu)\big|.
\end{aligned}$$

Definition 3. (See [30].) FOFBAMNNs (1) and (2) are said to be synchronized in finite time if there is a constant $\widetilde{T}_2$ such that $\lim_{\partial\to\widetilde{T}_2}\sum_{\vartheta=1}^{n}|\tilde{e}_\vartheta(\partial)|+\sum_{\nu=1}^{m}|\tilde{z}_\nu(\partial)|=0$ and $\sum_{\vartheta=1}^{n}|\tilde{e}_\vartheta(\partial)|+\sum_{\nu=1}^{m}|\tilde{z}_\nu(\partial)|\equiv 0$ for $\partial>\widetilde{T}_2$, where $\widetilde{T}_2$ is called the settling time.

Definition 4. (See [29].) FOFBAMNNs (1) and (2) are said to realize QS with error level $l>0$ if there exists $\widetilde{T}_1\geqslant 0$ such that $\|E(\partial)\|_2\leqslant l$ for all $\partial\geqslant\widetilde{T}_1$, where $E(\partial)=(\tilde{e}_1(\partial),\tilde{e}_2(\partial),\dots,\tilde{e}_n(\partial),\tilde{z}_1(\partial),\tilde{z}_2(\partial),\dots,\tilde{z}_m(\partial))^{\mathrm{T}}$.

3 Main results

3.1 FTS of FOFBAMNNs via hybrid controller

In this section, a hybrid controller is designed to implement FTS. In order to implement FTS between the FOFBAMNNs (1) and (2), we design the following hybrid controller:

$$u_\vartheta(\partial) = -\tilde{\xi}_1 \operatorname{sign}\bigl(\tilde{e}_\vartheta(\partial)\bigr)\bigl|\tilde{e}_\vartheta(\partial - \breve{\tau})\bigr| - \tilde{\delta}_1 \tilde{e}_\vartheta(\partial) - \tilde{\omega}_1 \frac{\tilde{e}_\vartheta(\partial)}{|\tilde{e}_\vartheta(\partial)|^{s_1}} - \tilde{\rho}_1 \frac{\tilde{e}_\vartheta(\partial)}{|\tilde{e}_\vartheta(\partial)|^{s_2}},$$
$$v_\nu(\partial) = -\tilde{\xi}_2 \operatorname{sign}\bigl(\tilde{z}_\nu(\partial)\bigr)\bigl|\tilde{z}_\nu(\partial - \breve{\tau})\bigr| - \tilde{\delta}_2 \tilde{z}_\nu(\partial) - \tilde{\omega}_2 \frac{\tilde{z}_\nu(\partial)}{|\tilde{z}_\nu(\partial)|^{s_1}} - \tilde{\rho}_2 \frac{\tilde{z}_\nu(\partial)}{|\tilde{z}_\nu(\partial)|^{s_2}}, \tag{4}$$

where $\vartheta \in \mathbb{N}_1^n$, $\nu \in \mathbb{M}_1^m$, $\tilde{\xi}_1$, $\tilde{\xi}_2$, $\tilde{\delta}_1$, $\tilde{\delta}_2$, $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\rho}_1$, $\tilde{\rho}_2$ are positive constants, $s_1 \geqslant 2$ and $s_1 < s_2 < 2s_1$ are tunable constants.

Theorem 1. *Let Assumption* 1 *hold, and if control gains* $\tilde{\xi}_1$, $\tilde{\xi}_2$, $\tilde{\delta}_1$, $\tilde{\delta}_2$ *satisfy*

$$\tilde{\delta}_1 \geqslant \max_{1\leqslant\vartheta\leqslant n}\left\{-p_\vartheta + \sum_{\nu=1}^m |\hat{b}_{\nu\vartheta}|\rho_\vartheta\right\}, \qquad \tilde{\xi}_1 \geqslant \max_{1\leqslant\vartheta\leqslant n}\left\{\sum_{\nu=1}^m \bigl(|\hat{m}_{\nu\vartheta}| + |\hat{w}_{\nu\vartheta}|\bigr)\rho_\vartheta\right\}, \tag{5}$$

$$\tilde{\delta}_2 \geqslant \max_{1\leqslant\nu\leqslant m}\left\{-q_\nu + \sum_{\vartheta=1}^n |\check{a}_{\vartheta\nu}|\xi_\nu\right\}, \qquad \tilde{\xi}_2 \geqslant \max_{1\leqslant\nu\leqslant m}\left\{\sum_{\vartheta=1}^n \bigl(|\check{\alpha}_{\vartheta\nu}| + |\check{\beta}_{\vartheta\nu}|\bigr)\xi_\nu\right\}, \tag{6}$$

then the FOFBAMNNs (1) *and* (2) *can realize FTS based on hybrid controller* (4)*. Additionally, the settling time* ∂^Δ *is given as* $\partial^\Delta \leqslant \widetilde{T}_1$*,*

$$\widetilde{T}_1 = \left[\frac{\Gamma(1+\varepsilon)}{\sigma_2 2^{(s_2-2s_1)/s_1} s_2}\left(\left(V^{s_1}(\partial_0) + \left(\frac{\sigma_2}{\sigma_1}\right)^{s_1/(s_2-s_1)}\right)^{s_2/s_1} - \left(\frac{\sigma_2}{\sigma_1}\right)^{s_2/(s_2-s_1)}\right)\right]^{1/\varepsilon}$$
$$+ \partial_0. \tag{7}$$

Proof. Construct the Lyapunov function

$$U(\partial) = \sum_{\vartheta=1}^n \bigl|\tilde{e}_\vartheta(\partial)\bigr| + \sum_{\nu=1}^m \bigl|\tilde{z}_\nu(\partial)\bigr|.$$

Utilizing Lemma 2, we derive

$$ {}_{\partial_0}^{c}D_{\partial}^{\varepsilon} U(\partial) \leqslant \sum_{\vartheta=1}^n \operatorname{sign}\bigl(\tilde{e}_\vartheta(\partial)\bigr)\, {}_{t_0}^{c}D_t^{\varepsilon} \tilde{e}_\vartheta(\partial) + \sum_{\nu=1}^m \operatorname{sign}\bigl(\tilde{z}_\nu(\partial)\bigr)\, {}_{t_0}^{c}D_t^{\varepsilon} \tilde{z}_\nu(\partial). \tag{8}$$

Substituting (3) and (4) into (8) yields

$$\begin{aligned}
&{}_{\partial_0}^{c}D_{\partial}^{\varepsilon} U(\partial) \\
&\quad\leqslant \sum_{\vartheta=1}^n \operatorname{sign}\bigl(\tilde{e}_\vartheta(\partial)\bigr)\Biggl[-p_\vartheta \tilde{e}_\vartheta(\partial) + \sum_{\nu=1}^m \check{a}_{\vartheta\nu}\Theta_\nu\bigl(\tilde{\Lambda}_\nu(\partial)\bigr) - \sum_{\nu=1}^m \check{a}_{\vartheta\nu}\Theta_\nu\bigl(\Lambda_\nu(\partial)\bigr) \\
&\qquad + \bigwedge_{\nu=1}^m \check{\alpha}_{\vartheta\nu}\Theta_\nu\bigl(\tilde{\Lambda}_\nu(\partial - \breve{\tau})\bigr) - \bigwedge_{\nu=1}^m \check{\alpha}_{\vartheta\nu}\Theta_\nu\bigl(\Lambda_\nu(\partial - \breve{\tau})\bigr) \\
&\qquad + \bigvee_{\nu=1}^m \check{\beta}_{\vartheta\nu}\Theta_\nu\bigl(\tilde{\Lambda}_\nu(\partial - \breve{\tau})\bigr) - \bigvee_{\nu=1}^m \check{\beta}_{\vartheta\nu}\Theta_\nu\bigl(\Lambda_\nu(\partial - \breve{\tau})\bigr) \\
&\qquad - \tilde{\xi}_1 \operatorname{sign}\bigl(\tilde{e}_\vartheta(\partial)\bigr)\bigl|\tilde{e}_\vartheta(\partial - \breve{\tau})\bigr| - \tilde{\delta}_1 \tilde{e}_\vartheta(\partial) - \tilde{\omega}_1 \frac{\tilde{e}_\vartheta(\partial)}{|\tilde{e}_\vartheta(\partial)|^{s_1}} - \tilde{\rho}_1 \frac{\tilde{e}_\vartheta(\partial)}{|\tilde{e}_\vartheta(\partial)|^{s_2}}\Biggr]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\nu=1}^{m}\operatorname{sign}\big(\tilde z_\nu(\partial)\big)\Bigg[-q_\nu\tilde z_\nu(\partial)+\sum_{\vartheta=1}^{n}\hat b_{\nu\vartheta}\Xi_\vartheta\big(\tilde\chi_\vartheta(\partial)\big)-\sum_{\vartheta=1}^{n}\hat b_{\nu\vartheta}\Xi_\vartheta\big(\chi_\vartheta(\partial)\big)\\
&+\bigwedge_{\vartheta=1}^{n}\hat m_{\nu\vartheta}\Xi_\vartheta\big(\tilde\chi_\vartheta(\partial-\breve\tau)\big)-\bigwedge_{\vartheta=1}^{n}\hat m_{\nu\vartheta}\Xi_\vartheta\big(\chi_\vartheta(\partial-\breve\tau)\big)\\
&+\bigvee_{\vartheta=1}^{n}\hat w_{\nu\vartheta}\Xi_\vartheta\big(\tilde\chi_\vartheta(\partial-\breve\tau)\big)-\bigvee_{\vartheta=1}^{n}\hat w_{\nu\vartheta}\Xi_\vartheta\big(\chi_\vartheta(\partial-\breve\tau)\big)\\
&-\tilde\xi_2\operatorname{sign}\big(\tilde z_\nu(\partial)\big)\big|\tilde z_\nu(\partial-\breve\tau)\big|-\tilde\delta_2\tilde z_\nu(\partial)-\tilde\omega_2\frac{\tilde z_\nu(\partial)}{|\tilde z_\nu(\partial)|^{s_1}}-\tilde\rho_2\frac{\tilde z_\nu(\partial)}{|\tilde z_\nu(\partial)|^{s_2}}\Bigg]\\
\leqslant&\sum_{\vartheta=1}^{n}\Bigg[-p_\vartheta\big|\tilde e_\vartheta(\partial)\big|+\sum_{\nu=1}^{m}|\breve a_{\vartheta\nu}|\xi_\nu\big|\tilde z_\nu(\partial)\big|+\sum_{\nu=1}^{m}\big(|\breve\alpha_{\vartheta\nu}|+|\breve\beta_{\vartheta\nu}|\big)\xi_\nu\big|\tilde z_\nu(\partial-\breve\tau)\big|\\
&-\tilde\xi_1\big|\tilde e_\vartheta(\partial-\breve\tau)\big|-\tilde\delta_1\big|\tilde e_\vartheta(\partial)\big|-\tilde\omega_1\big|\tilde e_\vartheta(\partial)\big|^{1-s_1}-\tilde\rho_1\big|\tilde e_\vartheta(\partial)\big|^{1-s_2}\Bigg]\\
&+\sum_{\nu=1}^{m}\Bigg[-q_\nu\big|\tilde z_\nu(\partial)\big|+\sum_{\vartheta=1}^{n}|\hat b_{\nu\vartheta}|\rho_\vartheta|\tilde e_\vartheta(\partial)\big|+\sum_{\vartheta=1}^{n}\big(|\hat m_{\nu\vartheta}|+|\hat w_{\nu\vartheta}|\big)\rho_\vartheta\big|\tilde e_\vartheta(\partial-\breve\tau)\big|\\
&-\tilde\xi_2\big|\tilde z_\nu(\partial-\breve\tau)\big|-\tilde\delta_2\big|\tilde z_\nu(\partial)\big|-\tilde\omega_2\big|\tilde z_\nu(\partial)\big|^{1-s_1}-\tilde\rho_2\big|\tilde z_\nu(\partial)\big|^{1-s_2}\Bigg]\\
=&\sum_{\vartheta=1}^{n}\Bigg(-p_\vartheta-\tilde\delta_1+\sum_{\nu=1}^{m}|\hat b_{\nu\vartheta}|\rho_\vartheta\Bigg)\big|\tilde e_\vartheta(\partial)\big|-\sum_{\vartheta=1}^{n}\tilde\omega_1\big|\tilde e_\vartheta(\partial)\big|^{1-s_1}\\
&+\sum_{\nu=1}^{m}\Bigg(-q_\nu-\tilde\delta_2+\sum_{\vartheta=1}^{n}|\breve a_{\vartheta\nu}|\xi_\nu\Bigg)\big|\tilde z_\nu(\partial)\big|-\sum_{\nu=1}^{m}\tilde\omega_2\big|\tilde z_\nu(\partial)\big|^{1-s_1}\\
&+\sum_{\vartheta=1}^{n}\Bigg(-\tilde\xi_1+\sum_{\nu=1}^{m}\big(|\hat m_{\nu\vartheta}|+|\hat w_{\nu\vartheta}|\big)\rho_\vartheta\Bigg)\big|\tilde e_\vartheta(\partial-\breve\tau)\big|\\
&+\sum_{\nu=1}^{m}\Bigg(-\tilde\xi_2+\sum_{\vartheta=1}^{n}\big(|\breve\alpha_{\vartheta\nu}|+|\breve\beta_{\vartheta\nu}|\big)\xi_\nu\Bigg)\big|\tilde z_\nu(\partial-\breve\tau)\big|\\
&-\sum_{\vartheta=1}^{n}\tilde\rho_1\big|\tilde e_\vartheta(\partial)\big|^{1-s_2}-\sum_{\nu=1}^{m}\tilde\rho_2\big|\tilde z_\nu(\partial)\big|^{1-s_2}. \qquad (9)
\end{aligned}
$$

From (5), (6), and (9) one has

$$
\begin{aligned}
{}_{\partial_0}^{c}D_{\partial}^{\varepsilon}U(\partial)\leqslant&-\sum_{\vartheta=1}^{n}\tilde\omega_1\big|\tilde e_\vartheta(\partial)\big|^{1-s_1}-\sum_{\vartheta=1}^{n}\tilde\rho_1\big|\tilde e_\vartheta(\partial)\big|^{1-s_2}\\
&-\sum_{\nu=1}^{m}\tilde\omega_2\big|\tilde z_\nu(\partial)\big|^{1-s_1}-\sum_{\nu=1}^{m}\tilde\rho_2\big|\tilde z_\nu(\partial)\big|^{1-s_2}. \qquad (10)
\end{aligned}
$$

For $s_1 \geqslant 1$, this yields

$$\sum_{\vartheta=1}^{n} \left|\tilde{e}_\vartheta(\partial)\right|^{1-s_1} \geqslant n\left(\sum_{\vartheta=1}^{n} \tilde{e}_\vartheta(\partial)\right)^{1-s_1} \tag{11}$$

and

$$\sum_{\nu=1}^{m} \left|\tilde{z}_\nu(\partial)\right|^{1-s_1} \geqslant m\left(\sum_{\nu=1}^{m} \tilde{z}_\nu(\partial)\right)^{1-s_1}. \tag{12}$$

Similarly, for $s_1 < s_2 < 2s_1$, this yields

$$\sum_{\vartheta=1}^{n} \left|\tilde{e}_\vartheta(\partial)\right|^{1-s_2} \geqslant n\left(\sum_{\vartheta=1}^{n} \tilde{e}_\vartheta(\partial)\right)^{1-s_2} \tag{13}$$

and

$$\sum_{\nu=1}^{m} \left|\tilde{z}_\nu(\partial)\right|^{1-s_2} \geqslant m\left(\sum_{\nu=1}^{m} \tilde{z}_\nu(\partial)\right)^{1-s_2}. \tag{14}$$

From (10)–(14) one has

$$\begin{aligned}{}_{\partial_0}^{c}D_{\partial}^{\varepsilon}U(\partial) \leqslant{}& -n\tilde{\omega}_1 V_1^{-(s_1-1)}(\partial) - n\tilde{\rho}_1 V_1^{-(s_2-1)}(\partial)\\ &- m\tilde{\omega}_2 V_2^{-(s_1-1)}(\partial) - m\tilde{\rho}_2 V_2^{-(s_2-1)}(\partial).\end{aligned}$$

Let $\sigma_1 = \min\{n\tilde{\omega}_1, m\tilde{\omega}_2\} > 0$, $\sigma_2 = \min\{n\tilde{\rho}_1, m\tilde{\rho}_2\} > 0$. One has

$$\begin{aligned}{}_{\partial_0}^{c}D_{\partial}^{\varepsilon}U(\partial) \leqslant{}& -\sigma_1 V_1^{-(s_1-1)}(\partial) - \sigma_2 V_1^{-(s_2-1)}(\partial)\\ &- \sigma_1 V_2^{-(s_1-1)}(\partial) - \sigma_2 V_2^{-(s_2-1)}(\partial)\\ &- \sigma_2\bigl(V_1^{-(s_2-1)}(\partial) + V_2^{-(s_2-1)}(\partial)\bigr)\\ \leqslant{}& -\sigma_1 V^{-(s_1-1)}(\partial) - \sigma_2 V^{-(s_2-1)}(\partial).\end{aligned}$$

According to Lemma 4, the FOFBAMNNs in (1) can implement FTS with the FOFBAMNNs in (2) by employing the hybrid controller (4), and the corresponding settling time $\partial^{\triangle}$ can be determined by (7). □

Remark 2. The hybrid controller (4) consists of three components: $-\tilde{\delta}_1\tilde{e}_\vartheta(\partial)$, which mitigates the quasi-linear growth within the linear term; $-\tilde{\xi}_1 \operatorname{sign}(\tilde{e}_\vartheta(\partial))|\tilde{e}_\vartheta(\partial - \breve{\tau})|$, which mitigates the impacts of the time-delay term; and $-\tilde{\omega}_1\tilde{e}_\vartheta(\partial)/|\tilde{e}_\vartheta(\partial)|^{s_1} - \tilde{\rho}_1\tilde{e}_\vartheta(\partial)/|\tilde{e}_\vartheta(\partial)|^{s_2}$, which ensures FTS between systems (1) and (2) with $\tilde{\omega}_1$ and $\tilde{\rho}_1$ adjustable for synchronization time. Compared to linear feedback controllers [30], this hybrid approach is more adaptable and effective, especially in handling nonlinearity and high uncertainty.

Remark 3. The settling times in Theorem 1 and Lemma 4 are derived using algebraic inequalities that are straightforward to compute, making them easy to estimate in both theoretical studies and practical applications. Moreover, compared to the methods for estimating settling times in [10] and [13], the approach presented in this paper provides more precise results.

3.2 QS of FOFBAMNNs via adaptive controller

In this section, to ensure QS between FOFBAMNNs (1) and (2), we propose the following adaptive control strategy:

$$\begin{aligned}
u_\vartheta(\partial) &= \begin{cases} -\gamma_\vartheta^\chi(\partial)e_\vartheta(\partial) - \frac{\operatorname{sign}(\tilde{e}_\vartheta(\partial))\Psi_\vartheta^\chi(\partial)\tilde{e}_\vartheta^2(\partial-\breve{\tau})}{|e_\vartheta(\partial)|} \\ \quad + \frac{\varepsilon_1 \operatorname{sign}(\tilde{e}_\vartheta(\partial))}{|\tilde{e}_\vartheta(\partial)|}, & |\tilde{e}_\vartheta(\partial)| \neq 0, \\ 0, & |\tilde{e}_\vartheta(\partial)| = 0, \end{cases} \\
{}_{t_0}^{c}D_t^\varepsilon \gamma_\vartheta^\chi(\partial) &= \lambda_\vartheta \tilde{e}_\vartheta^2(\partial) - \mu_1\big(\gamma_\vartheta^\chi(\partial) - r_{1\vartheta}^*\big), \\
{}_{t_0}^{c}D_t^\varepsilon \Psi_\vartheta^\chi(\partial) &= \theta_\vartheta \tilde{e}_\vartheta^2(\partial-\breve{\tau}) - \mu_2\big(\Psi_\vartheta^\chi(\partial) - r_{2\vartheta}^*\big), \\
v_\nu(\partial) &= \begin{cases} -\gamma_\nu^\Lambda(\partial)\tilde{z}_\nu(\partial) - \frac{\operatorname{sign}(\tilde{z}_\nu(\partial))\Psi_\nu^\Lambda(\partial)\tilde{z}_\nu^2(\partial-\breve{\tau})}{|\tilde{z}_\nu(\partial)|} \\ \quad + \frac{\varepsilon_2 \operatorname{sign}(\tilde{z}_\nu(\partial))}{|\tilde{z}_\nu(\partial)|}, & |\tilde{z}_\nu(\partial)| \neq 0, \\ 0, & |\tilde{z}_\nu(\partial)| = 0, \end{cases} \\
{}_{t_0}^{c}D_t^\varepsilon \gamma_\nu^\Lambda(\partial) &= \zeta_\nu \tilde{z}_\nu^2(\partial) - \mu_3\big(\gamma_\nu^\Lambda(\partial) - r_{3\nu}^*\big), \\
{}_{t_0}^{c}D_t^\varepsilon \Psi_\nu^\Lambda(\partial) &= \eta_\nu \tilde{z}_\nu^2(\partial-\breve{\tau}) - \mu_4\big(\Psi_\nu^\Lambda(\partial) - r_{4\nu}^*\big).
\end{aligned} \tag{15}$$

Theorem 2. *Let Assumption* 1 *be satisfied, and let* $\bar{\omega} = \min\{\bar{\omega}_1, \bar{\omega}_2\}$, $h_1, h_2 < 0$, $\varepsilon = n\varepsilon_1 + m\varepsilon_2$. *If*

$$\begin{aligned}
\bar{\omega}_1 &= \min_{\vartheta\in\mathbb{N}_1^n}\left\{p_\vartheta - \sum_{\nu=1}^m \frac{\xi_\nu}{2}\big(|\check{a}_{\vartheta\nu}| + |\breve{\alpha}_{\vartheta\nu}| + |\breve{\beta}_{\vartheta\nu}|\big) + r_{1\vartheta}^* - \sum_{\nu=1}^m \frac{\rho_\vartheta}{2}|\hat{b}_{\nu\vartheta}|\right\} > 0, \\
\bar{\omega}_2 &= \min_{\nu\in\mathbb{M}_1^m}\left\{q_\nu - \sum_{\vartheta=1}^n \frac{\rho_\vartheta}{2}\big(|\hat{b}_{\nu\vartheta}| + |\hat{m}_{\nu\vartheta}| + |\hat{w}_{\nu\vartheta}|\big) + r_{3\nu}^* - \sum_{\vartheta=1}^n \frac{\xi_\nu}{2}|\check{a}_{\vartheta\nu}|\right\} > 0, \\
h_1 &= \max_{\vartheta\in\mathbb{N}_1^n}\left\{-r_{2\vartheta}^* + \sum_{\nu=1}^m \frac{\rho_\vartheta}{2}\big(|\hat{m}_{\nu\vartheta}| + |\hat{w}_{\nu\vartheta}|\big)\right\} < 0, \\
h_2 &= \max_{\nu\in\mathbb{M}_1^m}\left\{-r_{4\nu}^* + \sum_{\vartheta=1}^n \frac{\xi_\nu}{2}\big(|\breve{\alpha}_{\vartheta\nu}| + |\breve{\beta}_{\vartheta\nu}|\big)\right\} < 0,
\end{aligned}$$

QS can be implemented between FOFBAMNNs (1) *and* (2) *under controller* (15).

Proof. Construct the Lyapunov function $U(\partial) = U_1(\partial) + U_2(\partial)$, where

$$\begin{aligned}
U_1(\partial) &= \sum_{\vartheta=1}^n \tilde{e}_\vartheta^2(\partial) + \sum_{\nu=1}^m \tilde{z}_\nu^2(\partial), \\
U_2(\partial) &= \sum_{\vartheta=1}^n \frac{\big(\gamma_\vartheta^\chi(\partial) - r_{1\vartheta}^*\big)^2}{2\lambda_\vartheta} + \sum_{\vartheta=1}^n \frac{(\Psi_\vartheta^\chi(\partial) - r_{2\vartheta}^*)^2}{2\theta_\vartheta} \\
&\quad + \sum_{\nu=1}^m \frac{(\gamma_\nu^\Lambda(\partial) - r_{3\nu}^*)^2}{2\zeta_\nu} + \sum_{\nu=1}^m \frac{(\Psi_\nu^\Lambda(\partial) - r_{4\nu}^*)^2}{2\eta_\nu}.
\end{aligned}$$

Utilizing Lemma 1, we obtain

$$
\begin{aligned}
&{}_{\partial_0}^{c}D_{\partial}^{\varepsilon}U(\partial) \\
&\quad \leqslant \sum_{\vartheta=1}^{n}\tilde{e}_{\vartheta}(\partial)\,{}_{t_0}^{c}D_{t}^{\varepsilon}\tilde{e}_{\vartheta}(\partial)+\sum_{\nu=1}^{m}\tilde{z}_{\nu}(\partial)\,{}_{t_0}^{c}D_{t}^{\varepsilon}\tilde{z}_{\nu}(\partial) \\
&\qquad +\sum_{\vartheta=1}^{n}\frac{1}{\lambda_{\vartheta}}\big(\gamma_{\vartheta}^{\chi}(\partial)-r_{1\vartheta}^{*}\big)\,{}_{t_0}^{c}D_{t}^{\varepsilon}\gamma_{\vartheta}^{\chi}(\partial)+\sum_{\vartheta=1}^{n}\frac{1}{\theta_{\vartheta}}\big(\varPsi_{\vartheta}^{\chi}(\partial)-r_{2\vartheta}^{*}\big)\,{}_{t_0}^{c}D_{t}^{\varepsilon}\varPsi_{\vartheta}^{\chi}(\partial) \\
&\qquad +\sum_{\nu=1}^{m}\frac{1}{\zeta_{\nu}}\big(\gamma_{\nu}^{\varLambda}(\partial)-r_{3\nu}^{*}\big)\,{}_{t_0}^{c}D_{t}^{\varepsilon}\gamma_{\nu}^{\varLambda}(\partial)+\sum_{\nu=1}^{m}\frac{1}{\eta_{\nu}}\big(\varPsi_{\nu}^{\varLambda}(\partial)-r_{4\nu}^{*}\big)\,{}_{t_0}^{c}D_{t}^{\varepsilon}\varPsi_{\nu}^{\varLambda}(\partial). \qquad (16)
\end{aligned}
$$

Substituting (3) and (15) into (16) yields

$$
\begin{aligned}
{}_{\partial_0}^{c}D_{\partial}^{\varepsilon}U(\partial) \leqslant{}& \sum_{\vartheta=1}^{n}\tilde{e}_{\vartheta}(\partial)\Bigg[-p_{\vartheta}\tilde{e}_{\vartheta}(\partial)+\sum_{\nu=1}^{m}\check{a}_{\vartheta\nu}\varTheta_{\nu}\big(\tilde{\varLambda}_{\nu}(\partial)\big)-\sum_{\nu=1}^{m}\check{a}_{\vartheta\nu}\varTheta_{\nu}\big(\varLambda_{\nu}(\partial)\big) \\
&+\bigwedge_{\nu=1}^{m}\check{\alpha}_{\vartheta\nu}\varTheta_{\nu}\big(\tilde{\varLambda}_{\nu}(\partial-\breve{\tau})\big)-\bigwedge_{\nu=1}^{m}\check{\alpha}_{\vartheta\nu}\varTheta_{\nu}\big(\varLambda_{\nu}(\partial-\breve{\tau})\big) \\
&+\bigvee_{\nu=1}^{m}\check{\beta}_{\vartheta\nu}\varTheta_{\nu}\big(\tilde{\varLambda}_{\nu}(\partial-\breve{\tau})\big)-\bigvee_{\nu=1}^{m}\check{\beta}_{\vartheta\nu}\varTheta_{\nu}\big(\varLambda_{\nu}(\partial-\breve{\tau})\big)\Bigg] \\
&+\sum_{\nu=1}^{m}\tilde{z}_{\nu}(\partial)\Bigg[-q_{\nu}\tilde{z}_{\nu}(\partial)+\sum_{\vartheta=1}^{n}\hat{b}_{\nu\vartheta}\varXi_{\vartheta}\big(\tilde{\chi}_{\vartheta}(\partial)\big)-\sum_{\vartheta=1}^{n}\hat{b}_{\nu\vartheta}\varXi_{\vartheta}\big(\chi_{\vartheta}(\partial)\big) \\
&+\bigwedge_{\vartheta=1}^{n}\hat{m}_{\nu\vartheta}\varXi_{\vartheta}\big(\tilde{\chi}_{\vartheta}(\partial-\breve{\tau})\big)-\bigwedge_{\vartheta=1}^{n}\hat{m}_{\nu\vartheta}\varXi_{\vartheta}\big(\chi_{\vartheta}(\partial-\breve{\tau})\big) \\
&+\bigvee_{\vartheta=1}^{n}\hat{w}_{\nu\vartheta}\varXi_{\vartheta}\big(\tilde{\chi}_{\vartheta}(\partial-\breve{\tau})\big)-\bigvee_{\vartheta=1}^{n}\hat{w}_{\vartheta\nu}\varXi_{\vartheta}\big(\chi_{\vartheta}(\partial-\breve{\tau})\big)\Bigg] \\
&+\sum_{\vartheta=1}^{n}\big(\gamma_{\vartheta}^{\chi}(\partial)-r_{1\vartheta}^{*}\big)\tilde{e}_{\vartheta}^{2}(\partial)+\sum_{\vartheta=1}^{n}\big(\varPsi_{\vartheta}^{\chi}(\partial)-r_{2\vartheta}^{*}\big)\tilde{e}_{\vartheta}^{2}(\partial-\breve{\tau}) \\
&+\sum_{\nu=1}^{m}\big(\gamma_{\nu}^{\varLambda}(\partial)-r_{3\nu}^{*}\big)\tilde{z}_{\nu}^{2}(\partial)+\sum_{\nu=1}^{m}\big(\varPsi_{\nu}^{\varLambda}(\partial)-r_{4\nu}^{*}\big)\tilde{z}_{\nu}^{2}(\partial-\breve{\tau}) \\
&+\sum_{\vartheta=1}^{n}\tilde{e}_{\vartheta}(\partial)u_{\vartheta}(\partial)+\sum_{\nu=1}^{m}\tilde{z}_{\nu}(\partial)v_{\nu}(\partial) \\
&-\sum_{\vartheta=1}^{n}\frac{\mu_1}{\lambda_{\vartheta}}\big(\gamma_{\vartheta}^{\chi}(\partial)-r_{1\vartheta}^{*}\big)^2-\sum_{\vartheta=1}^{n}\frac{\mu_2}{\theta_{\vartheta}}\big(\varPsi_{\vartheta}^{\chi}(\partial)-r_{2\vartheta}^{*}\big)^2 \\
&-\sum_{\nu=1}^{m}\frac{\mu_3}{\zeta_{\nu}}\big(\gamma_{\nu}^{\varLambda}(\partial)-r_{3\nu}^{*}\big)^2-\sum_{\nu=1}^{m}\frac{\mu_4}{\eta_{\nu}}\big(\varPsi_{\nu}^{\varLambda}(\partial)-r_{4\nu}^{*}\big)^2. \qquad (17)
\end{aligned}
$$

With Assumption 1 and basic inequality $2ab \leqslant a^2 + b^2$ for any $a, b > 0$, this yields

$$\begin{aligned}
&\sum_{\vartheta=1}^{n} \tilde{e}_\vartheta(\partial)\left(\sum_{\nu=1}^{m} \check{a}_{\vartheta\nu}\Theta_\nu\big(\tilde{\Lambda}_\nu(\partial)\big) - \sum_{\nu=1}^{m} \hat{a}_{\vartheta\nu}\Theta_\nu\big(\Lambda_\nu(\partial)\big)\right) \\
&\quad \leqslant \sum_{\vartheta=1}^{n}\sum_{\nu=1}^{m} |\check{a}_{\vartheta\nu}|\xi_\nu\big|\tilde{z}_\nu(\partial)\big|\big|\tilde{e}_\nu(\partial)\big| \\
&\quad \leqslant \sum_{\vartheta=1}^{n}\sum_{\nu=1}^{m} |\check{a}_{\vartheta\nu}|\frac{\xi_\nu}{2}\big(\tilde{e}_\vartheta^2(\partial) + \tilde{z}_\nu^2(\partial)\big).
\end{aligned} \tag{18}$$

Similarly, we derive

$$\begin{aligned}
&\sum_{\nu=1}^{m} \tilde{z}_\nu(\partial)\left(\sum_{\vartheta=1}^{n} \hat{b}_{\nu\vartheta}\Xi_\vartheta\big(\tilde{\chi}_\vartheta(\partial)\big) - \sum_{\vartheta=1}^{n} \hat{b}_{\nu\vartheta}\Xi_\vartheta\big(\chi_\vartheta(\partial)\big)\right) \\
&\quad \leqslant \sum_{\nu=1}^{m}\sum_{\vartheta=1}^{n} |\hat{b}_{\nu\vartheta}|\frac{\rho_\vartheta}{2}\big(\tilde{e}_\vartheta^2(\partial) + \tilde{z}_\nu^2(\partial)\big).
\end{aligned} \tag{19}$$

According to Assumption 1 and Lemma 6, we have

$$\begin{aligned}
&\tilde{e}_\vartheta(\partial)\left(\bigwedge_{\nu=1}^{m} \check{\alpha}_{\vartheta\nu}\Theta_\nu\big(\tilde{\Lambda}_\nu(\partial - \breve{\tau})\big) - \bigwedge_{\nu=1}^{m} \check{\alpha}_{\vartheta\nu}\Theta_\nu\big(\Lambda_\nu(\partial - \breve{\tau})\big)\right) \\
&\quad \leqslant \sum_{\nu=1}^{m} |\check{\alpha}_{\vartheta\nu}|\frac{\xi_\nu}{2}\big(\tilde{e}_\vartheta^2(\partial) + \tilde{z}_\nu^2(\partial - \breve{\tau})\big),
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\tilde{e}_\vartheta(\partial)\left(\bigvee_{\nu=1}^{m} \breve{\beta}_{\vartheta\nu}\Theta_\nu\big(\tilde{\Lambda}_\nu(\partial - \breve{\tau})\big) - \bigvee_{\nu=1}^{m} \breve{\beta}_{\vartheta\nu}\Theta_\nu\big(\Lambda_\nu(\partial - \breve{\tau})\big)\right) \\
&\quad \leqslant \sum_{\nu=1}^{m} |\breve{\beta}_{\vartheta\nu}|\frac{\xi_\nu}{2}\big(\tilde{e}_\vartheta^2(\partial) + \tilde{z}_\nu^2(\partial - \breve{\tau})\big),
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\tilde{z}_\nu(\partial)\left(\bigwedge_{\vartheta=1}^{n} \hat{m}_{\nu\vartheta}\Xi_\vartheta\big(\tilde{\chi}_\vartheta(\partial - \breve{\tau})\big) - \bigwedge_{\vartheta=1}^{n} \hat{m}_{\nu\vartheta}\Xi_\vartheta\big(\chi_\vartheta(\partial - \breve{\tau})\big)\right) \\
&\quad \leqslant \sum_{\vartheta=1}^{n} |\hat{m}_{\nu\vartheta}|\frac{\rho_\vartheta}{2}\big(\tilde{z}_\nu^2(\partial) + \tilde{e}_\vartheta^2(\partial - \breve{\tau})\big),
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\tilde{z}_\nu(\partial)\left(\bigwedge_{\vartheta=1}^{n} \hat{w}_{\nu\vartheta}\Xi_\vartheta\big(\tilde{\chi_\vartheta}(\partial - \breve{\tau})\big) - \bigwedge_{\vartheta=1}^{n} \hat{w}_{\nu\vartheta}\Xi_\vartheta\big(\chi_\vartheta(\partial - \breve{\tau})\big)\right) \\
&\quad \leqslant \sum_{\vartheta=1}^{n} |\hat{w}_{\nu\vartheta}|\frac{\rho_\vartheta}{2}\big(\tilde{z}_\nu^2(\partial) + \tilde{e}_\vartheta^2(\partial - \breve{\tau})\big).
\end{aligned} \tag{23}$$

Substituting (18)–(23) and (15) into (17), one has

$$
\begin{aligned}
&{}_{\partial_0}^{c}D_{\partial}^{\varepsilon}U(\partial)\\
&\quad\leqslant \sum_{\vartheta=1}^{n}\Biggl[-p_\vartheta+\sum_{\nu=1}^{m}\frac{\xi_\nu}{2}\bigl(|\breve{a}_{\vartheta\nu}|+|\breve{\alpha}_{\vartheta\nu}|+|\breve{\beta}_{\vartheta\nu}|\bigr)-r_{1\vartheta}^{*}+\sum_{\nu=1}^{m}\frac{\rho_\vartheta}{2}|\hat{b}_{\nu\vartheta}|\Biggr]\tilde{e}_\vartheta^2(\partial)\\
&\qquad+\sum_{\nu=1}^{m}\Biggl[-q_\nu+\sum_{\vartheta=1}^{n}\frac{\rho_\vartheta}{2}\bigl(|\hat{b}_{\nu\vartheta}|+|\hat{m}_{\nu\vartheta}|+|\hat{w}_{\nu\vartheta}|\bigr)-r_{3\nu}^{*}+\sum_{\vartheta=1}^{n}\frac{\xi_\nu}{2}|\breve{a}_{\vartheta\nu}|\Biggr]\tilde{z}_\nu^2(\partial)\\
&\qquad+\sum_{\nu=1}^{m}\Biggl[-r_{4\nu}^{*}+\sum_{\vartheta=1}^{n}\frac{\xi_\nu}{2}\bigl(|\breve{\alpha}_{\vartheta\nu}|+|\breve{\beta}_{\vartheta\nu}|\bigr)\Biggr]\tilde{z}_\nu^2(\partial-\breve{\tau})+\sum_{\vartheta=1}^{n}\varepsilon_1+\sum_{\nu=1}^{m}\varepsilon_2\\
&\qquad+\sum_{\vartheta=1}^{n}\Biggl[-r_{2\vartheta}^{*}+\sum_{\nu=1}^{m}\frac{\rho_\vartheta}{2}\bigl(|\hat{m}_{\nu\vartheta}|+|\hat{w}_{\nu\vartheta}|\bigr)\Biggr]\tilde{e}_\vartheta^2(\partial-\breve{\tau}).
\end{aligned}
$$

Then from the conditions of the theorem we can derive

$$
\begin{aligned}
{}_{\partial_0}^{c}D_{\partial}^{\varepsilon}U(\partial)&\leqslant-\sum_{\vartheta=1}^{n}\bar{\omega}_1\tilde{e}_\vartheta^2(\partial)-\sum_{\nu=1}^{m}\bar{\omega}_2\tilde{z}_\nu^2(\partial)+\varepsilon\\
&\leqslant-\bar{\omega}U_1(\partial)+\varepsilon.
\end{aligned}\tag{24}
$$

Utilizing Lemma 5 and according to (24), this yields

$$
U_1(\partial)\leqslant\biggl(U(\partial_0)-\frac{\varepsilon}{\bar{\omega}}\biggr)E_\varepsilon\bigl(-\bar{\omega}(\partial-\partial_0)^\varepsilon\bigr)+\frac{\varepsilon}{\bar{\omega}},\quad \partial\geqslant\partial_0+\biggl(\frac{\Gamma(\varepsilon)}{\bar{\omega}}\biggr)^{1/(1-\varepsilon)}.\tag{25}
$$

It can be derived from Definition 4 and (25) that

$$
\bigl\|E(\partial)\bigr\|_2\leqslant\sqrt{\biggl(U(\partial_0)-\frac{\varepsilon}{\bar{\omega}}\biggr)E_\varepsilon\bigl(-\bar{\omega}(\partial-\partial_0)^\varepsilon\bigr)+\frac{\varepsilon}{\bar{\omega}}}.
$$

Utilizing Lemma 3, one has

$$
\lim_{\partial\to+\infty}\bigl\|E(\partial)\bigr\|_2\leqslant\biggl(\frac{\varepsilon}{\bar{\omega}}\biggr)^{1/2},
$$

the proof is completed. □

Remark 4. Unlike adaptive controllers [16] and linear feedback controllers [30], this paper proposes a novel adaptive nonlinear feedback controller specifically designed to tackle the QS problem in FOFBAMNNs for the first time. As a result, controller (15) we developed is more efficient, widely applicable, and less conservative.

Remark 5. The adaptive nonlinear controller (15) herein includes adaptive nonlinear controller $u_{1\vartheta}(\partial)=-\gamma_\vartheta^\chi(\partial)e_\vartheta(\partial)$, adaptive delayed controller $u_{2\vartheta}(\partial)=-\operatorname{sign}(\tilde{e}_\vartheta(\partial))\times \Psi_\vartheta^\chi(\partial)\tilde{e}_\vartheta^2(\partial-\breve{\tau})/|e_\vartheta(\partial)|$, and nonlinear controller $u_{3\vartheta}(\partial)=\varepsilon_1\operatorname{sign}(\tilde{e}_\vartheta(\partial))/|\tilde{e}_\vartheta(\partial)|$. It should be noted that $u_{2\vartheta}(\partial)$ is constructed to counteract negative impact from time delays, while $u_{1\vartheta}(\partial)$ and $u_{3\vartheta}(\partial)$ are to ensure QS between systems (1) and (2).

4 Illustrative examples

Two examples are provided below to further demonstrate the effectiveness of the conclusions derived from the theorem.

Example 1. The two-dimensional FOFBAMNNs models are considered in the following:

$$\begin{aligned}
{}_{\partial_0}^{c}D_{\partial}^{\varepsilon}\chi_\vartheta(\partial) &= -p_\vartheta\chi_\vartheta(\partial) + \sum_{\nu=1}^{2}\breve{a}_{\vartheta\nu}\Theta_\nu\big(\Lambda_\vartheta(\partial)\big) + \bigwedge_{\nu=1}^{2}\breve{\alpha}_{\vartheta\nu}\Theta_\nu\big(\Lambda_\nu(\partial-\breve{\tau})\big) \\
&\quad + \bigvee_{\nu=1}^{2}\breve{\beta}_{\vartheta\nu}\Theta_\nu\big(\Lambda_\nu(\partial-\breve{\tau})\big) + \breve{I}_\vartheta(\partial), \\
{}_{\partial_0}^{c}D_{\partial}^{\varepsilon}\Lambda_\nu(\partial) &= -q_\nu\Lambda_\nu(\partial) + \sum_{\vartheta=1}^{2}\hat{b}_{\nu\vartheta}\Xi_\vartheta\big(\chi_\vartheta(\partial)\big) + \bigwedge_{\vartheta=1}^{2}m_{\nu\vartheta}\Xi_\vartheta\big(\chi_\vartheta(\partial-\breve{\tau})\big) \\
&\quad + \bigvee_{\vartheta=1}^{2}\hat{w}_{\nu\vartheta}\Xi_\vartheta\big(\chi_\vartheta(\partial-\breve{\tau})\big) + \breve{\mathcal{I}}_\nu(\partial),
\end{aligned} \tag{26}$$

where $\varepsilon = 0.98$, $\vartheta,\nu = 1,2$, $p_1 = p_2 = 0.01$, $q_1 = q_2 = 0.01$, $\breve{a}_{11} = -0.4$, $\breve{a}_{12} = -0.25$, $\breve{a}_{21} = 0.3$, $\breve{a}_{22} = -0.15$, $\hat{b}_{11} = -0.3$, $\hat{b}_{12} = -0.15$, $\hat{b}_{21} = -0.25$, $\hat{b}_{22} = -0.2$, $\breve{\alpha}_{11} = 0.03$, $\breve{\alpha}_{12} = -0.05$, $\breve{\alpha}_{21} = -0.03$, $\breve{\alpha}_{22} = 0.02$, $\hat{m}_{11} = 0.04$, $\hat{m}_{12} = -0.05$, $\hat{m}_{21} = 0.05$, $\hat{m}_{22} = -0.1$, $\breve{\beta}_{11} = 0.2$, $\breve{\beta}_{12} = -0.05$, $\breve{\beta}_{21} = -0.12$, $\breve{\beta}_{22} = 0.3$, $\hat{w}_{11} = -0.05$, $\hat{w}_{12} = -0.05$, $\hat{w}_{21} = 0.3$, $\hat{w}_{22} = -0.1$, $\breve{\tau} = 1$. Besides, the activation functions are selected as $\Theta_\nu(\cdot) = \Xi_\vartheta(\cdot) = \tanh(\cdot)$, $\breve{I}_\vartheta(\partial) = \breve{\mathcal{I}}_\nu(\partial) = 0$. By calculation, we have $\xi_1 = \xi_2 = \rho_1 = \rho_2 = 1$. All the initial value of system (26) are set as

$$\begin{aligned}
\chi_1(s) &= -2.01, & \Lambda_1(s) &= -2.15, \\
\chi_2(s) &= -1, & \Lambda_2(s) &= 1.5.
\end{aligned}$$

Figure 1 expresses the changes of state variables without a controller, the controlled FOFBAMNNs is depicted by

$$\begin{aligned}
{}_{t_0}^{c}D_{t}^{\varepsilon}\chi_\vartheta(\partial) &= -p_\vartheta\chi_\vartheta(\partial) + \sum_{\nu=1}^{2}\breve{a}_{\vartheta\nu}\Theta_\nu\big(\Lambda_\vartheta(\partial)\big) + \bigwedge_{\nu=1}^{2}\breve{\alpha}_{\vartheta\nu}\Theta_\nu\big(\Lambda_\nu(\partial-\breve{\tau})\big) \\
&\quad + \bigvee_{\nu=1}^{2}\breve{\beta}_{\vartheta\nu}\Theta_\nu\big(\Lambda_\nu(\partial-\breve{\tau})\big) + \breve{I}_\vartheta(\partial) + u_\vartheta(\partial), \\
{}_{t_0}^{c}D_{t}^{\varepsilon}\Lambda_\nu(\partial) &= -q_\nu\Lambda_\nu(\partial) + \sum_{\vartheta=1}^{2}\hat{b}_{\nu\vartheta}\Xi_\vartheta\big(\chi_\vartheta(\partial)\big) + \bigwedge_{\vartheta=1}^{2}m_{\nu\vartheta}\Xi_\vartheta\big(\chi_\vartheta(\partial-\breve{\tau})\big) \\
&\quad + \bigvee_{\vartheta=1}^{2}\hat{w}_{\nu\vartheta}\Xi_\vartheta\big(\chi_\vartheta(\partial-\breve{\tau})\big) + \breve{\mathcal{I}}_\nu(\partial) + v_\nu(\partial),
\end{aligned} \tag{27}$$

where $\vartheta,\nu = 1,2$, and the initial value is chosen as

$$\begin{aligned}
\tilde{\chi}_1(s) &= -2.06, & \tilde{\Lambda}_1(s) &= 2.7, \\
\tilde{\chi}_2(s) &= 2.1, & \tilde{\Lambda}_2(s) &= -2.3.
\end{aligned}$$

The other parameters of FOFBAMNNs (27) are the same as those of (26). To ensure FTS, controller (4) is designed with its parameters configured as follows: $\tilde{\xi}_1 = \tilde{\xi}_2 = 0.75$, $\tilde{\delta}_1 = \tilde{\delta}_2 = 17$, $\tilde{\omega}_1 = \tilde{\omega}_2 = 0.08$, $\tilde{\rho}_1 = \tilde{\rho}_2 = 0.08$, $s_1 = 2.01$, $s_2 = 2.1$. By calculation, we can verify that (5)–(6) are satisfied. Then systems (26) and (27) can reach FTS. Additionally, the setting time $\partial^\triangle$ is estimated as

$$\partial^\triangle \leqslant \left[\frac{\Gamma(1+\varepsilon)}{\sigma_2 2^{(s_2-2s_1)/s_1} s_2}\left(\left(V^{s_1}(\partial_0) + \left(\frac{\sigma_2}{\sigma_1}\right)^{s_1/(s_2-s_1)}\right)^{s_2/s_1} - \left(\frac{\sigma_2}{\sigma_1}\right)^{s_2/(s_2-s_1)}\right)\right]^{1/\varepsilon} + \partial_0 = 7.8556.$$

Figure 2 illustrates the variations in state variables when applying controller (4), indicating that systems (26) and (27) can implement FTS under controller (4). The error curves for the drive–response system, $\tilde{e}_1(\partial)$, $\tilde{e}_2(\partial)$, $\tilde{z}_1(\partial)$, and $\tilde{z}_2(\partial)$, are depicted in Fig. 3. As seen in Fig. 3, $\tilde{e}_1(\partial)$ and $\tilde{e}_2(\partial)$ both become zero, along with $\tilde{z}_1(\partial)$ and $\tilde{z}_2(\partial)$ when $\partial > 0.5$, which provides additional verification of Theorem 1.

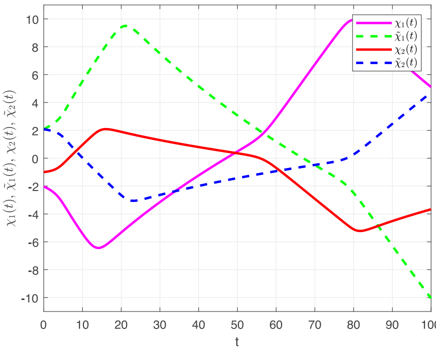
Example 2. For the delayed FOFBAMNNs (26) and (27), we choose: $\varepsilon = 0.98$, $\vartheta, \nu = 1, 2$, $p_1 = p_2 = 0.3$, $q_1 = q_2 = 0.4$, $\breve{a}_{11} = 0.1$, $\breve{a}_{12} = -0.25$, $\breve{a}_{21} = 0.2$, $\breve{a}_{22} = -0.15$, $\hat{b}_{11} = -0.3$, $\hat{b}_{12} = -0.15$, $\hat{b}_{21} = 0.15$, $\hat{b}_{22} = -0.2$, $\breve{\alpha}_{11} = 0.13$, $\breve{\alpha}_{12} = -0.05$, $\breve{\alpha}_{21} = -0.15$, $\breve{\alpha}_{22} = 0.02$, $\hat{m}_{11} = 0.14$, $\hat{m}_{12} = -0.15$, $\hat{m}_{21} = 0.15$, $\hat{m}_{22} = -0.5$, $\breve{\beta}_{11} = 0.14$, $\breve{\beta}_{12} = -0.05$, $\breve{\beta}_{21} = -0.12$, $\breve{\beta}_{22} = 0.6$, $\hat{w}_{11} = -0.15$, $\hat{w}_{12} = -0.05$, $\hat{w}_{21} = 0.16$, $\hat{w}_{22} = -0.3$, $\breve{\tau} = 1$. Besides, the activation functions are selected as $\Theta_\nu(\cdot) = \Xi_\vartheta(\cdot) = \tanh(\cdot)$, $\breve{I}_\vartheta(\partial) = \breve{\mathcal{I}}_\nu(\partial) = 0$. By calculation, we have $\xi_1 = \xi_2 = \rho_1 = \rho_2 = 1$. All initial values of systems (26) and (27) are reset as follows:

$$\begin{aligned} &\chi_1(s) = 0.01, \qquad \chi_2(s) = 1, \qquad \Lambda_1(s) = 0.05, \qquad \Lambda_2(s) = -0.5, \\ &\tilde{\chi}_1(s) = 0.06, \qquad \tilde{\chi}_2(s) = 0.1, \qquad \tilde{\Lambda}_1(s) = 0.0.7, \qquad \tilde{\Lambda}_2(s) = -0.3. \end{aligned}$$

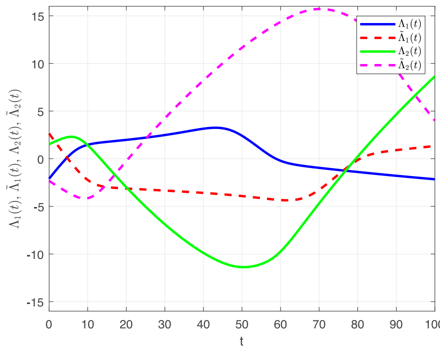
Figure 4 depicts the phase trajectories of system (26) without controller. We can see from Fig. 4 that system (26) has chaotic behavior. The behavior of the system is difficult to predict and is highly sensitive to initial conditions. Figure 5 displays the states of systems (26) and (27) without a controller. From the figure we can clearly observe that the trajectories of the two systems do not converge over time, indicating that synchronization is not achieved.

To implement QS, for controller (15), we choose $\varepsilon_1 = \varepsilon_2 = 0.01$, $\lambda_1 = \lambda_2 = 0.2$, $\theta_1 = \theta_2 = 0.2$, $\zeta_1 = \zeta_2 = 0.1$, $\eta_1 = \eta_2 = 0.1$, $\mu_1 = \mu_2 = 0.6$, $\mu_3 = \mu_4 = 1.5$, $r^*_{11} = 2$, $r^*_{12} = 2.2$, $r^*_{21} = 2.4$, $r^*_{22} = 2.5$, $r^*_{31} = 0.65$, $r^*_{32} = 0.6$, $r^*_{41} = 0.68$, $r^*_{42} = 0.7$. By calculation, it can be obtained that $\bar{\omega}_1 = 1.705$, $\bar{\omega}_2 = 0.07$, $h_1 = -2$, $h_2 = -0.34$, $\varepsilon = 0.04$ that satisfies the theorem conditions. Hence, systems (26) and (27) can achieve QS with controller (15). Additionally, the error bond is estimated as $(\varepsilon/\bar{\omega})^{1/2} \approx 0.7559$.

Figure 6 displays the curves of state variables with controller (15). It is easy to see that synchronization between systems (26) and (27) is not possible. Figure 7 shows the error norm curves, which fluctuate within a narrow range over time. That is, the error functions

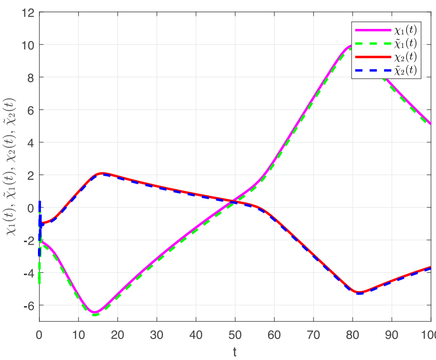


(a) $\chi_\vartheta(\partial)$ and $\tilde{\chi}_\vartheta(\partial)$

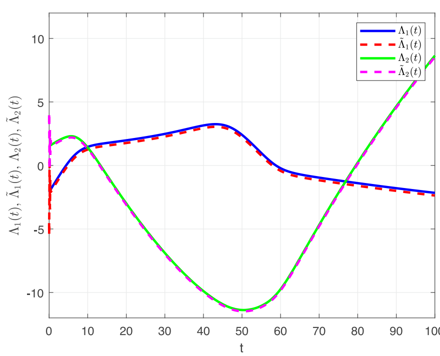


(b) $\Lambda_\nu(\partial)$ and $\tilde{\Lambda}_\nu(\partial)$

Figure 1. Trajectory of state variables without a controller.



(a) $\chi_\vartheta(\partial)$ and $\tilde{\chi}_\vartheta(\partial)$



(b) $\Lambda_\nu(\partial)$ and $\tilde{\Lambda}_\nu(\partial)$

Figure 2. Trajectory of the state variables based under controller (4).

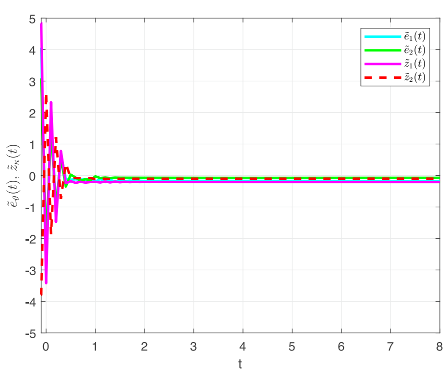
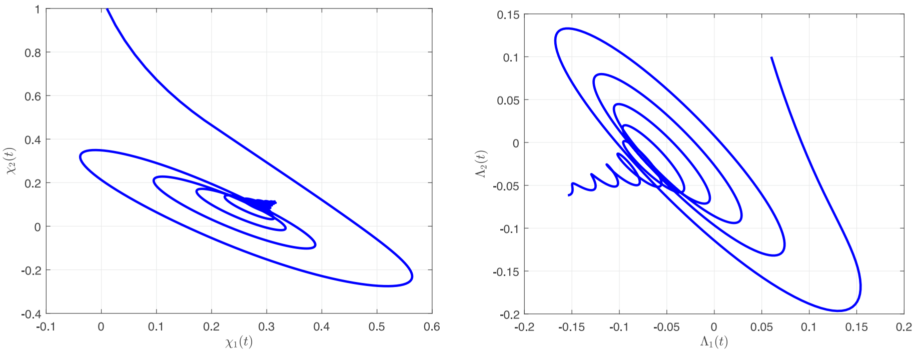
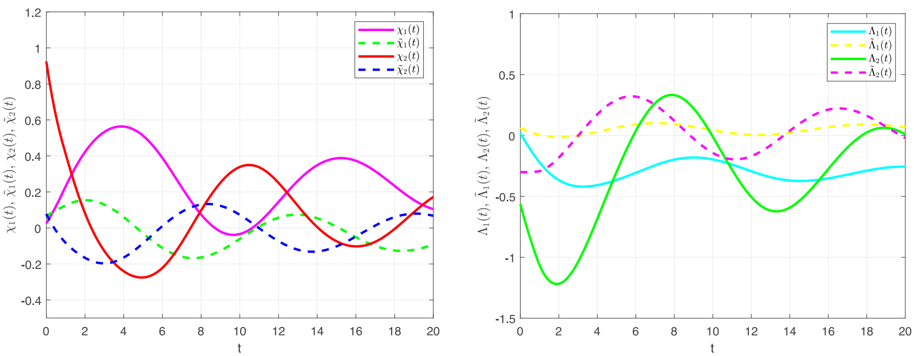


Figure 3. Synchronization errors $\tilde{e}_\vartheta(\partial)$ and $\tilde{z}_\nu(\partial)$ under controller (4).



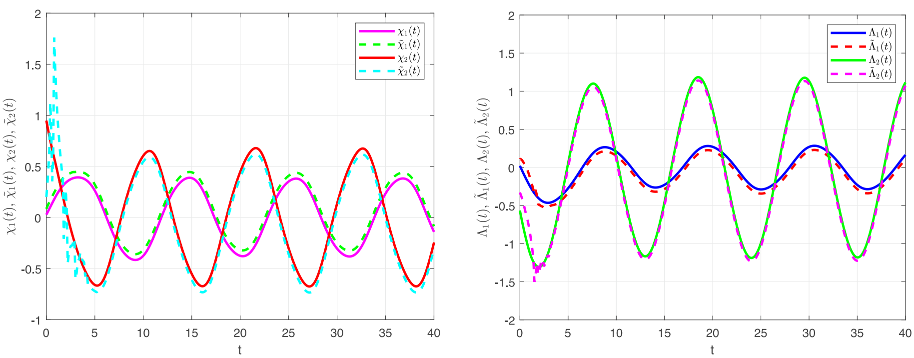
(a) $\chi_1(t)$ and $\chi_2(t)$ (b) $\Lambda_1(t)$ and $\Lambda_2(t)$

Figure 4. Phase trajectories of FOFBAMNN (26).



(a) $\chi_\vartheta(\partial)$ and $\tilde{\chi}_\vartheta(\partial)$ (b) $\Lambda_\nu(\partial)$ and $\tilde{\Lambda}_\nu(\partial)$

Figure 5. Trajectory of state variables without a controller.



(a) $\chi_\vartheta(\partial)$ and $\tilde{\chi}_\vartheta(\partial)$ (b) $\Lambda_\nu(\partial)$ and $\tilde{\Lambda}_\nu(\partial)$

Figure 6. Trajectory of the state variables based under controller (15).

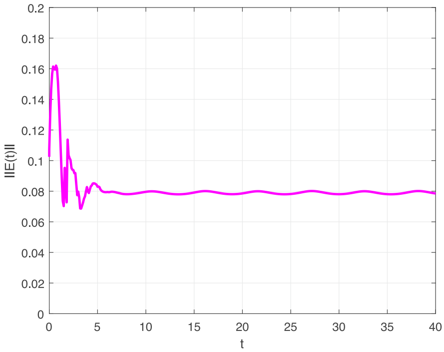
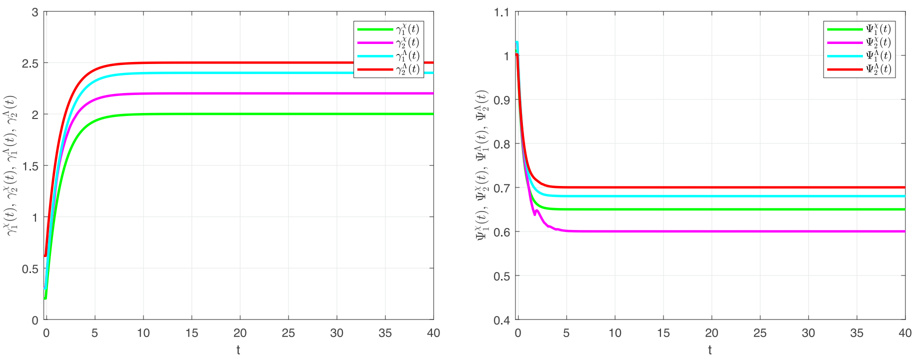


Figure 7. Synchronization error with controller (15) between systems (26) and (27).



(a) $\gamma_\vartheta^\chi(\partial)$ and $\gamma_\nu^\Lambda(\partial)$ (b) $\Psi_\vartheta^x(\partial)$ and $\Psi_\nu^\Lambda(\partial)$

Figure 8. The evolution of adaptive feedback gains, $\vartheta, \nu = 1, 2$.

tends to be stable. Figures 6 and 7 illustrate that systems (26) and (27) can achieve QS under controller (15). Figure 8 illustrates the evolution of the adaptive feedback gains of controller (15).

5 Conclusion

This paper focuses on addressing the FTS and QS problems in FOFBAMNNs. Unlike previous studies that used adaptive controllers [16] or linear feedback controllers [30], this work proposes two novel controllers specifically designed for FTS and QS. Based on the finite-time theorem, Lyapunov theory, and inequality techniques, the paper establishes several criteria to achieve FTS and QS. In addition, the QS error analysis can be extended to the CS of complex-valued FOFBAMNNs with time-varying delays and parameter uncertainties. Two examples are provided to validate the theoretical results. It is also noted that in many cases, accurately measuring state delays is impractical, making the development of finite-time convergence results under time delays an important topic for future research.

Conflicts of interest. The authors declare no conflicts of interest.

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