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Background risk model in presence of heavy tails under dependence

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Abstract. In this paper, we examine two problems on applied probability, which are directly connected with the dependence in presence of heavy tails. The first problem is related to max-sum equivalence of the randomly weighted sums in bivariate setup. Introducing a new dependence, called generalized tail asymptotic independence, we establish the bivariate max-sum equivalence under a rather general dependence structure when the primary random variables follow distributions from the intersection of the dominatedly varying and the long-tailed distributions. Based on this max-sum equivalence, we provide a result about the asymptotic behavior of two kinds of ruin probabilities over a finite-time horizon in a bivariate renewal risk model with constant interest rate. The second problem is related to the asymptotic behavior of the tail distortion risk measure in a static portfolio called background risk model. In opposite to other approaches on this topic, we use a general enough assumption that is based on multivariate regular variation.

Keywords: joint tail behavior, randomly weighted sums, tail distortion risk measure, bivariate renewal risk model, interdependence, multivariate regular variation.

1 Introduction

1.1 Concepts and motivation

In last decades, the distributions with heavy tails play a crucial role in applied probability, and especially in risk theory and risk management; see [17, 18, 22], etc.

Simultaneously, the dependence modeling among risks seems to have equally important impact on insurance applications, while keeps the mathematical interest with respect to generalizations either of some independent results or of some counterexamples in which the independent results cannot be generalized. Hence, we observe that the study of dependent models, combined with the presence of heavy tails, presents a useful tool in applications and, at the same time, a strong mathematical support.

In this paper, we will explore the concept of interdependence in the sense of a complex dependence between two sequences of random variables whose distributions are from the heavy-tailed class. In order to make it clear, we depict in two ways the interdependence that are used in this paper and in the frame of the new results.

At first, we understand the interdependence as a structure between two finite sequences of primary random variables $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$, which are heavy-tailed distributed. Each sequence contains dependent components, but simultaneously the two sequences are also mutually dependent. In this sense, we introduce a new dependence structure, called generalized tail asymptotic independent, symbolically, GTAI, in Definition 1 below, which belongs to the family of second-order asymptotic independence. Next, in Section 2, we establish the max-sum equivalence of randomly weighted sums within bivariate framework, where random weights $\Theta_1, \dots, \Theta_n$, $\Delta_1, \dots, \Delta_m$ are bounded from above, nonnegative, and nondegenerate to zero. These weights are arbitrarily dependent on each other and independent of $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ under dominatedly varying and long-tailed distributions for the primary variables. Namely, we establish the asymptotic relation

$$\mathbf{P}\left[\sum_{i=1}^{n} \Theta_i X_i > x,\ \sum_{j=1}^{m} \Delta_j Y_j > y\right] \sim \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{P}[\Theta_i X_i > x,\ \Delta_j Y_j > y] \tag{1}$$

as $x \wedge y \to \infty$. The study of relation (1) through this kind of interdependence covers a gap in the literature, and it includes many of the already existing results as it is discussed in Section 2. Next, in Section 3, the establishment of the asymptotic equivalence in (1) helps to provide asymptotic expressions for two types of ruin probability over finite-time horizon within the frame of bivariate, continuous-time risk model. This model assumes a constant force of interest and a common renewal counting process for the two lines of business. In Section 4, we meet a second type of interdependence. In this section, we focus on asymptotic behavior of a risk measure, called the tail distortion risk measure, symbolically, TDRM, with respect to a model known as background risk model. Accordingly, the following types of quantities play a direct role:

$$\sum_{i=1}^{n} \Theta_i X_i. \tag{2}$$

They are called randomly weighted sums. The interdependence is now expressed by the dependence among the components of vectors $\Theta = (\Theta_1, \dots, \Theta_n)$ and $\mathbf{X} = (X_1, \dots, X_n)$, and simultaneously between the vectors Θ and $\mathbf{X}$. Such dependence structures were studied in the literature under the framework of multivariate regular variation for the distribution of $\mathbf{X}$. However, in larger classes of heavy tailed distributions for the components of $\mathbf{X}$, the interdependence effect appears rarely. The reason lies in the difficulty to find max-sum equivalence for the randomly weighted sums, namely,

$$\mathbf{P}\left[\sum_{i=1}^{n} \Theta_i X_i > x\right] \sim \sum_{i=1}^{n} \mathbf{P}[\Theta_i X_i > x] \tag{3}$$

as $x \wedge y \to \infty$. We refer to [18, 34] for papers that studied relation (3) under a variety of dependence structures and several classes of heavy-tailed distributions. In the paper [7], relation (3) was established for the first time through interdependence in the case $n = 2$ within distribution classes larger than regular variation.

We remain in the frame of regular variation for some of $\Theta_1 X_1, \dots, \Theta_n X_n$, but under the relaxed assumption that $\Theta\mathbf{X}$ follows a multivariate regular varying distribution. We allow a wide spectrum of dependencies among the components of the sum in relation (2). Furthermore, our assumption for the products is easily verified when one of Θ and $\mathbf{X}$ follows multivariate regularly varying distribution through several applications of Breiman's theorem on multivariate setup.

Finally, in the last section, we study the asymptotic behavior of TDRM in a backgroung risk model, which satisfies the general assumption that $\Theta\mathbf{X}$ follows MRV.

1.2 Notations

In this subsection, after some necessary notations, we introduce the preliminary material for the heavy-tailed distribution classes. Let us denote $\mathbf{x} = (x_1, \dots, x_n)$, the scalar product $c\mathbf{x} = (cx_1, \dots, cx_n)$, $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, $x^+ = x \vee 0$, $x- = (-x \vee 0)$, $\lfloor x \rfloor$ is the integer part of x, and e_i represents the vector whose all the components are 0, except the ith that is 1. By $\mathbf{1}_{\{A\}}$ we denote the indicator function on the set A. For two positive functions f and g, we write $f(x) = O(g(x))$ as $x \to \infty$ if

$$\limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty$$

and $f(x) = o(g(x))$ as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

We write $f(x) \sim cg(x)$ as $x \to \infty$ for some $c \in (0, \infty)$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = c.$$

We write $f(x) \asymp g(x)$ if both $f(x) = O(g(x))$ and $g(x) = O(f(x))$ hold as $x \to \infty$. All the previous asymptotic notations hold for $x \wedge y \to \infty$ when we have positive, bivariate functions. For example, we write $f(x, y) \sim cg(x, y)$ as $x \wedge y \to \infty$ with $c \in (0, \infty)$ if

$$\lim_{x\wedge y\to\infty} \frac{f(x, y)}{g(x, y)} = c.$$

Let us denote by $V(x) = \mathbf{P}(Z \leqslant x)$ the distribution of the random variable Z and by $\overline{V}(x) := 1 - V(x) = \mathbf{P}(Z > x)$ its tail.

Let us now consider the classes of heavy-tailed distributions and their properties. We assume that all the distributions have infinite right endpoint that means $\overline{V}(x) > 0$ for all $x > 0$. We say that distribution V has heavy tail, and we write $V \in \mathcal{H}$ if for any $\epsilon > 0$, the following relation is true:

$$\int_{-\infty}^{\infty} \mathrm{e}^{\epsilon x}\, V(\mathrm{d}x) = \infty.$$

We say that distribution V has long tail (symbolically, $V \in \mathcal{L}$) if for any (or, equivalently, for some) $a > 0$, we have

$$\lim_{x\to\infty} \frac{\overline{V}(x-a)}{\overline{V}(x)} = 1.$$

The class $\mathcal{L}$ represents a subclass of heavy-tailed distributions $\mathcal{L} \subset \mathcal{H}$. If $V \in \mathcal{L}$, then there exists some function $a : [0,\infty) \to (0,\infty)$ such that $a(x) \to \infty$, $a(x) = o(x)$, and $\overline{V}(x \pm a(x)) \sim \overline{V}(x)$ as $x \to \infty$. This $a(\cdot)$ is called insensitivity function with respect to V.

We say that distribution V belongs to the class of subexponential distributions and write $V \in \mathcal{S}$ if for all (or, equivalently, for some) $n = 2, 3, \dots,$

$$\lim_{x\to\infty} \frac{\overline{V^{n*}}(x)}{\overline{V}(x)} = n,$$

where V^{n*} is the n-fold convolution of distribution V with itself. The classes $\mathcal{S}$, $\mathcal{L}$, and $\mathcal{H}$ were introduced by Chistyakov.

We say that V has dominatedly varying tail and write $V \in \mathcal{D}$ if

$$\limsup_{x\to\infty} \frac{\overline{V}(tx)}{\overline{V}(x)} < \infty$$

for all (or, equivalently, for some) $0 < t < 1$. Let us make clear that $\mathcal{D} \subsetneq \mathcal{H}$ and $\mathcal{D} \not\subseteq \mathcal{S}$, $\mathcal{S} \not\subseteq \mathcal{D}$. However, $\mathcal{D} \cap \mathcal{L} \equiv \mathcal{D} \cap \mathcal{S} \subset \mathcal{S}$. Now, we remind some properties of regular variation. A random variable Z with distribution V is regularly varying with index $\alpha \in (0,\infty)$, and we write $V \in \mathcal{R}_{-\alpha}$ if

$$\lim_{x\to\infty} \frac{\overline{V}(tx)}{\overline{V}(x)} = t^{-\alpha}$$

for any $t > 0$. The following inclusion is true:

$$\mathcal{R} := \bigcup_{\alpha>0} \mathcal{R}_{-\alpha} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H};$$

see, for example, [17, p. 21, Remark 2.1]. Let us now consider the limits

$$\overline{V}_*(t) := \liminf_{x\to\infty} \frac{\overline{V}(tx)}{\overline{V}(x)}, \qquad \overline{V}^*(t) := \limsup_{x\to\infty} \frac{\overline{V}(tx)}{\overline{V}(x)}$$

for all $t > 1$.

For some distribution V, the upper and lower Matuszewska indexes are given by

$$J_V^+ := \inf\left\{-\frac{\log \overline{V}_*(t)}{\log t}, \, t > 1\right\}, \qquad J_V^- := \sup\left\{-\frac{\log \overline{V}^*(t)}{\log t}, \, t > 1\right\},$$

respectively. For these indexes, the following relations hold: $V \in \mathcal{D}$ if and only if $0 \leqslant J_V^- \leqslant J_V^+ < \infty$, and if $V \in \mathcal{R}_{-\alpha}$, then $J_V^- = J_V^+ = \alpha$; see [17, Sect. 2.4].

The class of regularly varying distributions have many closure properties; see, for example, [17]. One of this properties is the asymptotic behavior of the tail of product convolution, which is the popular Breiman's theorem. In [10], we find the following result.

If Z and Θ are two independent random variables with distribution of Z from class $\mathcal{R}_{-a}$, then for some $\alpha \in (0, \infty)$ and Θ, which is nonnegative, nondegenerate to zero, and such that $\mathbf{E}[\Theta^{\alpha+\epsilon}] < \infty$ for some $\epsilon > 0$,

$$\mathbf{P}(\Theta Z > x) \sim \mathbf{E}(\Theta^\alpha)\mathbf{P}(Z > x)$$

as $x \to \infty$, which further means that the distribution of the product ΘZ belongs to $\mathcal{R}_{-a}$.

Now, we can go to the extension of regular variation in random vectors. Let $\mathbf{X}$ be a random vector in the space $[0, \infty]^n$. We remind that $\mathbf{X}$ follows the multivariate regularly varying distribution, symbolically, MRV, if there exists a function $b : \mathbb{R}_+ \to \mathbb{R}_+$ and a nondegenerate to zero Radon measure μ such that for every μ-continuous Borel, namely, with $\mu(\partial B) = 0$ where ∂B represents the border of B, set $B \subseteq [0, \infty]^n \setminus \{\mathbf{0}\}$, we obtain that

$$\lim_{x\to\infty} x\mathbf{P}\left[\frac{\mathbf{X}}{b(x)} \in B\right] = \mu(B). \tag{4}$$

For the normalizing function $b(\cdot)$, we have that $b(\cdot) \in \mathcal{R}_{1/\alpha}$ as is indicated in [23, Sect. 2.4]. We write $\mathbf{X} \in \mathrm{MRV}(\alpha, b, \mu)$. This measure μ is homogeneous, namely, for any Borel set $B \subseteq [0, \infty]^n \setminus \{\mathbf{0}\}$, we obtain

$$\mu(tB) = t^{-\alpha}\mu(B)$$

for any $t > 0$. Another representation of (4) is in the following form:

$$\lim_{x\to\infty} \frac{1}{\overline{V}(x)}\mathbf{P}\left[\frac{\mathbf{X}}{x} \in B\right] = \mu(B)$$

for a distribution $V \in \mathcal{R}_{-a}$.

MRV is a well-known multivariate distribution class with rich properties. We refer to [23] for several treatments and to [4, 15, 19] for applications on risk theory and risk management.

It is worth to mention that recently there were some attempts to extend the heavy-tailed distributions to multivariate setup; see, for example, [16, 24] for such kind of approaches and survey of classes.

Now, let us remind the strong asymptotic independence that we need later; see [21, Asm. A]. We should notice that in [21], the convergence in the next dependence structure was defined in the general case as $(x, y) \to (\infty, \infty)$, but for sake of compactness of the text, we use here the convergence $x \wedge y \to \infty$. Let X and Y be two real-valued random variables with distributions F and G, respectively. We say that X and Y are strongly asymptotically independent, symbolically, SAI, if

$$\begin{aligned} \mathbf{P}(X^- > x,\ Y > y) &= O\big[F(-x)\overline{G}(y)\big], \\ \mathbf{P}(X > x,\ Y^- > y) &= O\big[\overline{F}(x)G(-y)\big], \\ \mathbf{P}(X > x,\ Y > y) &\sim C\overline{F}(x)\overline{G}(y) \end{aligned}$$

as $x \wedge y \to \infty$ for some constant $C > 0$.

Remark 1. It is easy to see that SAI contains the independence as a special case. In the case where X and Y are nonnegative (or, generally, bounded from below), X and Y are SAI if

$$\mathbf{P}(X > x,\ Y > y) \sim C\overline{F}(x)\overline{G}(y)$$

as $x \wedge y \to \infty$. The SAI covers a wide spectrum of dependence as, for example, Ali–Mikhail–Haq, Farlie–Gumbel–Morgenstern, and Frank copulas; see [21].

2 Generalized tail asymptotic independence

Next, we establish the following relation:

$$\mathbf{P}\big(X_n(\Theta) > x,\ Y_m(\Delta) > y\big) \sim \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{P}(\Theta_i X_i > x,\ \Delta_j Y_j > y) \tag{5}$$

as $x \wedge y \to \infty$, where

$$X_n(\Theta) := \sum_{i=1}^{n} \Theta_i X_i, \qquad Y_m(\Delta) := \sum_{j=1}^{m} \Delta_j Y_j$$

with $\{\Theta_i, \Delta_j,\ i, j \in \mathbb{N}\}$, arbitrarily dependent nonnegative random variables, called random weights, and the primary random variables $\{(X_i, Y_i),\ i \in \mathbb{N}\}$ are such that X_i and Y_i are SAI (but X_i and Y_j are independent for any $i \neq j$) with $\mathbf{P}(X_i > x) = \overline{F}_i(x) \in \mathcal{D} \cap \mathcal{L}$ and $\mathbf{P}(Y_j > x) = \overline{G}_j(x) \in \mathcal{D} \cap \mathcal{L}$.

The study of the joint tail behavior of the tail of two randomly weighted sums provides a realistic framework for the insurance applications since most insurance companies maintain several portfolios, which are subject to dependence environment; see for relation (5) under several heavy-tailed distribution classes and several dependence structures in [20, 25, 31]. Indeed, we find mostly two forms of dependence structure. Firstly, $\{\Theta_i, \Delta_j\}$ are arbitrarily dependent, while $\{(X_i, Y_i)\}$ are independent random vectors within each random vector appears some dependence structure. Secondly, $\{\Theta_i, \Delta_j\}$ are arbitrarily dependent, and in each sequence $\{X_i\}$ and $\{Y_i\}$, there appears some dependence structure, while the two sequences $\{X_i\}$ and $\{Y_i\}$ are independent. In this paper, we combine these two approaches through the following definition. In next definition, we use the random variables $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ that follow distributions with supports, which are not bounded from above.

Definition 1. Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ be real valued random variables. Then we say that $X_1, \dots, X_n$, $Y_1, \dots, Y_m$ are generalized tail asymptotic independent, symbolically, GTAI, if both following relations hold:

$$\lim_{x_i \wedge x_k \wedge y_j \to \infty} \mathbf{P}\bigl(|X_i| > x_i \bigm| X_k > x_k,\ Y_j > y_j\bigr) = 0 \tag{6}$$

for all $1 \leqslant k \neq i \leqslant n$, $j = 1, \dots, m$, and

$$\lim_{x_i \wedge y_j \wedge y_k \to \infty} \mathbf{P}\bigl(|Y_j| > y_j \bigm| X_i > x_i,\ Y_k > y_k\bigr) = 0 \tag{7}$$

for all $1 \leqslant j \neq k \leqslant m$, $i = 1, \dots, n$.

Remark 2. This dependence structure allows dependence between $X_1, \dots, X_n$, between $Y_1, \dots, Y_m$, and dependence between X_i and Y_j (not only for $i = j$). In this paper, we restrict ourselves in the case with X_i, Y_i to be SAI dependent for the same i, and X_i, Y_j independent for any $i \neq j$. We have to note that the GTAI structure belongs to the family of "second-order asymptotic independence" that means that the probability of three or more extremal events is negligible with respect to the probability of two extremal events, namely, one in each sequence.

Notice that if X_i and Y_j are independent for any $i, j \in \mathbb{N}$ (i.e., the two sequences are independent), then relationships (6), (7) can be written as follows:

$$\lim_{x_i \wedge x_k \to \infty} \mathbf{P}\bigl(|X_i| > x_i \bigm| X_k > x_k\bigr) = 0 \tag{8}$$

for all $1 \leqslant i \neq k \leqslant n$, and

$$\lim_{y_j \wedge y_k \to \infty} \mathbf{P}\bigl(|Y_j| > y_j \bigm| Y_k > y_k\bigr) = 0 \tag{9}$$

for all $1 \leqslant j \neq k \leqslant m$. Through (8) and (9) we obtain the definition of tail asymptotic independence of $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$, respectively, introduced in [14]. We wonder if the results of our paper can be identified using instead of GTAI the TAI over the $X_1, \dots, X_n, Y_1 \dots, Y_m$ as we find under similar frame in [6]. The reply is no because, in spite of the presence of "interdependence" in both cases, GTAI studies second-order asymptotic independence events, while TAI studies only first-order asymptotic independence events. Hence, in Theorem 1 below, we demonstrate the "insensitivity" with respect to dependence in a more extremal event compared to the TAI case.

As follows from the last remark, if we choose two mutually independent sequences, where each one has tail asymptotic independent terms, then the structure GTAI is satisfied. We present now two examples, which contain interdependence among the two sequences and also satisfy the GTAI structure. For sake of simplicity, we restrict ourselves on nonnegative random variables with $n = m = 2$.

Example 1. Let X_1, X_2, Y_1, Y_2 be nonnegative random variables, and let Z_1, Z_2, Z_3, Z_4 be random variables with $Z_i \in \{X_1, X_2, Y_1, Y_2\}$, where $Z_i \neq Z_j$ for any $1 \leqslant i \neq j \leqslant 4$. Let z_1, z_2, z_3, z_4 with $z_i \in \{x_1, x_2, y_1, y_2\}$ admit $z_i = z_j$ for any $1 \leqslant i \neq j \leqslant 4$. We assume that Z_1, Z_2, Z_3, Z_4 are widely upper orthant-dependent; see [28]. Namely, for any integer $n = 1, \dots, 4$, there exists a positive number $g_u(n)$ such that for any $z_i \in \mathbb{R}$ with $i = 1, \dots, n$, we have

$$\mathbf{P}\left[\bigcap_{i=1}^{n}\{Z_i > z_i\}\right] \leqslant g_u(n) \prod_{i=1}^{n} \mathbf{P}[Z_i > z_i].$$

Further, for any $1 \leqslant i \neq j \leqslant 4$ we assume that

$$\lim_{z_i \wedge z_j \to \infty} \frac{\mathbf{P}[Z_i > z_i,\, Z_j > z_j]}{\mathbf{P}[Z_i > z_i]\mathbf{P}[Z_j > z_j]} > 0,$$

for example, if Z_i, Z_j are SAI with $C = 0$ (see in [21]), then the last condition is not valid. From these conditions and with $n = 3$ on the first relation, the GTAI structure follows directly.

Except the advantage to imply the GTAI structure, in the next example, we get an idea about the dependence frames that satisfy the conditions of Theorem 1 below.

Example 2. Under the notation of Example 1, we consider that Z_1, Z_2, Z_3, Z_4 are SAI for any two of them, namely, say for any Z_i, Z_j, where $i \neq j$, there exists a constant $C_{ij} > 0$ such that $\mathbf{P}[Z_i > z_i,\, Z_j > z_j] \sim C_{ij}\mathbf{P}[Z_i > z_i]\mathbf{P}[Z_j > z_j]$ as $z_i \wedge z_j \to \infty$. Further, we consider that there are SAI for any three of them, namely, say for any Z_i, Z_j, Z_k, where $i \neq j \neq k$, there exists a constant $C_{ijk} > 0$ such that

$$\mathbf{P}[Z_i > z_i,\, Z_j > z_j,\, Z_k > z_k] \sim C_{ijk}\mathbf{P}[Z_i > z_i]\mathbf{P}[Z_j > z_j]\mathbf{P}[Z_k > z_k]$$

as $z_i \wedge z_j \wedge z_k \to \infty$. From the triple SAI structure there is directly implied the GTAI.

Now, we can give the first assumption for the main result of the section.

Assumption 1. We assume that the following random variables $X_1, \dots, X_n, Y_1, \dots, Y_m$ are GTAI and the random weights $\Theta_1, \dots, \Theta_n, \Delta_1, \dots, \Delta_m$ are nonnegative and nondegenerate to zero random variables that follow distributions whose supports are bounded from above and independent of $X_1, \dots, X_n, Y_1, \dots, Y_m$.

In the next lemma, we see that under Assumption 1, the GTAI structure remains invariant with respect to products.

Lemma 1. *Under Assumption* 1*, we obtain that all products* $\Theta_1 X_1, \dots, \Theta_n X_n$*,* $\Delta_1 Y_1, \dots, \Delta_m Y_m$ *are* GTAI.

Proof. By definition of GTAI, for any $\epsilon > 0$, there exist some constant $\kappa_0 > 0$ such that for any $x_i \wedge x_k \wedge y_j > \kappa_0$, the relation $\mathbf{P}(|X_i| > x_i \mid X_k > x_k, Y_j > y_j) < \epsilon$ holds for all $1 \leqslant i \neq k \leqslant n$, $j = 1, \dots, m$. Let $\mathbf{P}(\Theta_i \leqslant b_i) = 1$, $\mathbf{P}(\Delta_j \leqslant d_j) = 1$, where $b_i, d_j > 0$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$. Then, for x_i, x_k, y_j sufficiently large, namely, $x_i/b_i > \kappa_0$, $x_k/b_k > \kappa_0$, and $y_j/d_j > \kappa_0$, we have that

$$\begin{aligned}
&\mathbf{P}\bigl(|\Theta_i X_i| > x_i,\ \Theta_k X_k > x_k,\ \Delta_j Y_j > y_j\bigr)\\
&\quad = \int_0^{b_i}\int_0^{b_k}\int_0^{d_j} \mathbf{P}\biggl[|X_i| > \frac{x_i}{c_i},\ X_k > \frac{x_k}{c_k},\ Y_j > \frac{y_j}{\lambda_j}\biggr]\mathbf{P}(\Theta_i \in \mathrm{d}c_i,\ \Theta_k \in \mathrm{d}c_k,\ \Delta_j \in \mathrm{d}\lambda_j)\\
&\quad = \int_0^{b_i}\int_0^{b_k}\int_0^{d_j} \mathbf{P}\biggl[|X_i| > \frac{x_i}{c_i}\ \Big|\ X_k > \frac{x_k}{c_k},\ Y_j > \frac{y_j}{\lambda_j}\biggr]\mathbf{P}\biggl[X_k > \frac{x_k}{c_k},\ Y_j > \frac{y_j}{\lambda_j}\biggr]\\
&\qquad\qquad \times \mathbf{P}(\Theta_i \in \mathrm{d}c_i,\ \Theta_k \in \mathrm{d}c_k,\ \Delta_j \in \mathrm{d}\lambda_j)\\
&\quad \leqslant \epsilon\mathbf{P}(\Theta_k X_k > x_k,\ \Delta_j Y_j > y_j).
\end{aligned}$$

By arbitrariness of $\epsilon > 0$ we have the first relation of GTAI. The symmetrical relation, namely,

$$\mathbf{P}\bigl(|\Delta_j Y_j| > y_j,\ \Theta_i X_i > x_i,\ \Delta_k Y_k > y_k\bigr) \leqslant \epsilon\mathbf{P}(\Theta_i X_i > x_i,\ \Delta_k Y_k > y_k)$$

for any $1 \leqslant j \neq k \leqslant m$, $i = 1, \dots, n$, can be easily obtained through similar arguments, and this completes the proof. □

Assumption 2. Let $\{(X_i, Y_i),\ i \in \mathbb{N}\}$ be some sequence of random vectors with marginal distributions $F_i \in \mathcal{D} \cap \mathcal{L}$ and $G_i \in \mathcal{D} \cap \mathcal{L}$, respectively, for all $i \in \mathbb{N}$. Assume that X_i and Y_i are SAI for the same i with constant $C_i > 0$, and X_i, Y_j are independent for any $i \neq j$.

The next lemma plays crucial role in the proof of Theorem 1.

Lemma 2. *Under Assumptions* 1 *and* 2, *we find*

$$\mathbf{P}\bigl(\Theta_i X_i > x,\ \Delta_j Y_j > y,\ \Theta_k |X_k| > a(x)\bigr) = o\bigl(\mathbf{P}(\Theta_i X_i > x,\ \Delta_j Y_j > y)\bigr)$$

as $x \wedge y \to \infty$ for any $1 \leqslant i \neq k \leqslant n$ with $j = 1, \dots, m$, where $a(x) > 0$ is such that $a(x) \to \infty$, $a(x) = o(x)$.

Proof. We can see that

$$\begin{aligned}
&\mathbf{P}\bigl(\Theta_i X_i > x,\ \Delta_j Y_j > y,\ \Theta_k |X_k| > a(x)\bigr)\\
&\quad = \mathbf{P}\bigl(\Theta_k |X_k| > a(x)\ \big|\ \Theta_i X_i > x,\ \Delta_j Y_j > y\bigr)\mathbf{P}\bigl(\Theta_i X_i > x,\ \Delta_j Y_j > y\bigr)\\
&\quad = o\bigl[\mathbf{P}\bigl(\Theta_i X_i > x,\ \Delta_j Y_j > y)\bigr]
\end{aligned}$$

as $x \wedge y \to \infty$, where at the last step, we used Lemma 1. □

Now, we can present the first main result.

Theorem 1. *Under Assumptions* 1 *and* 2, *for every pair $(n, m) \in \mathbb{N}^2$, we obtain* (5) *as $x \wedge y \to \infty$.*

Proof. Let us follow the line of [20, Thm. 1]. We consider the following events:

$$A_x^{\pm} := \left\{ \bigvee_{i=1}^{n} \Theta_i X_i > x \pm a(x) \right\}, \qquad A_y^{\pm} := \left\{ \bigvee_{j=1}^{m} \Delta_j Y_j > y \pm a(y) \right\},$$

where $a(x) > 0$ is such that $a(x) \to \infty$, $a(x) = o(x)$, and $a \in \mathcal{R}_0$. Now, let us define the probabilities

$$I_1(x,y) := \mathbf{P}\big(X_n(\Theta) > x,\ Y_m(\Delta) > y,\ A_x^-,\ A_y^-\big),$$
$$I_2(x,y) := \mathbf{P}\big(X_n(\Theta) > x,\ Y_m(\Delta) > y,\ (A_x^-)^c\big),$$
$$I_3(x,y) := \mathbf{P}\big(X_n(\Theta) > x,\ Y_m(\Delta) > y,\ (A_y^-)^c\big).$$

Hence, we can see that

$$\mathbf{P}\big(X_n(\Theta) > x,\ Y_m(\Delta) > y\big) \leqslant I_1(x,y) + I_2(x,y) + I_3(x,y). \tag{10}$$

Therefore, for the upper bound of the probability in the left-hand side of (10), it remains to estimate $I_1(x,y)$, $I_2(x,y)$, and $I_3(x,y)$.

$$\begin{aligned} I_1(x,y) &\leqslant \mathbf{P}(A_x^-, A_y^-) = \mathbf{P}\left[\bigvee_{i=1}^{n} \Theta_i X_i > x - a(x),\ \bigvee_{j=1}^{m} \Delta_j Y_j > y - a(y) \right] \\ &\leqslant \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}\big[\Theta_i X_i > x - a(x),\ \Delta_j Y_j > y - a(y)\big] \\ &\sim \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}(\Theta_i X_i > x,\ \Delta_j Y_j > y) \end{aligned}$$

as $x \wedge y \to \infty$, where at the last step, we used [20, Lemma 3(ii)] (as far as X_i and Y_i are SAI). Next,

$$\begin{aligned} I_2(x,y) &= \mathbf{P}\left[X_n(\Theta) > x,\ Y_m(\Delta) > y,\ \bigvee_{i=1}^{n} \Theta_i X_i > \frac{x}{n},\ \bigvee_{j=1}^{m} \Delta_j Y_j > \frac{y}{m},\ (A_x^-)^c \right] \\ &= \mathbf{P}\Bigg[X_n(\Theta) > x,\ Y_m(\Delta) > y,\ \bigvee_{i=1}^{n} \Theta_i X_i > \frac{x}{n}, \\ &\qquad \bigvee_{j=1}^{m} \Delta_j Y_j > \frac{y}{m},\ \bigvee_{k=1}^{n} \Theta_k X_k \leqslant x - a(x) \Bigg] \\ &\leqslant \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{1=k\neq i}^{n} \mathbf{P}\left[\Theta_i X_i > \frac{x}{n},\ \Delta_j Y_j > \frac{y}{m},\ \Theta_k X_k > \frac{a(x)}{n} \right] \\ &= o\left(\sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}(\Theta_i X_i > x,\ \Delta_j Y_j > y) \right) \end{aligned}$$

as $x \wedge y \to \infty$, where at the last step, we used Lemma 2 and [20, Lemma 3(i)] due to $F, G \in \mathcal{D} \cap \mathcal{L} \subsetneq \mathcal{D}$.

Symmetrically, we find

$$I_3(x,y) = o\left(\sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}(\Theta_i X_i > x,\ \Delta_j Y_j > y) \right)$$

as $x \wedge y \to \infty$. So from (10) we obtain

$$\mathbf{P}\bigl(X_n(\Theta) > x,\ Y_m(\Delta) > y\bigr) \lesssim \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}(\Theta_i X_i > x,\ \Delta_j Y_j > y)$$

as $x \wedge y \to \infty$.

For the lower bound of $\mathbf{P}(X_n(\Theta) > x,\ Y_m(\Delta) > y)$, we get the following inequality:

$$\mathbf{P}\bigl(X_n(\Theta) > x,\ Y_m(\Delta) > y\bigr) \geqslant \mathbf{P}\bigl(X_n(\Theta) > x,\ Y_m(\Delta) > y,\ A_x^+,\ A_y^+\bigr).$$

Applying Bonferroni inequality twice, we obtain

$$\begin{aligned}
&\mathbf{P}\bigl(X_n(\Theta) > x,\ Y_m(\Delta) > y,\ A_x^+,\ A_y^+\bigr) \\
&\quad \geqslant \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}\bigl(X_n(\Theta) > x,\ Y_m(\Delta) > y,\ \Theta_i X_i > x + a(x),\ \Delta_j Y_j > y + a(y)\bigr) \\
&\qquad - \sum_{1 \leqslant i < k \leqslant n} \sum_{j=1}^{m} \mathbf{P}\bigl(\Theta_i X_i > x + a(x),\ \Theta_k X_k > x + a(x),\ \Delta_j Y_j > y + a(y)\bigr) \\
&\qquad - \sum_{i=1}^{n} \sum_{1 \leqslant j < k \leqslant m} \mathbf{P}\bigl(\Theta_i X_i > x + a(x),\ \Delta_j Y_j > y + a(y),\ \Delta_k Y_k > y + a(y)\bigr), \qquad (11)
\end{aligned}$$

and further, by [20, Lem. 3 (ii)] and by Lemma 1 the last two terms in (11) are asymptotically negligible with respect to $\sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}(\Theta_i X_i > x,\ \Delta_j Y_j > y)$ as $x \wedge y \to \infty$. For the first term of right-hand side in (11), we find a lower bound

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}\bigl(\Theta_i X_i > x + a(x),\ \Delta_j Y_j > y + a(y)\bigr) \\
&\qquad - \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}\left[\Theta_i X_i > x + a(x),\ \Delta_j Y_j > y + a(y),\ \sum_{k=1}^{n} \Theta_k X_k \leqslant x\right] \\
&\qquad - \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}\left[\Theta_i X_i > x + a(x),\ \Delta_j Y_j > y + a(y),\ \sum_{k=1}^{m} \Delta_k Y_k \leqslant y\right] \\
&\quad \geqslant \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}\bigl(\Theta_i X_i > x + a(x),\ \Delta_j Y_j > y + a(y)\bigr)
\end{aligned}$$

$$-\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{i\neq k=1}^{n}\mathbf{P}\left[\Theta_i X_i > x + a(x),\ \Delta_j Y_j > y + a(y),\ \Theta_k X_k < -\frac{a(x)}{n}\right]$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{j\neq k=1}^{m}\mathbf{P}\left[\Theta_i X_i > x + a(x),\ \Delta_j Y_j > y + a(y),\ \Delta_k Y_k < -\frac{a(y)}{m}\right],$$

which by Lemma 2 and [20, Lemma 3(ii)] is asymptotically greater than the double sum

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}(\Theta_i X_i > x,\ \Delta_j Y_j > y)$$

as $x \wedge y \to \infty$, and thus we have the asymptotic relation (5). □

Remark 3. We proved asymptotic estimation (5) when $X_1, \dots, X_n,\ Y_1, \dots, Y_m$ are GTAI as it comes by Definition 1. $X_i,\ Y_i$ are SAI, and $\Theta_i,\ \Delta_i$ are upper bounded and nonnegative random variables. This restriction is relatively small compared to the extension made in terms of dependence and at the same time reasonable in these models since Θ_i and Δ_j depict the discount factors.

Let us remind that $C_i > 0$ is the constant from SAI condition on each pair (X_i, Y_i), $i = 1, \dots, n \wedge m$; see Assumption 2. Now, applying [20, Lemma 3 (iii)] in Theorem 1, we have the following consequence.

Corollary 1. *Under the conditions of Theorem 1 with the restriction that $F_i \in \mathcal{R}_{-\alpha_i}$ and $G_j \in \mathcal{R}_{-\alpha'_j}$ with $\alpha_i, \alpha'_j \in [0, \infty)$ for any $i, j \in \mathbb{N}$, we obtain*

$$\begin{aligned}&\mathbf{P}\big(X_n(\Theta) > x,\ Y_m(\Delta) > y\big)\\ &\quad\sim \sum_{i=1}^{n}\sum_{1=j\neq i}^{m}\mathbf{E}\big[\Theta_i^{\alpha_i}\Delta_j^{\alpha'_j}\big]\overline{F}_i(x)\overline{G}_j(y) + \sum_{i=1}^{n\wedge m} C_i\mathbf{E}\big[\Theta_i^{\alpha_i}\Delta_i^{\alpha'_i}\big]\overline{F}_i(x)\overline{G}_i(y)\end{aligned} \tag{12}$$

as $x \wedge y \to \infty$.

Remark 4. We write $X_n^{\pm}(\Theta) := \sum_{i=1}^{n}\Theta_i X_i^{\pm},\ Y_m^{\pm}(\Delta) := \sum_{j=1}^{m}\Delta_j Y_j^{\pm}$. We see that the corresponding maximums for $X_n(\Theta)$ and $Y_m(\Delta)$, namely, $\bigvee_{i=1}^{n} X_i(\Theta)$ and $\bigvee_{j=1}^{m} Y_j(\Delta)$, satisfy the inequalities

$$\begin{aligned}\mathbf{P}\big(X_n(\Theta) > x, Y_m(\Delta) > y\big) &\leqslant \mathbf{P}\left[\bigvee_{i=1}^{n} X_i(\Theta) > x,\ \bigvee_{j=1}^{m} Y_j(\Delta) > y\right]\\ &\leqslant \mathbf{P}\big(X_n^{+}(\Theta) > x,\ Y_m^{+}(\Delta) > y\big).\end{aligned}$$

Therefore, Theorem 1 and Corollary 1 are also satisfied, namely, relations (5) and (12) after replacement of the pair $(X_n(\Theta), Y_m(\Delta))$ with $(\bigvee_{i=1}^{n} X_i(\Theta),\ \bigvee_{j=1}^{m} Y_j(\Delta))$.

Finally, we present an application on bidimensional discrete-time risk model. Recently, the discrete-time, one-dimensional models have attracted attention by many researchers; see [22, 32]. On the other hand, more researchers study the multivariate risk models because it is rarely for an insurance company to operate with one line of business; see [8, 9, 15] among others.

We limit us in only two portfolios and discrete time, where X_i and Y_i depict the net loss in ith period in the first and second line of business, respectively, Θ_i and Δ_i, they continue to be discount factors of the ith period.

Therefore, the stochastic surplus process of insurer at time $n \in \mathbb{N}$ is described by

$$\mathbf{S}_n := (S_{1n}, S_{2n}) = \left(x - \sum_{i=1}^{n} \Theta_i X_i,\ y - \sum_{j=1}^{n} \Delta_j Y_j \right),$$

where x and y are initial capitals in each line of business. Hence, one type of ruin probability is given as

$$\psi(x, y, n) := \mathbf{P}\left[\bigvee_{i=1}^{n} X_i(\Theta) > x,\ \bigvee_{j=1}^{n} Y_j(\Delta) > y \right]$$

for any $n \in \mathbb{N}$. This ruin probability depicts the probability that both portfolios have been with negative surplus during the n first periods, but not necessarily simultaneously.

Corollary 2.

(i) *Under the conditions of Theorem* 1*, we obtain*

$$\psi(x, y, n) \sim \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{P}(\Theta_i X_i > x,\ \Delta_j Y_j > y)$$

as $x \wedge y \to \infty$.

(ii) *Under the conditions of Corollary* 1*, we obtain*

$$\psi(x, y, n) \sim \sum_{i=1}^{n} \sum_{1=j\neq i}^{n} \mathbf{E}\bigl[\Theta_i^{\alpha_i} \Delta_j^{\alpha'_j}\bigr] \overline{F}_i(x) \overline{G}_j(y) + \sum_{i=1}^{n} C_i \mathbf{E}\bigl[\Theta_i^{\alpha_i} \Delta_i^{\alpha'_i}\bigr] \overline{F}_i(x) \overline{G}_i(y)$$

as $x \wedge y \to \infty$.

Proof. Directly from Remark 4, the definition of ruin probability, Theorem 1, and Corollary 1, respectively. □

3 Application on bidimensional renewal risk model

Recently, the bidimensional risk model has gained popularity due to its improved ability to address practical insurance problems, while at the same time providing a relatively flexible mathematical framework compared to multidimensional models; see [6, 21, 30], etc. All these papers studied various models, not necessarily renewal ones, and explored different dependence structures: either among claims from the two business lines, between interarrival times and claims, or both. However, we find interdependence among the two lines of business only in [6] with TAI claims. Inspired by the simple and concise risk model found in [30] and [21], we attempt a partial extension using Theorem 1.

The discounted bivariate surplus process $(U_1(t), U_2(t))^{\mathrm{T}}$ for $t \geqslant 0$ has the form

$$\begin{pmatrix} U_1(t) \\ U_2(t) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \int_{0-}^{t} \mathrm{e}^{-r\,s} C_1(\mathrm{d}s) \\ \int_{0-}^{t} \mathrm{e}^{-r\,s} C_2(\mathrm{d}s) \end{pmatrix} - \begin{pmatrix} D_r^{(1)}(t) \\ D_r^{(2)}(t) \end{pmatrix}, \tag{13}$$

where (x, y) is the vector of the initial capitals for the two lines of business; $r \geqslant 0$ is the constant interest rate; $D_r^{(1)}(t) := \sum_{i=1}^{N(t)} X_i \mathrm{e}^{-rT_i}$, $D_r^{(2)}(t) := \sum_{j=1}^{N(t)} Y_j \mathrm{e}^{-rT_j}$ are the discounted aggregate claims of each line up to time $t \geqslant 0$; $\{(C_1(t), C_2(t)),\, t \geqslant 0\}$ is the premiums accumulation process for the two business lines, which represent non-decreasing càdlàg paths with $(C_1(0), C_2(0)) = (0, 0)$; and $\{(X_i, Y_i),\, i \in \mathbb{N}\}$ is the sequence of claims, which arrive at the time moments $\{T_i,\, i \in \mathbb{N}\}$ that represents a renewal counting process

$$N(t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{T_i \leqslant t\}}$$

for any $t \geqslant 0$ with finite renewal mean

$$\lambda(t) = \mathbf{E}\bigl[N(t)\bigr] = \sum_{i=i}^{\infty} \mathbf{P}[T_i \leqslant t].$$

$\{N(t),\, t \geqslant 0\}$ represents a homogeneous renewal process, namely, $\{\theta_i,\, n \in \mathbb{N}\}$ with $\theta_1 = T_1$ and $\theta_i = T_i - T_{i-1}$ for any integer $i \geqslant 2$, representing the interarrival times between two successive arrival times, is a sequence of independent and identically distributed, positive random variables.

In risk model (13), we can examine several kinds of ruin probability over finite-time horizon of length $T > 0$ such that it satisfies $\lambda(T) > 0$. Let us define the ruin probability as follows:

$$\psi_*(x, y; T) := \mathbf{P}\bigl[T_* \leqslant T \bigm| \bigl(U_1(0), U_2(0)\bigr) = (x, y)\bigr], \tag{14}$$

where $*$ is either "max" or "and", and as T_*, we consider

$$\begin{aligned} T_{\max} &:= \inf\Bigl\{t > 0\colon\ \bigl[U_1(t) \vee U_2(t)\bigr] < 0\Bigr\}, \\ T_{\text{and}} &:= \inf\Bigl\{t > 0\colon\ \inf_{0 \leqslant s \leqslant t} U_1(s) < 0,\ \inf_{0 \leqslant s \leqslant t} U_2(s) < 0\Bigr\}. \end{aligned} \tag{15}$$

By relations (14) and (15) we understand that $\psi_{\max}$ depicts the probability that both portfolios get simultaneously negative surplus in the interval $[0, T]$, while ψ_{and} depicts the probability that both portfolios get negative surplus in the interval $[0,T]$, but not necessarily simultaneously. Hence, we observe that

$$\psi_{\max}(x,y;T) \leqslant \psi_{\text{and}}(x,y;T). \tag{16}$$

The following result represents a partial generalization of [21, Thm. 1.1]. Although we consider more general dependence structures, by restricting the claim distribution class to $\mathcal{D}\cap\mathcal{L} \subsetneq \mathcal{S}$, the convergence is as $x\wedge y \to \infty$ instead of $(x,y)\to(\infty,\infty)$.

We can observe that in this risk model, except the dependence structures appearing in the conditions on the claims, the two business lines are also dependent through the common renewal process. We note that $C>0$ is the common constant of SAI property for any (X,Y) since these random pairs are identically distributed.

Theorem 2. *Let consider the bivariate renewal risk model* (13) *with $r \geqslant 0$. We assume that $\{(X_i,Y_i),\ i\in\mathbb{N}\}$, $\{(C_1(t),C_2(t)),\ t\geqslant 0\}$, and $\{N(t),\ t\geqslant 0\}$ are mutually independent and the sequence of pairs (X_i,Y_i) satisfies the conditions of Theorem* 1 *under the restriction that (X_i,Y_i) are identically distributed random pairs with marginal distribution F and G, respectively. Then, for any finite $T>0$ such that $\lambda(T)>0$, we have that as $x\wedge y\to\infty$,*

$$\begin{aligned}\psi_{\max}(x,y;T) &\sim \psi_{\text{and}}(x,y;T)\\ &\sim \iint\limits_{s,t\geqslant 0,\ s+t\leqslant T} \bigl[\overline{F}\bigl(x\mathrm{e}^{r(t+s)}\bigr)\overline{G}\bigl(y\mathrm{e}^{rt}\bigr)+\overline{F}\bigl(x\mathrm{e}^{rt}\bigr)\overline{G}\bigl(y\mathrm{e}^{r(t+s)}\bigr)\bigr]\,\lambda(\mathrm{d}s)\,\lambda(\mathrm{d}t)\\ &\quad + C\int\limits_0^{\mathrm{T}} \overline{F}\bigl(x\mathrm{e}^{rt}\bigr)\overline{G}\bigl(y\mathrm{e}^{rt}\bigr)\,\lambda(\mathrm{d}t) =: \varDelta(x,y;T).\end{aligned}$$

Before presenting the proof of the main result, we need a preliminary lemma that has it own merit as it provides the joint tail of the discounted aggregate claims.

Lemma 3. *Under the conditions of Theorem* 2*,*

$$\mathbf{P}\bigl[D_r^{(1)}(T)>x,\ D_r^{(2)}(T)>y\bigr] \sim \varDelta(x,y;T) \tag{17}$$

as $x\wedge y\to\infty$.

Proof. For any $m\in\mathbb{N}$ and $x\wedge y\geqslant 0$, we obtain

$$\begin{aligned}&\mathbf{P}\bigl[D_r^{(1)}(T)>x,\ D_r^{(2)}(T)>y\bigr]\\ &\quad = \mathbf{P}\Biggl[\sum_{i=1}^{N(T)} X_i\mathrm{e}^{-rT_i}>x,\ \sum_{j=1}^{N(T)} Y_j\mathrm{e}^{-rT_j}>y\Biggr]\\ &\quad = \Biggl(\sum_{n=1}^{m}+\sum_{n=m+1}^{\infty}\Biggr)\mathbf{P}\Biggl[\sum_{i=1}^{n} X_i\mathrm{e}^{-rT_i}>x,\ \sum_{j=1}^{n} Y_j\mathrm{e}^{-rT_j}>y,\ N(T)=n\Biggr]\\ &\quad =: I_1(x,y;T)+I_2(x,y;T).\end{aligned} \tag{18}$$

For the second term with any $p > J_F^+ \vee J_G^+$, by Markov's inequality, the SAI dependence, and the fact that $F, G \in \mathcal{D} \cap \mathcal{L} \subsetneq \mathcal{D}$ we can find some constant $L > 0$ such that

$$\begin{aligned}
I_2(x, y; T) &\leqslant \left(\sum_{n=m+1}^{\lfloor (x\wedge y)/L \rfloor} + \sum_{n=\lfloor (x\wedge y)/L \rfloor + 1}^{\infty} \right) \mathbf{P}\left[\sum_{i=1}^{n} X_i > x,\ \sum_{j=1}^{n} Y_j > y \right] \\
&\quad \times \mathbf{P}\big[N(T) = n\big] \\
&\leqslant \sum_{n=m+1}^{\lfloor (x\wedge y)/L \rfloor} n^2 \mathbf{P}\left[X_i > \frac{x}{n},\ Y_j > \frac{y}{n} \right] \mathbf{P}\big[N(T) = n\big] + \mathbf{P}\left[N(T) > \frac{x \wedge y}{L} \right] \\
&\leqslant \sum_{n=m+1}^{\lfloor (x\wedge y)/L \rfloor} K, n^{2(p+1)} \overline{F}(x) \overline{G}(y) \mathbf{P}\big[N(T) = n\big] \\
&\quad + \left(\frac{x \wedge y}{L} \right)^{-2(p+1)} \mathbf{E}\big[\big[N(T)\big]^{2(p+1)} \mathbf{1}_{\{N(T) > (x\wedge y)/L\}} \big] \\
&\lesssim K \overline{F}(x) \overline{G}(y) \mathbf{E}\big[\big[N(T)\big]^{2(p+1)} \mathbf{1}_{\{N(T) > m\}} \big]
\end{aligned}$$

as $x \wedge y \to \infty$, where the constant K can be found in [20, Lemma 3(i)]. At the last step, we used that, since $F, G \in \mathcal{D}$ and $p > J_F^+ \vee J_G^+$, then it follows

$$(x \wedge y)^{-2p} = o\big[\overline{F}(x) \overline{G}(y)\big] \tag{19}$$

as $x \wedge y \to \infty$. Indeed, we see that

$$(x \wedge y)^{-p} \leqslant x^{-p} = o\big[\overline{F}(x)\big], \qquad (x \wedge y)^{-p} \leqslant y^{-p} = o\big[\overline{G}(y)\big]$$

as $x \to \infty$ and $y \to \infty$, respectively, from where we obtain (19).

We observe that

$$\int_0^T \mathbf{P}\big[X \mathrm{e}^{-rt} > x,\ Y \mathrm{e}^{-rt} > y\big]\, \lambda(\mathrm{d}t) \sim C \int_0^T \overline{F}\big(x \mathrm{e}^{rt}\big) \overline{G}\big(y \mathrm{e}^{rt}\big)\, \lambda(\mathrm{d}t)$$

because $T > 0$ if finite. Furthermore,

$$\begin{aligned}
&\int_0^T \mathbf{P}\big[X \mathrm{e}^{-rt} > x,\ Y \mathrm{e}^{-rt} > y\big]\, \lambda(\mathrm{d}t) \\
&\qquad \geqslant \mathbf{P}\big[X \mathrm{e}^{-rT} > x,\ Y \mathrm{e}^{-rT} > y\big] \lambda(T) \asymp \overline{F}(x) \overline{G}(y) \lambda(T)
\end{aligned} \tag{20}$$

as $x \wedge y \to \infty$, where at the last step, we used again [20, Lemma 3(i)]. So, we find that

$$\begin{aligned}
&\lim_{m\to\infty} \limsup_{x\wedge y\to\infty} \frac{I_2(x, y; T)}{\int_0^T \mathbf{P}[X \mathrm{e}^{-rt} > x,\ Y \mathrm{e}^{-rt} > y]\, \lambda(\mathrm{d}t)} \\
&\quad \leqslant \frac{K}{M} \lim_{m\to\infty} \frac{1}{\lambda(T)} \mathbf{E}\big[\big[N(T)\big]^{2(p+1)} \mathbf{1}_{\{N(T) > m\}} \big] = 0,
\end{aligned}$$

where the constant $M > 0$ stems from relation (20), while at the last step, we used [26, Lemma 3.2]. Therefore, we obtain that

$$\lim_{m\to\infty} \limsup_{x\wedge y\to\infty} \frac{I_2(x,y;T)}{C\int_0^T \overline{F}(x\mathrm{e}^{rt})\overline{G}(y\mathrm{e}^{rt})\,\lambda(\mathrm{d}t)} = 0. \tag{21}$$

Now, we estimate $I_1(x,y;T)$. At first, through the dominated convergence theorem and Theorem 1, because $0 \leqslant \mathrm{e}^{-rT_i} \leqslant 1$, we obtain

$$\begin{aligned}
&\mathbf{P}\Biggl[\sum_{i=1}^{n} X_i\mathrm{e}^{-rT_i} > x,\ \sum_{j=1}^{n} Y_j\mathrm{e}^{-rT_j} > y,\ N(T)=n\Biggr] \\
&\quad= \mathop{\int\cdots\int}_{\substack{0\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant T\\ t_{n+1}>T}} \mathbf{P}\Biggl[\sum_{i=1}^{n} X_i\mathrm{e}^{-rt_i} > x,\ \sum_{j=1}^{n} Y_j\mathrm{e}^{-rt_j} > y\Biggr]\mathbf{P}[T_1\in\mathrm{d}t_1,\dots,T_{n+1}\in\mathrm{d}t_{n+1}] \\
&\quad\sim \mathop{\int\cdots\int}_{\substack{0\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant T\\ t_{n+1}>T}} \Biggl(\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{P}\bigl[X_i\mathrm{e}^{-rt_i} > x,\ Y_j\mathrm{e}^{-rt_j} > y\bigr]\Biggr)\mathbf{P}[T_1\in\mathrm{d}t_1,\dots,T_{n+1}\in\mathrm{d}t_{n+1}] \\
&\quad\sim \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{P}\bigl[X_i\mathrm{e}^{-rT_i} > x,\ Y_j\mathrm{e}^{-rT_j} > y,\ N(T)=n\bigr]
\end{aligned} \tag{22}$$

as $x\wedge y\to\infty$. Therefore, from relations (18) and (22) we conclude

$$\begin{aligned}
I_1(x,y;T) &\sim \sum_{n=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{P}\bigl[X_i\mathrm{e}^{-rT_i} > x,\ Y_j\mathrm{e}^{-rT_j} > y,\ N(T)=n\bigr] \\
&= \Biggl(\sum_{n=1}^{\infty} - \sum_{n=m+1}^{\infty}\Biggr)\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{P}\bigl[X_i\mathrm{e}^{-rT_i} > x,\ Y_j\mathrm{e}^{-rT_j} > y,\ N(T)=n\bigr] \\
&=: \sum_{l=3}^{4} I_l(x,y;T).
\end{aligned} \tag{23}$$

Further, we follow the line from [30, Lemma 3.4], but for convenience, we present here the full argument. For the first term, we obtain

$$\begin{aligned}
I_3(x,y;T) &= \sum_{i=1}^{\infty}\sum_{n=i}^{\infty}\sum_{j=1}^{n} \mathbf{P}\bigl[X_i\mathrm{e}^{-rT_i} > x,\ Y_j\mathrm{e}^{-rT_j} > y,\ N(T)=n\bigr] \\
&= \sum_{i=1}^{\infty}\sum_{n=i}^{\infty}\Biggl(\sum_{j=1}^{i-1} + \sum_{j=i} + \sum_{j=i+1}^{n}\Biggr)\mathbf{P}\bigl[X_i\mathrm{e}^{-rT_i} > x,\ Y_j\mathrm{e}^{-rT_j} > y,\ N(T)=n\bigr] \\
&= \sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty} \mathbf{P}\bigl[X_i\mathrm{e}^{-rT_i} > x,\ Y_j\mathrm{e}^{-rT_j} > y,\ T_i\leqslant T\bigr]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{\infty}\mathbf{P}\big[X_i\mathrm{e}^{-rT_i} > x,\ Y_i\mathrm{e}^{-rT_i} > y,\ T_i \leqslant T\big]\\
&+\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\mathbf{P}\big[X_i\mathrm{e}^{-rT_i} > x,\ Y_j\mathrm{e}^{-rT_j} > y,\ T_j \leqslant T\big]\\
=:&\sum_{k=1}^{3} I_{3k}(x,y;T). \qquad (24)
\end{aligned}
$$

For the first term $I_{31}(x,y;T)$, taking into account that $\{N(t),\ t \geqslant 0\}$ represents a homogeneous renewal process, $(T_i - T_j)$ is independent of T_j, and furthermore, $(T_i - T_j) \stackrel{\mathrm{d}}{=} T_{i-j}$, where the equality here means equality in distribution. Hence, we find

$$
\begin{aligned}
&I_{31}(x,y;T)\\
&\quad=\sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty}\mathbf{P}\big[X_i\mathrm{e}^{-r(T_j+(T_i-T_j))} > x,\ Y_j\mathrm{e}^{-rT_j} > y,\ T_j + (T_i - T_j) \leqslant T\big]\\
&\quad=\sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty}\iint\limits_{s,t\geqslant 0,\ s+t\leqslant T}\mathbf{P}\big[X_i\mathrm{e}^{-r(t+s)} > x,\ Y_j\mathrm{e}^{-rt} > y\big]\,\mathbf{P}[T_{i-j}\in \mathrm{d}s]\,\mathbf{P}[T_j \in \mathrm{d}t]\\
&\quad=\iint\limits_{s,t\geqslant 0,\ s+t\leqslant T}\overline{F}\big(x\mathrm{e}^{r(t+s)}\big)\overline{G}\big(y\mathrm{e}^{rt}\big)\,\lambda(\mathrm{d}s)\,\lambda(\mathrm{d}t). \qquad (25)
\end{aligned}
$$

By symmetry we obtain

$$
\begin{aligned}
&I_{33}(x,y;T)\\
&\quad=\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\mathbf{P}\big[X_i\mathrm{e}^{-rT_i} > x,\ Y_j\mathrm{e}^{-r(T_i+(T_j-T_i))} > y,\ T_i + (T_j - T_i) \leqslant T\big]\\
&\quad=\iint\limits_{s,t\geqslant 0,\ s+t\leqslant T}\overline{F}\big(x\mathrm{e}^{rt}\big)\overline{G}\big(y\mathrm{e}^{r(t+s)}\big)\,\lambda(\mathrm{d}s)\,\lambda(\mathrm{d}t). \qquad (26)
\end{aligned}
$$

Finally, by the SAI dependence between X_i and Y_i we have

$$
\begin{aligned}
I_{32}(x,y;T) &= \sum_{i=1}^{\infty}\int_0^{\mathrm{T}}\mathbf{P}\big[X_i\mathrm{e}^{-rt} > x,\ Y_i\mathrm{e}^{-rt} > y\big]\,\mathbf{P}[T_i \in \mathrm{d}t]\\
&\sim C\int_0^{\mathrm{T}}\overline{F}\big(x\mathrm{e}^{rt}\big)\overline{G}\big(y\mathrm{e}^{rt}\big)\,\lambda(\mathrm{d}t) =: C\,I(x,y,F,G,T) \qquad (27)
\end{aligned}
$$

as $x \wedge y \to \infty$. So, by relations (24)–(27) we find

$$
I_3(x,y;T) \sim \varDelta(x,y;T) \qquad (28)
$$

as $x \wedge y \to \infty$.

Now, we estimate $I_4(x,y;T)$. It is easy to see that for sufficiently large $x\wedge y$, we get

$$\begin{aligned} I_4(x,y;T) &= \sum_{n=m+1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{P}\bigl[X_i\mathrm{e}^{-rT_i}>x,\ Y_j\mathrm{e}^{-rT_j}>y,\ N(T)=n\bigr] \\ &\leqslant \sum_{n=m+1}^{\infty}\sum_{i=1}^{n}\Biggl(\sum_{i\neq j=1}^{n}+\sum_{i=j=1}^{n}\Biggr)\mathbf{P}\bigl[X_i\mathrm{e}^{-rT_1}>x,\ Y_j\mathrm{e}^{-rT_1}>y,\ T_n\leqslant T\bigr] \\ &= \sum_{n=m+1}^{\infty}\sum_{i=1}^{n}\Biggl(\sum_{i\neq j=1}^{n}+\sum_{i=j=1}^{n}\Biggr)\int_0^{\mathrm{T}}\mathbf{P}\bigl[X_i\mathrm{e}^{-rt}>x,\ Y_j\mathrm{e}^{-rt}>y\bigr] \\ &\qquad\qquad \times\mathbf{P}\bigl[N(T-t)\geqslant n-1\bigr]\mathbf{P}[T_1\in\mathrm{d}t] \\ &\leqslant K\sum_{n=m+1}^{\infty}\bigl[n(n-1)+2n\bigr]\int_0^{T}\overline{F}\bigl(x\mathrm{e}^{r,t}\bigr)\overline{G}\bigl(y\mathrm{e}^{rt}\bigr) \\ &\qquad\qquad \times\mathbf{P}\bigl[N(T-t)\geqslant n-1\bigr]\mathbf{P}[T_1\in\mathrm{d}t] \\ &\leqslant K\sum_{n=m+1}^{\infty}(n^2+n)\mathbf{P}\bigl[N(T)\geqslant n-1\bigr]\int_0^{T}\overline{F}\bigl(x\mathrm{e}^{r,t}\bigr)\overline{G}\bigl(y\mathrm{e}^{rt}\bigr)\mathbf{P}[T_1\in\mathrm{d}t], \end{aligned}$$

where the constant $K>0$ comes from [20, Lemma 3(i)]. Hence, for any large enough m and $x\wedge y>0$, we can find some small enough $\varepsilon>0$ such that $I_4(x,y;T)\leqslant\varepsilon K\times I(x,y,F,G,T)\leqslant\varepsilon K I_3 x,y;T)$, and letting that m tend to infinity, we can take $\varepsilon\downarrow 0$. Thus by relations (23), (28) and last asymptotic inequality we find

$$I_1(x,y;T)\sim I_3(x,y;T)\sim\Delta(x,y;T) \tag{29}$$

as $x\wedge y\to\infty$. Therefore, by (18), (21), and (29) we reach relation (17). □

Proof. Proof of Theorem 2 For the ψ_{and}, by Lemma 3 we find

$$\begin{aligned} \psi_{\mathrm{and}}(x,y;T) &= \mathbf{P}\Bigl[\inf_{0<t\leqslant T}U^{(1)}(t)<0,\ \inf_{0<t\leqslant T}U^{(2)}(t)<0\Bigr] \\ &\leqslant \mathbf{P}\bigl[D_r^{(1)}(T)>x,\ D_r^{(2)}(T)>y\bigr]\sim\Delta(x,y;T) \end{aligned} \tag{30}$$

as $x\wedge y\to\infty$. On the other hand, for the lower bound of $\psi_{\max}$, following the line from [30, Thm. 2.1], we obtain

$$\begin{aligned} \psi_{\max}(x,y;T) &= \mathbf{P}\Bigl[\inf_{0<t\leqslant T}\bigl\{U^{(1)}(t)\vee U^{(2)}(t)\bigr\}<0\Bigr] \\ &\geqslant \mathbf{P}\Biggl[D_r^{(1)}(T)-\int_0^{T}\mathrm{e}^{-rs}C_1(\mathrm{d}s)>x,\ D_r^{(2)}(T)-\int_0^{T}\mathrm{e}^{-rs}C_2(\mathrm{d}s)>y\Biggr] \end{aligned}$$

$$= \int_0^\infty \int_0^\infty \mathbf{P}\bigl[D_r^{(1)}(T) > x + u,\ D_r^{(2)}(T) > y + z\bigr] H(\mathrm{d}u, \mathrm{d}z)$$
$$\sim \int_0^\infty \int_0^\infty \varDelta(x + u,\ y + z; T) H(\mathrm{d}u, \mathrm{d}z)$$

as $x \wedge y \to \infty$, where by H we denote the distribution of the

$$\Biggl(\int_0^T \mathrm{e}^{-rs} C_1(\mathrm{d}s),\ \int_0^T \mathrm{e}^{-rs} C_2(\mathrm{d}s) \Biggr),$$

and at the last step, we used Lemma 3. Therefore, due to the fact that $F, G \in \mathcal{D} \cap \mathcal{L} \subsetneq \mathcal{L}$, we obtain $\varDelta(x + u,\ y + z; T) \sim \varDelta(x, y; T)$ as $x \wedge y \to \infty$, which in combination of the last relation. By dominated convergence theorem we find

$$\psi_{\max}(x, y; T) \gtrsim \varDelta(x, y; T) \tag{31}$$

as $x \wedge y \to \infty$. From relations (30), (31), and (16) we get the desired result. $\square$

4 Tail distortion risk measures

Let us remind the Hadamard product of two nonnegative, n-variate random vectors $\varTheta$ and $\mathbf{X}$. Here the vector $\varTheta = (\varTheta_1 \dots, \varTheta_n)$ has nonnegative and nondegenerate to zero components and represents the systemic risk factors, while the nonnegative random vector $\mathbf{X} = (X_1, \dots, X_n)$ describes the losses of n portfolios, namely, the random variable X_i represents the loss of the ith portfolio over a concrete time horizon with $i = 1, \dots, n$. Hence, the product $\varTheta\,\mathbf{X}$ corresponds to discounted claim of the n portfolios over a concrete time horizon.

In order to make the model more realistic, we allocate the initial capital into n lines of business, in general, of different amounts. Thus, we need nonrandom, positive weights $w_1, \dots, w_n$ with $\sum_{i=1}^n w_i = 1$. Then the discounted aggregate loss of portfolio is presented as

$$\varTheta\mathbf{X}(\mathbf{w}) := \sum_{i=1}^n w_i \varTheta_i X_i.$$

This model is called background risk model, and it can describe the systemic risk. For more details about background risk model, see [1] and [27]. We intent to study a risk measure, called tail distortion risk measure, symbolically, TDRM, in the background risk model $\varTheta\,\mathbf{X}(\mathbf{w})$ under a condition; see Assumption 3 below, which permits several forms of dependence, among the systemic risk factors $\varTheta$, among the losses $\mathbf{X}$ and between the $\varTheta$ and $\mathbf{X}$ simultaneously. Before the definitions, we give a survey of the current literature on the topic.

In [5], the distortion tail risk measure was examined in a similar background risk model. In [5], asymptotic results are developed for the tail distortion risk measure TDRM of the quantity $\Theta\,\mathbf{X}(\mathbf{w})$ and the weights w_i for $i = 1, \dots, n$, defined as above, with Θ to be one-dimensional, and therefore, is a common systemic risk factor for all business lines and independent of $\mathbf{X}$. Also, they assume left-continuous distortion function. In [33], TDRM was studied for right-continuous distortion function in the $X_n(\Theta)$ model, where multivariate systemic risk factors were permitted. In this work, there were used either MRV structure for the vector $\mathbf{X}$ or some general enough structures of asymptotic independence for its components (with regularly varying tails), which can provide a direct asymptotic expression. However, the vectors Θ and $\mathbf{X}$ are still independent.

In this work, we study the $\Theta\,\mathbf{X}(\mathbf{w})$ model that, by means of Assumption 3, allows arbitrary dependence among each vectors components, and also allows a dependence between Θ and $\mathbf{X}$, which is a mathematical generalization with obvious practical impact. Furthermore, Assumption 3 also allows the domination, with respect to tail heaviness, of the systemic risk factors versus the losses, while in [5, 33], only the opposite is allowed, namely, the domination of tails of losses versus the tails of systemic risk factors. The last case is far from realistic during periods of economic instability.

Assumption 3. Let $\Theta\,\mathbf{X} \in \mathrm{MRV}(\alpha, V, \mu)$ for some $\alpha \in (0, \infty)$.

Remark 5. Assumption 3 is satisfied in many cases, where we have $\mathbf{X} \in \mathrm{MRV}(\alpha, V, \mu^*)$ (or $\Theta \in \mathrm{MRV}(\alpha, V, \mu^*)$) for some Radon measure μ^* through multivariate versions of Breiman's theorem. For example, under some moment conditions for the components of Θ (respectively for the components of $\mathbf{X}$), in [2], we find that in case of independent Θ and $\mathbf{X}$, the MRV structure remains in the product $\Theta\,\mathbf{X}$ with a new Random measure, but with same regular variation index and same normalization function. Later, in [13], we have an extension of the result in dependent Θ and $\mathbf{X}$. For the special case where $\Theta = \Theta\,\mathbf{1}$, under a rather weak dependence structure between Θ and $\mathbf{X}$ suggested in [19], we can find in [9] that the product $\Theta\,\mathbf{X}$ keeps the MRV structure with same regular variation index and same normalizing function.

Remark 6. An important property for the following proofs is the closure of the MRV class with respect to linear combinations. Namely, if we have a nonnegative random vector $\mathbf{X} \in \mathrm{MRV}(\alpha, V, \mu^*)$, then the distribution of all the nonnegative, nondegenerate to zero linear combinations $\sum_{i=1}^{n} l_i X_i$ belongs to the class $\mathcal{R}_{-\alpha}$ with the same index α; see, for example, [23, Sect. 7.3.1]. Hence, under Assumption 3, we obtain that $\Theta\mathbf{X}(\mathbf{w}) \in \mathcal{R}_{-\alpha}$.

Now, in the classic one-dimensional background risk model, we study the asymptotic behavior of tail distortion risk measure, which is more general than conditional tail expectation. The following class of risk measures was introduced in [29]. For a given nondecreasing function $g : [0, 1] \to [0, 1]$ such that $g(0) = 0$, $g(1) = 1$ and for any nonnegative random variable X with distribution F, the measure

$$\rho_g[X] = \int_0^\infty g\bigl[\overline{F}(x)\bigr]\,\mathrm{d}x$$

is called distorted risk measure, and the function g is called distortion function. It is well known that

$$\mathrm{VaR}_p(Z) := \inf\{x \in \mathbb{R}\colon V(x) \geqslant p\}, \qquad \mathrm{CTE}_p(Z) = \mathbf{E}\bigl[Z \bigm| Z > \mathrm{VaR}_p(Z)\bigr]$$

are distortion risk measures for some $p \in (0,1)$; see, for example, [35]. In order to stress on the risk tail introduced by [35], the definition of TDRM follows.

Definition 2. Let $g : [0,1] \to [0,1]$ be nondecreasing such that $g(0) = 0$ and $g(1) = 1$. Then the tail distortion risk measure of a nonnegative random variable X with distribution F is given by

$$\rho_g\bigl[X \bigm| X > \mathrm{VaR}_p(X)\bigr] = \int_0^\infty g\bigl[\overline{F}_{X|X>\mathrm{VaR}_p(X)}(x)\bigr]\,\mathrm{d}x,$$

where $\overline{F}_{X|X>y}(x) = \mathbf{P}(X > x \mid X > y)$.

It is well known that the tail distortion risk measure of a continuous random variable X is a distortion risk measure. Furthermore, if the distortion function is the identical function $g(x) = x$, then the tail distortion risk measure coincides with the conditional tail expectation.

From [35] we find that if $X \in \mathcal{R}_{-\alpha}$ with $\alpha > 0$ and, additionally,

$$\int_1^\infty g\biggl(\frac{1}{y^{\alpha-k}}\biggr)\,\mathrm{d}y < \infty$$

for some $0 < k < \alpha$, then $\rho_g[X \mid X > \mathrm{VaR}_p(X)] \sim C_\alpha(g)\,\mathrm{VaR}_p(X)$ as $p \to 1$, where

$$C_\alpha(g) := \int_0^1 y^{-1/\alpha}\,g(\mathrm{d}y) = 1 + \int_1^\infty g\biggl(\frac{1}{y^\alpha}\biggr)\,\mathrm{d}y.$$

This last result was shown for any distortion function without continuity requirement.

From [12] we find for the normalization function B_X defined as quantile of $1/\overline{F}$,

$$B_X(s) := \biggl(\frac{1}{\overline{F}}\biggr)^{\leftarrow}(s) = F^{\leftarrow}\biggl(1 - \frac{1}{s}\biggr) \tag{32}$$

for any $s \geqslant 1$, and for some right-continuous distortion function g, the following representation holds:

$$\rho_g\bigl[X \bigm| X > \mathrm{VaR}_p(X)\bigr] = \int_0^1 B_X\biggl(\frac{1}{y(1-p)}\biggr)\,g(\mathrm{d}y) \tag{33}$$

for any nonnegative random variable X with distribution F.

It is clear that the previous asymptotic expressions in Section 4 depends on the distortion function only though the constant $C_\alpha(g)$, and therefore, the quantiles of random variables remain clear of distortion. This helps in practical applications since the only thing we need to apply the model when the distortion function is varying for the same risks is to calculate the integral.

Further, we study the background risk under Assumption 3. So, in this model, we allow dependence among the losses of the n lines of business and dependence among the systemic risk factors with the losses.

Theorem 3. *Let the product $\Theta\mathbf{X}$ satisfy Assumption 3 and condition*

$$\int_1^\infty g\left(\frac{1}{y^{\alpha/(1+\zeta)}}\right) \mathrm{d}y < \infty \tag{34}$$

for some $\zeta > 0$. Then

$$\rho_g\bigl[\Theta\mathbf{X}(\mathbf{w}) \bigm| \Theta\mathbf{X}(\mathbf{w}) > \mathrm{VaR}_p\left(\Theta, \mathbf{X}(\mathbf{w})\right)\bigr] \sim C_\alpha(g)\frac{\gamma_{\mathbf{w}}^{1/\alpha}}{\Gamma_\alpha}\sum_{i=1}^n \mathrm{VaR}_p(\Theta_i X_i) \tag{35}$$

as $p \to 1$ with

$$\gamma_{\mathbf{w}} := \lim_{x\to\infty} \frac{\mathbf{P}(\Theta\mathbf{X}(\mathbf{w}) > x)}{\mathbf{P}[\sum_{i=1}^n \Theta_i X_i > x]}, \qquad \Gamma_\alpha := \sum_{i=1}^n \gamma_{e_i}^{1/\alpha}. \tag{36}$$

Proof. From Assumption 3 we obtain $\Theta\mathbf{X} \in \mathrm{MRV}(\alpha, V, \mu)$, therefore, $\Theta\mathbf{X}(\mathbf{w}) \in \mathcal{R}_{-\alpha}$. From relation (33) and the fact that

$$\mathrm{VaR}_p\bigl(\Theta\mathbf{X}(\mathbf{w})\bigr) = B_{\Theta\mathbf{X}(\mathbf{w})}\left(\frac{1}{1-p}\right) \tag{37}$$

(see (32)) we can show

$$\frac{1}{B_{\Theta\mathbf{X}(\mathbf{w})}(\frac{1}{1-p})}\int_0^1 B_{\Theta\mathbf{X}(\mathbf{w})}\left(\frac{1}{y(1-p)}\right) g(\mathrm{d}y) \sim C_\alpha(g) \tag{38}$$

as $p \to 1$. Indeed, from [11, Thm. B.2.18] we get that there exists $\tilde{p}$, $0 < \tilde{p} < 1$, which depends on $\zeta > 0$, such that for $\tilde{p} \leqslant p < 1$ and $0 < y < 1$, we have that

$$\begin{aligned}\left|\frac{B_{\Theta\mathbf{X}(\mathbf{w})}(\frac{1}{y(1-p)}) - B_{\Theta\mathbf{X}(\mathbf{w})}(\frac{1}{1-p})}{\frac{1}{\alpha}B_{\Theta\mathbf{X}(\mathbf{w})}(\frac{1}{1-p})} - \frac{y^{-1/\alpha}-1}{1/\alpha}\right| &= \left|\frac{B_{\Theta\mathbf{X}(\mathbf{w})}(\frac{1}{y(1-p)})}{\frac{1}{\alpha}B_{\Theta\mathbf{X}(\mathbf{w})}(\frac{1}{1-p})} - \frac{y^{-1/\alpha}}{1/\alpha}\right| \\ &\leqslant y^{-(1+\zeta)/\alpha}.\end{aligned} \tag{39}$$

Further, we have

$$\left|\frac{B_{\Theta\mathbf{X}(\mathbf{w})}(\frac{1}{y(1-p)})}{\frac{1}{\alpha}B_{\Theta\mathbf{X}(\mathbf{w})}(\frac{1}{1-p})}\right| - \frac{y^{-1/\alpha}}{1/\alpha} \leqslant \left|\frac{B_{\Theta\mathbf{X}(\mathbf{w})}(\frac{1}{y(1-p)})}{\frac{1}{\alpha}B_{\Theta\mathbf{X}(\mathbf{w})}(\frac{1}{1-p})} - \frac{y^{-1/\alpha}}{1/\alpha}\right|. \tag{40}$$

Hence, from relations (39) and (40) we obtain

$$\left|\frac{B_{\Theta\mathbf{X}(\mathbf{w})}\big(\frac{1}{y\,(1-p)}\big)}{B_{\Theta\mathbf{X}(\mathbf{w})}\big(\frac{1}{1-p}\big)}\right| \leqslant y^{-1/\alpha} + \frac{y^{-(1+\zeta)/\alpha}}{\alpha}. \tag{41}$$

Since the integral in (34) converges for some $\zeta > 0$, it follows that

$$\int_0^1 y^{-1/\alpha}\, g(\mathrm{d}y) \leqslant \int_0^1 y^{-(1+\zeta)/\alpha}\, g(\mathrm{d}y) < \infty,$$

whence we obtain

$$\int_0^1 \left(y^{-1/\alpha} + \frac{y^{-(1+\zeta)/\alpha}}{\alpha}\right) g(\mathrm{d}y) < \infty.$$

Therefore, from (41) and by dominated convergence theorem we find

$$\begin{aligned}\lim_{p\to 1} \int_0^1 \frac{B_{\Theta\mathbf{X}(\mathbf{w})}\big(\frac{1}{y\,(1-p)}\big)}{B_{\Theta\mathbf{X}(\mathbf{w})}\big(\frac{1}{1-p}\big)}\, g(\mathrm{d}y) &= \int_0^1 \lim_{p\to 1} \frac{B_{\Theta\mathbf{X}(\mathbf{w})}\big(\frac{1}{y\,(1-p)}\big)}{B_{\Theta\mathbf{X}(\mathbf{w})}\big(\frac{1}{1-p}\big)}\, g(\mathrm{d}y) \\ &= \int_0^1 y^{-1/\alpha}\, g(\mathrm{d}y) = C_\alpha(g),\end{aligned}$$

where the penultimate equality follows from $B_{\Theta\mathbf{X}(\mathbf{w})}(\cdot) \in \mathcal{R}_{1/\alpha}$; see (32) and [23, Sect. 2.4]. Therefore, relation (38) is true.

Furthermore, taking into account relation (36) we obtain

$$\lim_{p\uparrow 1} \frac{\sum_{i=1}^n \mathrm{VaR}_p(\Theta_i X_i)}{\mathrm{VaR}_p(\Theta\mathbf{X}(\mathbf{w}))} = \lim_{x\to\infty} \left(\frac{\sum_{i=1}^n \mathbf{P}[\Theta_i\, X_i > x]}{\mathbf{P}[\sum_{i=1}^n w_i \Theta_i\, X_i > x]}\right)^{1/\alpha} = \frac{\Gamma_\alpha}{\gamma_{\mathbf{w}}^{1/\alpha}}. \tag{42}$$

However, we also obtain

$$\rho_g\big[\Theta\mathbf{X}(\mathbf{w}) \bigm| \Theta\mathbf{X}(\mathbf{w}) > \mathrm{VaR}_p\big(\Theta\mathbf{X}(\mathbf{w})\big)\big] = \int_0^1 B_{\Theta\mathbf{X}(\mathbf{w})}\left(\frac{1}{y(1-p)}\right) g(\mathrm{d}y),$$

while by (37) and (38) we find

$$\int_0^1 B_{\Theta\mathbf{X}(\mathbf{w})}\left(\frac{1}{y(1-p)}\right) g(\mathrm{d}y) \sim C_\alpha(g)\, \mathrm{VaR}_p\big(\Theta\mathbf{X}(\mathbf{w})\big)$$

as $p \to 1$, and consequently, from relation (42) and the last two expressions we find relation (35). □

Further, we provide a simple corollary, which serves as an extension of the result from [5, Thm. 3.1] in case when the systemic risk factor is independent of the risk vector $\mathbf{X} \in \mathrm{MRV}$.

Corollary 3.

(i) *Let* $\mathbf{X} \in \mathrm{MRV}(\alpha, V, \mu^*)$ *and* Θ *be independent of* $\mathbf{X}$. *We assume that there exists some* $\varepsilon > 0$ *such that* $\mathbf{E}(\Theta_i^{\alpha+\varepsilon}) < \infty$ *for any* $i = 1, \dots, n$. *If the integral in* (34) *converges, then*

$$\rho_g\big[\Theta\mathbf{X}(\mathbf{w}) \bigm| \Theta\mathbf{X}(\mathbf{w}) > \mathrm{VaR}_p\big(\Theta\mathbf{X}(\mathbf{w})\big)\big] \sim C_\alpha(g)\frac{\gamma_{\mathbf{w}}^{1/\alpha}}{\Gamma_\alpha}\sum_{i=1}^{n}\big[\mathbf{E}\big(\Theta_i^\alpha\big)\big]^{1/\alpha}\mathrm{VaR}_p(X_i) \tag{43}$$

as $p \to 1$.

(ii) *Let* $\Theta \in \mathrm{MRV}(\alpha, V, \mu^*)$ *and* $\mathbf{X}$ *be independent of* Θ. *We assume that there exists some* $\varepsilon > 0$ *such that* $\mathbf{E}(X_i^{\alpha+\varepsilon}) < \infty$ *for any* $i = 1, \dots, n$. *If the integral in* (34) *converges, then*

$$\rho_g\big[\Theta\,\mathbf{X}(\mathbf{w}) \bigm| \Theta\,\mathbf{X}(\mathbf{w}) > \mathrm{VaR}_p\left(\Theta\,\mathbf{X}(\mathbf{w})\right)\big] \sim C_\alpha(g)\frac{\gamma_{\mathbf{w}}^{1/\alpha}}{\Gamma_\alpha}\sum_{i=1}^{n}\big[\mathbf{E}\big(X_i^\alpha\big)\big]^{1/\alpha}\mathrm{VaR}_p(\Theta_i) \tag{44}$$

as $p \to 1$.

Proof. (i) Firstly, we have $\Theta\,\mathbf{X} \in MRV(\alpha,\, V,\, \mu)$ from [2]. Hence, by Theorem 3 we obtain (35). Because Θ_i and X_i are independent, from [3] we find

$$\mathrm{VaR}_p(\Theta_i X_i) \sim \big[\mathbf{E}\big(\Theta_i^\alpha\big)\big]^{1/\alpha}\mathrm{VaR}_p(X_i) \tag{45}$$

as $p \to 1$. Hence from relations (44) and (35) we obtain (43).

(ii) We follow a entirely similar argumentation as in item (i). □

5 Conclusion

In the corresponding literature, we find a confusion with respect to term multivariate heavy tails. The reason for this, is that the multivariate distribution tail cannot be determined uniquely. Having in mind the single big jump principle, the d-dimensional, nonweighted form of (5) is focused on the following joint distribution tail:

$$\mathbf{P}\left[\sum_{i=1}^{n_1} X_i^{(1)} > x_1,\ \dots,\ \sum_{i=1}^{n_d} X_i^{(d)} > x_d\right] \sim \sum_{i=1}^{n_1}\cdots\sum_{i=1}^{n_d}\mathbf{P}\big[X_i^{(1)} > x_1,\ \dots,\ X_i^{(d)} > x_d\big] \tag{46}$$

as $\wedge_{i=1}^{d} x_i \to \infty$. Along with approach of the multivariate subexponentiality by [24], we find in [16], under different conditions, the relation

$$\mathbf{P}\Biggl[\sum_{i=1}^{n} \mathbf{X}^{(i)} \in xA\Biggr] \sim \sum_{i=1}^{n} \mathbf{P}\bigl[\mathbf{X}^{(i)} \in xA\bigr] \tag{47}$$

as $x \to \infty$, where $\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(n)}$ represent d-dimensional random vectors, and A is some rare set. Although the single big jump approximation in (47) has several good properties (for example, most of the closure properties of the one-dimensional distribution classes hold also for the multivariate ones, and further, some results can be generalized to multidimensional setup), there exist still some basic drawbacks.

The main drawback is that the set A does not represent the joint distribution tail; see [16, Remark 2.2]. This leads to a "linear" approximation of the single big jump in the sense that, in order to happen this representation, there should be from 1 to d, out of a total of $n \times d$, random variables that are sufficiently "large". As consequence, the approximation of relation (47) cannot represent the probabilities $\psi_{\max}$ and ψ_{and}, which are among the four most popular ruin probabilities in multivariate risk models.

In opposite direction, relation (46) studied in Section 2 with $d = 2$ can solve the problem with these two ruin probabilities since it is focused on the joint distribution tail. Furthermore, we can say that relation (46) follows a "nonlinear" approach of the single big jump since requires d sufficiently "large" random variables, namely, one big jump for each line of business if you look from the risk theory aspect. In this sense, the two approaches, described by (46) and (47), work in complementary modes.

Finally, we want to mention that MRV, although can give solutions with respect to $\psi_{\max}$ and ψ_{and}, satisfies only the linear single big jump, namely, relation (47). From this point of view, the study of (46) leads to revision of multivariate heavy-tailed distributions, even of the well-established ones, like MRV. The events described by relation (47) do NOT put emphasis in the dimension, in opposite with the events described by relation (46).

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