Nonlinear Analysis: Modelling and Control, Online First, 1–19
https://doi.org/10.15388/namc.2025.30.43742

Vilnius University Press

Existence of nontrivial solutions for a class of $2n$-order ODE*

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Received: February 20, 2025 / **Revised:** July 15, 2025 / **Published online:** October 7, 2025

Abstract. We consider a semilinear $2n$-order problem with nonconstant coefficients. We deduce existence results by using variational methods in two directions. We primarily treat the existence when the nonlinearity has asymptotic linear behaviour at infinity and is either asymptotically sublinear or linear at zero. Secondly, we discuss the superlinear case at infinity and prove three existence results showing that our problem has at least one or two nonzero solutions.

Keywords: higher-order equations, variational methods, critical points.

1 Introduction

In this paper, we focus on the following boundary value problem:

$$\begin{aligned} &u^{(2n)} - \big(A(x)u''\big)'' - \big(B(x)u'\big)' + C(x)u = f(x,u) \quad \text{in } \Omega = (0,L), \\ &u = u'' = \cdots = u^{(2n-2)} = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{P}$$

where $A \in C^2(\bar{\Omega})$, $B \in C^1(\bar{\Omega})$, and $C \in C^0(\bar{\Omega})$ are some given functions, f is a continuous function on $\overline{\Omega} \times \mathbb{R}$, and $n \in \mathbb{N}$.

We deal with existence of solution for problem (P) under different conditions and suitable behaviour of the nonlinear term f.

Several authors have obtained existence and multiplicity results to (P) in the fourth-order case, i.e., $n = 2$ (see, for instance, [6] and [7]), or the sixth-order case, i.e., $n = 3$, but under different assumptions, for example, sign restrictions on $f(x,u)$ or $F(x,u) = \int_0^u f(x,s)\,\mathrm{d}s$.

*The paper is supported by PRIN 2022 – Progetti di Ricerca di rilevante Interesse Nazionale "Nonlinear differential problems with applications to real phenomena" (2022ZXZTN2).

[1]The author is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM)

[2]The author is a participant of "INdAM-GNAMPA Project", codice CUP E53C23001670001.

[3]The author is a participant of "INdAM-GNAMPA Project", codice CUP E5324001950001.

We mention the fourth-order case treated in [12] and [20], where functions A, B, and C are constants with $A = 0$, $B > 0$, and $C < 0$; $f(x,s) = b(x)s^3$, $b > 0$, and in the paper [4], where existence results are obtained under the assumptions $A = 0$,

$$\max\left\{-\frac{B}{\pi^2}, -\frac{C}{\pi^4}, -\frac{B}{\pi^2} - \frac{C}{\pi^4}\right\} < 1$$

and

$$F(x,s) \geqslant 0 \quad \text{for every } (x,s) \in \bigl([0,3/8] \cup [5/8,1]\bigr) \times [0,d]$$

for some positive constant d.

In the sixth-order case, existence and multiplicity results were obtained in [21] when $f(s) = s^3$, $A^2 < 4B$ (A, B positive constants), and $C = 1$ in Ω, and in [13], when $C > 0$, $f(x,s) = b(x)s^3$, where b is an even continuous $2L$-periodic function. A more general existence and multiplicity result was given in [17] by using variational methods and the Brézis and Nirenberg's linking theorems in the case

$$\frac{F(x,s)}{s^2} \to +\infty \quad \text{uniformly with respect to } x \text{ as } |s| \to \infty,$$

but under the restriction $F(x,s) \geqslant 0$.

In [18], the authors studied the existence of positive solutions of a general sixth-order nonlinear boundary value problem using the Krein–Rutman theorem and the Global Bifurcation Theory under the restriction $f > 0$.

In the paper of Bonanno and Livrea [10], problem (P), where $n = 3$ and $L = 1$, was treated by requiring an oscillation on $f(x,\cdot)$ at infinity. Under some additional restrictions, including

$$F(x,s) \geqslant 0 \quad \text{for every } (x,s) \in \bigl([0,5/12] \cup [7/21,1]\bigr) \times \mathbb{R},$$

the authors obtained that the problem admits an unbounded sequence of classical solutions.

It is worth mentioning an existence result due to Bonanno and et al. [5] in the case when $n = 3$, $f(s) = \lambda g(s)$ that holds without any sign restriction of g; see [5, Cor. 2]. The authors proved the result under some restriction on $\lambda > 0$ and under the following condition (here G is the potential of g):

$$\limsup_{s \to 0^+} \frac{G(s)}{s^2} = +\infty.$$

The purpose of this paper is to establish existence results for the semilinear $2n$-order problem (P) when $n \geqslant 4$. For simplicity, we shall consider the case when n is even. Similar results can be established when n is odd; see, for instance, [15].

In Section 2, we introduce some useful preliminaries and explain the variational set.

In Section 3, we prove that problem (P) has nontrivial solutions if f is asymptotically linear at infinity, i.e.,

$$\lim_{|s| \to \infty} \frac{f(x,s)}{s} = L_1(x) \quad \text{uniformly a.e. in } \Omega, \tag{H$_1$}$$

and if f is sublinear at zero, i.e.,

$$\lim_{s\to 0^+}\frac{f(x,s)}{s^\alpha}=L_2(x) \quad \text{and} \quad \lim_{s\to 0^+}F(x,s)=0 \quad \text{uniformly in a.e. } \Omega, \tag{H$_2$}$$

where $L_1, L_2 \in L^\infty(\Omega)$, $L_2(x) \geqslant 0$, $\|L_2\|_{L^\infty(\Omega)} > 0$, $0 < \alpha < 1$ are satisfied; see Theorem 1.

We also treat the linear case at zero (Theorem 2), i.e., we assume that instead of (H$_2$), f satisfies

$$\lim_{|s|\to 0}\frac{f(x,s)}{s}=L_3(x) \quad \text{uniformly a.e. in } \Omega. \tag{H$_3$}$$

In Section 4, we discuss the superlinear case at infinity and prove three existence results. The first result (Theorem 5) is a general one (exponential nonlinearities are allowed; see Example 2) and shows that under suitable conditions on the nonlinearity, problem (P) has at least one nonzero solution. The second result (Theorem 6) provides the existence of at least two nonzero solutions under the Ambrosetti–Rabinowitz condition.

2 Auxiliary results and variational settings

We consider the Hilbert space

$$H(\Omega)=\left\{u\in H^n(\Omega)\;\middle|\; u^{(2j)}=0,\ 0\leqslant j<\frac{n}{2} \text{ on } \partial\Omega\right\},$$

which, for n even, becomes

$$H(\Omega)=\left\{u\in H^n(\Omega)\;\middle|\; u=u''=\cdots=u^{(n-2)}=0 \text{ on } \partial\Omega\right\}$$

endowed with the standard inner product

$$(u,v)_{H^n(\Omega)}=\int\limits_\Omega \left(uv+u'v'+u''v''+\cdots+u^{(n)}v^{(n)}\right)\mathrm{d}x$$

and standard norm

$$\|u\|_{H^n(\Omega)}=(u,u)^{1/2}_{H^n(\Omega)}.$$

Now, we point out some useful inequalities.

Lemma 1. *For each $u\in H^n(\Omega)$, it results*

$$\|u\|_{C^{n-1}(\bar\Omega)}\leqslant \eta\|u\|_{H^n(\Omega)}, \tag{1}$$

where $\eta := (2\max\{1/L^2,1\})^{1/2}$.

Proof. By formula (3.8) in [11] it is easy to prove that for each $u \in H^n(\Omega)$, one has

$$\|u^{(n-i)}\|_\infty^2 \leqslant \max\left\{\frac{1}{L^2}, 1\right\}\left(\|u^{(n-i)}\|_{L^2(\Omega)}^2 + \|u^{(n-i+1)}\|_{L^2(\Omega)}^2\right)$$

for each $i \in \{1, 2, \dots, n\}$. This relation leads to

$$\|u\|_{C^{n-1}(\bar{\Omega})}^2 = \max_{0 \leqslant i \leqslant n-1} \|u^{(i)}\|_\infty^2 \leqslant \eta^2 \|u\|_{H^n(\Omega)}^2. \qquad \square$$

Remark 1. It is clear that constant η in (1) also describes the constant of the embedding $H^n(\Omega) \hookrightarrow C^0(\bar{\Omega})$.

Lemma 2. *The following relations hold true for any* $u \in H(\Omega)$:

$$\int_\Omega \left(u^{(i)}\right)^2 \mathrm{d}x \leqslant \left(\frac{L}{\pi}\right)^2 \int_\Omega \left(u^{(i+1)}\right)^2 \mathrm{d}x, \quad i = 0, 1, 2, \dots, n-1, \tag{2}$$

$$\int_\Omega u^2 \,\mathrm{d}x \leqslant \left(\frac{L}{\pi}\right)^{2i} \int_\Omega \left(u^{(i)}\right)^2 \mathrm{d}x, \quad i = 1, 2, \dots, n. \tag{3}$$

Proof. The case $i = 0$ is the usual Poincaré inequality. If $i = 2, 4, \dots, n-2$, then relation (2) follows directly from Poincaré's inequality. Hence, (2) is proved for $i = 0, 2, 4, \dots, n-2$.

Suppose now that $i = 1, 3, \dots, n-1$.

Integrating by parts, using Hölder's inequality and (2) with i replaced by $i-1$ (bear in mind that $i-1$ is even), we get

$$\begin{aligned}\int_\Omega \left(u^{(i)}\right)^2 \mathrm{d}x &= -\int_\Omega u^{(i-1)}u^{(i+1)} \,\mathrm{d}x \leqslant \left(\int_\Omega \left(u^{(i-1)}\right)^2 \mathrm{d}x\right)^{1/2} \left(\int_\Omega \left(u^{(i+1)}\right)^2 \mathrm{d}x\right)^{1/2} \\ &\leqslant \left(\left(\frac{L}{\pi}\right)^2 \int_\Omega \left(u^{(i)}\right)^2 \mathrm{d}x\right)^{1/2} \left(\int_\Omega \left(u^{(i+1)}\right)^2 \mathrm{d}x\right)^{1/2}\end{aligned}$$

from which

$$\int_\Omega \left(u^{(i)}\right)^2 \mathrm{d}x \leqslant \left(\frac{L}{\pi}\right)^2 \int_\Omega \left(u^{(i+1)}\right)^2 \mathrm{d}x.$$

Inequality (3) follows from (2). $\square$

Remark 2. An immediate consequence of Lemma 2 is the inequality

$$\frac{1}{\tilde{C}(L, n)} \|u\|_{H^n(\Omega)} \leqslant \|u^{(n)}\|_{L^2(\Omega)} \leqslant \|u\|_{H^n(\Omega)} \tag{4}$$

with $\tilde{C}(L, n) := \left(\sum_{i=0}^{n-1} (L/\pi)^{2(n-i)} + 1\right)^{1/2}$ and $u \in H(\Omega)$.

Indeed, from (2) one has

$$\|u^{(i)}\|^2_{L^2(\Omega)} \leqslant \left(\frac{L}{\pi}\right)^{2(n-i)} \|u^{(n)}\|^2_{L^2(\Omega)}$$

for each $i \in \{0, 1, 2, \dots, n-1\}$, and so (4) is easily obtained.

Remark 3. Using (4), we get that the scalar product

$$(u, v)_{H(\Omega)} = \int_\Omega u^{(n)} v^{(n)} \, \mathrm{d}x$$

induces the norm on $H(\Omega)$

$$\|u\|^* = (u, u)^{1/2}_{H(\Omega)}$$

equivalent to the norm $\|\cdot\|_{H^n(\Omega)}$ in the space $H(\Omega)$.

Lemma 3. *Let $u \in H(\Omega)$. Then we have the estimate*

$$\int_\Omega u^2 \, \mathrm{d}x \leqslant \left(\frac{L}{\pi}\right)^{2n} \|u^{(n)}\|^2_{L^2(\Omega)}. \tag{5}$$

Proof. We get inequality (5) by rewriting inequality (3) with $i = n$. □

In the sequel, we will put

$$\begin{aligned} A^- &:= \min_{x\in[0,L]} A(x), & A^+ &:= \max_{x\in[0,L]} A(x), \\ B^- &:= \min_{x\in[0,L]} B(x), & B^+ &:= \max_{x\in[0,L]} B(x), \\ C^- &:= \min_{x\in[0,L]} C(x), & C^+ &:= \max_{x\in[0,L]} C(x). \end{aligned}$$

The following extension of Lemma 8 in [20] is one of the main tools.

Lemma 4. *Let $u \in H(\Omega)$. Suppose that one of the following relations holds true*:

(i) $A, B, C > 0$ *in* $\overline{\Omega}$.

$$(A^+)^2 < 4B^-, \ \frac{(A^+)^2}{4B^-} \leqslant C^- - 1, \tag{6}$$

$$(A^+)^2 < 4C^-, \ B^- \geqslant C^-, \ \frac{(A^+)^2}{4C^-} \leqslant C^- - 1, \tag{7}$$

$$A^+ \leqslant B^-/2, \ C^- \geqslant 1, \quad 2A^+ < n\alpha \quad \textit{for some } \alpha \in (0, 1). \tag{8}$$

(ii) $A < 0$, $B, C > 0$ *in* $\overline{\Omega}$.

$$\frac{(B^-)^2}{-A^+} \leqslant 4C^-, \tag{9}$$

$$\frac{(B^-)^2}{-4A^+} - 4A^+ \leqslant C^- - 1. \tag{10}$$

(iii) $A = 0$, $B < 0$, $C > 0$ *in* $\overline{\Omega}$.

$$C^- - 1 \geqslant \frac{B^- - nB^-}{n}\left(\frac{-2B^-}{\alpha n}\right)^{1/n} \quad \textit{for some } \alpha \in (0,1), \tag{11}$$

$$(B^-)^2 < 2C^-, \qquad \frac{(B^-)^2}{2C^-} \leqslant C^- - 1. \tag{12}$$

(iv) *A, B, C functions of arbitrary sign.*

$$\max\bigl\{A^+K^{n-2},\ A^+K^{n-2} - B^-K^{n-1},\ A^+K^{n-2} - B^-K^{n-1} - C^-K^n\bigr\} < 1,$$

where $K = (L/\pi)^2$.

Then there exists a constant k *such that*

$$\int_\Omega \bigl[\bigl(u^{(n)}\bigr)^2 - A(x)(u'')^2 + B(x)(u')^2 + C(x)u^2\bigr]\,\mathrm{d}x \geqslant k\|u\|^2_{H^n(\Omega)}. \tag{13}$$

Proof. The proof follows from Lemmas 2.3, 2.5, and 2.6 in [14] if we are under assumptions (6), (7), (9), (10), and (12).

Case (iv) is proven in Proposition 2.2 of [10] when $n = 3$. Our general case follows by using similar arguments.

To prove inequality (13) under hypothesis (8), we can use the same technique based on the Fourier transform presented in the proof of Lemma 2.5 in [14]. Hence, it suffices to show that there exists $\alpha \in (0,1)$ such that

$$A^+\xi^4 \leqslant \alpha\xi^{2n} + B^-\xi^2 + C^- - 1 \quad \forall \xi \in \mathbb{R}, \tag{14}$$

which is equivalent to show that there exists $\alpha \in (0,1)$ such that

$$A^+t^2 \leqslant \alpha t^n + B^-t + C^- - 1 \quad \forall t \geqslant 0.$$

Let $\varphi : [0,\infty) \to \mathbb{R}$,

$$\varphi(t) = \alpha t^n + B^-t + C^- - 1 - A^+t^2.$$

$B^- \geqslant 2A^+$ implies that $\varphi' \geqslant 0$ on $[0,1]$. We see that

$$\varphi'(t) = B^- + n\alpha t\left(t^{n-2} - \frac{2A^+}{n\alpha}\right).$$

Due to the inequality

$$1 - \frac{2A^+}{n} \geqslant 0,$$

it follows that $\varphi' \geqslant 0$ on $[1,\infty)$. Hence, $\varphi' \geqslant 0$ on $[0,\infty)$. Since $\varphi(0) \geqslant 0$, we get (14).

To prove the result under hypothesis (11), we show that there exists $\alpha \in (0,1)$ such that

$$-B^-\xi^2 \leqslant \alpha\xi^{2n} + C^- - 1 \quad \forall \xi \in \mathbb{R},$$

i.e., we have to prove that

$$-B^- t \leqslant \alpha t^n + C^- - 1 \quad \forall t \in [0, \infty). \tag{15}$$

We define $\varphi : [0, \infty) \to \mathbb{R}$,

$$\varphi(t) = \alpha t^n + B^- t + C^- - 1,$$

and see that φ has a global minimum at $t_0 = (-B^-/(n\alpha))^{1/(n-1)} > 0$.

To prove (15), we have to show that $\varphi(t_0) \geqslant 0$, which means that

$$t_0\left(\alpha \frac{-B^-}{n\alpha} + B^-\right) + C^- - 1 \geqslant 0,$$

which is equivalent to (11) and hence is true. □

In order to prove existence when f is asymptotically linear at infinity, we also need the next elementary result, which is stated without proof.

Lemma 5. *The polynomial $Q^*(\xi) = \xi^{2n} - A^+\xi^4 + B^-\xi^2 + C^-$ is bounded from below by a strictly positive constant δ (and hence, $Q(\xi) = \xi^{2n} - A\xi^4 + B\xi^2 + C$ is bounded by the same constant) on the interval $[\pi/L, \infty)$, where δ can be chosen as follows (n is even):*

(i) $C > 0$ *in* $\overline{\Omega}$.

- *If $A \leqslant 0$, $B \geqslant 0$ in $\overline{\Omega}$, then we can take $\delta = (\pi/L)^{2n}$.*
- *If $A \leqslant 0$, $B < 0$ in $\overline{\Omega}$, the following two situations may occur:*
  1. *Q^* has no positive roots. Then we can choose $\delta = \sqrt[n-1]{(B^-/C^-)^2}$.*
  2. *Q^* has exactly two positive roots $0 < \xi_0 < \xi_1$. Suppose there is a natural number $k \geqslant 1$ such that $k\pi/L < \xi_0$ and $(k+1)\pi/L > \xi_1$. In this case, we can choose*

$$\delta = \min\left\{Q^*\left(\frac{\pi}{L}\right), \dots, Q^*\left(\frac{k\pi}{L}\right), Q^*\left(\frac{(k+1)\pi}{L}\right)\right\}. \tag{16}$$

- *If $A \geqslant 0$, $B < 0$ in $\overline{\Omega}$ the following three situations may occur:*
  1. *Q^* has no positive roots (case when $Q^* > 0$). We can choose $\delta = Q^*(\nu^2)$, where ν is the only root of $nt^{n-1}2A^+t + B^- = 0$.*
  2. *Q^* has exactly one positive root ξ_0. Suppose that there is a natural number $k \geqslant 1$ such that $k\pi/L < \xi_0$ and $(k+1)\pi/L > \xi_0$. We can take δ as in (16).*
  3. *Q^* has exactly two positive roots $0 < \xi_0 < \xi_1$. Suppose that there is a natural number $k \geqslant 1$ such that $k\pi/L < \xi_0$ and $(k+1)\pi/L > \xi_1$. Then δ is given by relation (16).*
- *If $A \geqslant 0$, $B = 0$ in $\overline{\Omega}$ the following two situations may occur:*
  1. *Q^* has no positive roots (case when $Q^* > 0$). Then $\delta = Q^*(\nu)$, where $\nu = \sqrt[n]{A^+/2}$.*

2. *Q^* has exactly two positive roots $0 < \xi_0 < \xi_1$. Suppose that there is a natural number $k \geqslant 1$ such that $k\pi/L < \xi_0$ and $(k+1)\pi/L > \xi_1$. Then δ is given by relation* (16).

- *If $A, B > 0$ in $\overline{\Omega}$ the following two situations may occur*:
  1. *Q^* has no positive roots. Then $\delta = Q^*(C^-)$.*
  2. *Q^* has exactly two positive roots $0 < \xi_0 < \xi_1$. Suppose that there is a natural number $k \geqslant 1$ such that $k\pi/L < \xi_0$ and $(k+1)\pi/L > \xi_1$. Then δ is given by relation* (16).

(ii) *$C < 0$ in $\overline{\Omega}$.*

- *If $A < 0$ in $\overline{\Omega}$, then Q^* has exactly one positive root ξ_0. Assuming that $\pi/L > \xi_0$, we can take $\delta = Q^*(\pi/L)$.*
- *If $A, B > 0$ in $\overline{\Omega}$, the following three situations may occur*:
  1. *Q^* has exactly one positive root ξ_0. Assuming that $\pi/L > \xi_0$, we can take $\delta = Q^*(\pi/L)$.*
  2. *Q^* has two positive roots $0 < \xi_0 < \xi_1$.*
     (a) *$Q^* \leqslant 0$ on $(0, \xi_1)$. Supposing $\pi/L > \xi_0$, we can take $\delta = Q^*(\pi/L)$.*
     (b) *$Q^* < 0$ on $(0, \xi_0)$ and $Q^* > 0$ on (ξ_0, ξ_1). Assuming that there is a natural number $k \geqslant 1$ such that $k\pi/L < \xi_0$ and $(k+1)\pi/L > \xi_1$, we can take δ as in* (16).
  3. *Q^* has three positive roots $0 < \xi_0 < \xi_1 < \xi_2$. Suppose that there is a natural number $k \geqslant 1$ such that $k\pi/L \in (\xi_0, \xi_1)$ and $(k+1)\pi/L > \xi_2$. Then δ is given by relation* (16)

Remark 4.

(i) Of course, if $A \leqslant 0$, $B, C \geqslant 0$ in $\overline{\Omega}$, then Lemma 4 is always true, i.e., there is nothing to prove.

(ii) Lemma 4 shows that

$$\|u\| = \left(\int\limits_{\Omega} \left(\left(u^{(n)}\right)^2 - A(x)(u'')^2 + B(x)(u')^2 + C(x)u^2 \right) \mathrm{d}x \right)^{1/2}$$

is a norm equivalent to the norms $\|\cdot\|_{H^n(\Omega)}$ and $\|\cdot\|^*$ on the space $H(\Omega)$.

We also recall the meaning of a weak solution to (P).

Definition 1. A weak solution of (P) is a function $u \in H(\Omega)$ such that

$$\int\limits_{\Omega} \left(u^{(n)} v^{(n)} - A(x)u''v'' + B(x)u'v' + C(x)uv - f(x,u)v \right) \mathrm{d}x = 0 \quad \forall v \in H(\Omega). \tag{17}$$

A classical solution of (P) is a function $u \in C^{2n}(\overline{\Omega})$ that satisfies (P).

Remark 5. Since f is a continuous function on $\overline{\Omega} \times \mathbb{R}$, it follows that a weak solution of (P) belongs to $C^{2n}(\overline{\Omega})$ (to prove, imitate the proof in [20, p. 493]). Also, this regularity allows us to obtain the last part of the boundary conditions, i.e., $u^{(i)} = 0$ on $\partial\Omega$ for $i = n, n+2, \dots, 2n-2$, and to conclude that any weak solution is a classical solution.

Indeed, if we choose $v \in H(\Omega)$ and substitute it in (17), we get by integrating by parts that

$$u^{(n)}v^{(n-1)}\big|_0^L + u^{(n+2)}v^{(n-3)}\big|_0^L + u^{(n+4)}v^{(n-5)}\big|_0^L + \cdots + u^{(2n-2)}v'\big|_0^L + \int\limits_\Omega \big(u^{(2n)} - \big(A(x)u''\big)'' - \big(B(x)u'\big)' + C(x)u - f(x,u)\big)v\,\mathrm{d}x = 0.$$

Since v is arbitrary, it follows that $u^{(i)} = 0$ on $\partial\Omega$ for $i = n, n+2, \dots, 2n-2$ and that u is a classical solution to (P).

In order to clarify the variational structure of problem (P), we introduce the functionals $\Phi, \Psi : H(\Omega) \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2}\|u\|^2, \quad \Psi(u) = \int\limits_0^L F\big(x, u(x)\big)\,\mathrm{d}x \quad \forall u \in H(\Omega), \tag{18}$$

where $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$ for every $(x,t) \in [0,L] \times \mathbb{R}$.

Using standard arguments, one can verify that Φ and Ψ are continuously Gâteaux differentiable, in particular, for every $u, v \in H(\Omega)$,

$$\Phi'(u)(v) = \int\limits_0^L \big(u^{(n)}(x)v^{(n)}(x) - A(x)u''(x)v''(x) + B(x)u'(x)v'(x) + C(x)u(x)v(x)\big)\,\mathrm{d}x$$

and

$$\Psi'(u)(v) = \int\limits_0^L f\big(x, u(x)\big)v(x)\,\mathrm{d}x.$$

Moreover, we observe that the functional $J : H(\Omega) \to \mathbb{R}$, defined by

$$J(u) = \Phi(u) - \Psi(u) = \frac{1}{2}\int\limits_\Omega \big(\big(u^{(n)}\big)^2 - A(x)(u'')^2 + B(x)(u')^2 + C(x)u^2\big)\,\mathrm{d}x - \int\limits_\Omega F(x,u)\,\mathrm{d}x,$$

is the energy functional related to problem (P), i.e., its critical points are weak solutions of problem (P).

Now, we recall some general definitions. Let $(X, \|\cdot\|)$ be a Banach space, its dual space is X^*. Let $J : X \to \mathbb{R}$ be a Gâteaux differentiable functional.

Definition 2. We say that J satisfies the Palais–Smale condition (for short, (PS) condition) if any sequence $\{u_m\}_{m\in\mathbb{N}} \subseteq X$ such that

(P_1) $\{J(u_m)\}_{m\in\mathbb{N}}$ is bounded,
(P_2) $\{J'(u_m)\}_{m\in\mathbb{N}}$ converges to 0 in X^*

admits a convergent subsequence in X.

Definition 3. For a fixed $r \in \mathbb{R}$, we say that $J = \Phi - \Psi$ satisfies $(\mathrm{PS})^{[r]}$ condition if any sequence $\{u_m\}$ such that

(P_1) $\{J(u_m)\}_{m\in\mathbb{N}}$ is bounded,
(P_2) $\{J'(u_m)\}_{m\in\mathbb{N}}$ converges to 0 in X^*,
(P_3) $\Phi(u_m) < r$ for all $m \in \mathbb{N}$

has a convergent subsequence.

Remark 6. By standard computations, J satisfies the (PS) and $(\mathrm{PS})^{[r]}$, $r > 0$, conditions; see [17, Lemma 7]) and [2, Remark 2.1], respectively.

3 Asymptotic linear case

Our first existence result is the following theorem.

Theorem 1. *Suppose* (H_1) *and* (H_2) *are fulfilled. If one of the conditions of Lemma* 5 *is satisfied, then the boundary value problem* (P) *has at least one nontrivial solution, provided*

$$\delta > \frac{2}{L}\|L_1\|_{L^\infty(\Omega)}\left(\frac{L}{\pi}\right)^{2n}. \tag{19}$$

Proof. We first establish that $J(u)$ is bounded from below by a negative constant.

By (H_1) we see that there exist $M > 0$ and $\varepsilon = \varepsilon(M) > 0$ such that for all $|s| > M$ and for all $x \in \Omega$,

$$\frac{f(x,s)}{s} \leqslant \|L_1\|_{L^\infty(\Omega)} + \varepsilon.$$

Hence, by the continuity of F there exists $C > 0$ such that

$$\begin{aligned} -F(x,s) &= -\int_0^1 f(x,st)s\,\mathrm{d}t \\ &\geqslant -\frac{(\|L_1\|_{L^\infty(\Omega)} + \varepsilon)s^2}{2} - C \quad \forall (x,s) \in \Omega \times \mathbb{R}. \end{aligned} \tag{20}$$

We recall that the set of functions $\{\sin(m\pi x/L),\ m \in \mathbb{N},\ m \geqslant 1\}$ is a complete orthogonal basis in $H(\Omega)$.

Hence, any $u \in H(\Omega)$ can be written

$$u(x) = \sum_{m=1}^{\infty} c_m \sin\frac{m\pi x}{L}, \quad c_m \in \mathbb{R},$$

and its norm in $H(\Omega)$ is given by

$$\left(\|u\|^*\right)^2 = \sum_{m=1}^{\infty} c_m^2.$$

We get

$$J(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega} F(x,u)\,\mathrm{d}x = \frac{L}{4}\sum_{m=1}^{\infty} c_m^2 Q\left(\frac{m\pi}{L}\right) - \int_{\Omega} F(x,u)\,\mathrm{d}x$$
$$\geqslant \frac{L}{4}\sum_{m=1}^{\infty} c_m^2 Q^*\left(\frac{m\pi}{L}\right) - \int_{\Omega} F(x,u)\,\mathrm{d}x.$$

Using Lemma 5, which shows that there exists a constant $\delta > 0$ such that $Q^* > \delta$ on $[\pi/L, \infty)$, (20), (5), and the fact that $\varepsilon(M) \to 0$ as $M \to \infty$, we have that

$$J(u) \geqslant \frac{\delta L}{4}\left(\|u\|^*\right)^2 - \frac{(\|L_1\|_{L^\infty(\Omega)} + \varepsilon)}{2}\int_{\Omega} u^2\,\mathrm{d}x - CL$$
$$\geqslant \frac{1}{2}(\|u\|^*)^2\left(\frac{\delta L}{2} - \left(\|L_1\|_{L^\infty(\Omega)} + \varepsilon\right)\left(\frac{L}{\pi}\right)^{2n}\right) - CL,$$

which, by the equivalence of $\|\cdot\|$ and $\|\cdot\|^*$, shows that J is bounded from below if (19) is satisfied.

By assumption (H_2) we get that $\mu^* = \inf_{H(\Omega)} J(u) < 0$, and hence, the solution we are going to find in the sequel is nontrivial. For a proof, see the paper [14].

We now prove the existence of a solution.

Ekeland's variational principle (see, for instance, [16, Thm. 2.2]) ensures that there exists a minimizing sequence $\{u_m\}_m$ in $H(\Omega)$ such that

$$J(u_m) \to \mu^* \quad \text{and} \quad J'(u_m) \to 0 \quad \text{as } m \to \infty.$$

Since J satisties (PS) condition, we get a subsequence $\{u_m\}_m$ strongly convergent to u_0 in $H(\Omega)$. Finally, we get that there exists $u_0 \in H(\Omega)$ such that $J'(u_0) = 0$, $J(u_0) < 0$, i.e., problem (P) has at least a nontrivial solution. □

Now, we solve the case when f is linear at both zero and infinity.

Theorem 2. *Suppose that assumptions* (H_1) *and* (H_3) *are fulfilled, where* $L_1, L_3 \in L^\infty(\Omega)$*,* $\|L_1\|_{L^\infty(\Omega)} > 0$*,* $\|L_3\|_{L^\infty(\Omega)} > \Lambda_{n,1} > 0$*, the coefficients* A *and* B *are constant, and* $C = 0$*. If condition* (iv) *of Lemma* 4 *with either* $A < 0$*,* $B \in \mathbb{R}$ *or* $A, B > 0$ *holds, then the boundary value problem* (P) *has at least one nontrivial solution, provided* (19) *is satisfied. Here* $\Lambda_{n,1}$ *is the first eigenvalue of* (P) *in relation to the norm* $\|\cdot\|$ *on* $H(\Omega)$*.*

Proof. Since Lemma 5 holds in the case $C < 0$ with $A < 0$, $B \in \mathbb{R}$ or $A, B > 0$, it is clear that it also holds if $C = 0$. Hence, we can follow the arguments presented in Theorem 1 to conclude the existence of a solution.

It remains to show that the solution is nontrivial, i.e., we must find $e \in H(\Omega)$ such that $J(e) < 0$.

From (H_1) we get that there exists $M > 0$ and $\delta_1 > 0$ such that for all $|s| > M$ and for all $x \in \Omega$,

$$\frac{f(x,s)}{s} \geqslant \|L_1\|_{L^\infty(\Omega)} - \delta_1.$$

As in the proof of Theorem 1, we get that

$$F(x,s) \geqslant \frac{(\|L_1\|_{L^\infty(\Omega)} - \delta_1)s^2}{2} \quad \forall x \in \Omega,\ |s| > M. \tag{21}$$

In a similar manner, using (H_3), there exists $\varepsilon > 0$ and $\delta_2 > 0$ such that

$$F(x,s) \geqslant \frac{(\Lambda_{n,1} + \delta_2)s^2}{2} \quad \forall x \in \Omega,\ |s| < \varepsilon. \tag{22}$$

Using (21), (22), and the fact that F is a continuous function, we can find $\delta > 0$ such that for sufficiently large K, the following inequality holds:

$$F(x,s) \geqslant \frac{(\Lambda_{n,1} + \delta)s^2}{2} - Ks^4 \quad \forall (x,s) \in \Omega \times \mathbb{R}. \tag{23}$$

Now we choose $e = s\varphi_1 \in H(\Omega)$, where φ_1 is the eigenfunction that corresponds to the eigenvalue $\Lambda_{n,1}$.

By (23) we get

$$\begin{aligned} J(e) &= \frac{1}{2}\|e\|^2 - \int_\Omega F(x,e)\,\mathrm{d}x \\ &\leqslant \frac{s^2}{2}\|\varphi_1\|^2 - \frac{s^2}{2}(\Lambda_{n,1} + \delta)\int_\Omega \big(\varphi_1(x)\big)^2\,\mathrm{d}x + Ks^4\int_\Omega \big(\varphi_1(x)\big)^4\,\mathrm{d}x. \end{aligned}$$

Since φ_1 is an eigenfunction, i.e.,

$$\int_\Omega \big(\varphi_1(x)\big)^2\,\mathrm{d}x = \frac{1}{\Lambda_{n,1}}\|\varphi_1\|^2,$$

it follows that

$$J(e) \leqslant -\frac{\delta s^2}{2\Lambda_{n,1}}\|\varphi_1\|^2 + Ks^4\int_\Omega \big(\varphi_1(x)\big)^4\,\mathrm{d}x.$$

Choosing s sufficiently small, we conclude that there exists $e \in H(\Omega)$ such that $J(e) < 0$ and the proof follows. □

Finally, we give an application of Theorem 2.

Example 1. Let $l, l_2 > 0$, $l_1 > \Lambda_{n,1} > 0$. Then the continuous function

$$f_1(s) = \begin{cases} \frac{ls^2 + l_1 s}{1 + l_2 s}, & s \geqslant 0, \\ l_1 s \ln(\arctan s + \mathrm{e}), & s < 0, \end{cases}$$

satisfies the requirements of Theorem 2. Hence, the boundary value problem (P) with f replaced by f_1 has at least one nontrivial solution if (19) holds true.

4 Superlinear case

In this section, we are going to present multiplicity results. We will use a particular function belonging to $H(\Omega)$, which is presented below.

Let $n \geqslant 2$ and $L > 0$. We consider $\alpha > 2/L$ and $d > 0$ and denote by $(\bar{a}_1, \bar{a}_3, \dots, \bar{a}_{2n-3}, \bar{a}_{2n-2})$ the solution of the following linear system of n equations and n unknowns $a_1, a_3, \dots, a_{2k-1}, \dots, a_{2n-3}, a_{2n-2}$:

$$\sum_{k=1}^{n-1} a_{2k-1} + a_{2n-2} = d,$$
$$\sum_{k=1}^{n-1} (2k-1)a_{2k-1} + (2n-2)a_{2n-2} = 0,$$
$$\sum_{k=2}^{n-1} (2k-1)(2k-2)a_{2k-1} + (2n-2)(2n-3)a_{2n-2} = 0,$$
$$\sum_{k=2}^{n-1} (2k-1)(2k-2)(2k-3)a_{2k-1} + (2n-2)(2n-3)(2n-4)a_{2n-2} = 0,$$
$$\dots,$$
$$\sum_{k=n-2}^{n-1} \frac{(2k-1)!}{(2k-n+1)!} a_{2k-1} + \frac{(2n-2)!}{n!} a_{2n-2} = 0,$$
$$\sum_{k=n-2}^{n-1} \frac{(2k-1)!}{(2k-n)!} a_{2k-1} + \frac{(2n-2)!}{(n-1)!} a_{2n-2} = 0.$$

A straightforward computation shows that the function

$$v(x) := \sum_{k=1}^{n-1} \bar{a}_{2k-1} \alpha^{2k-1} x^{2k-1} + \bar{a}_{2n-2} \alpha^{2n-2} x^{2n-2}$$

satisfies the following conditions:

$$v\left(\frac{1}{\alpha}\right) = d, \quad v'\left(\frac{1}{\alpha}\right) = 0, \quad v''\left(\frac{1}{\alpha}\right) = 0, \quad v'''\left(\frac{1}{\alpha}\right) = 0, \quad \dots,$$
$$v^{(n-2)}\left(\frac{1}{\alpha}\right) = 0, \quad v^{(n-1)}\left(\frac{1}{\alpha}\right) = 0.$$

Moreover, it is simple to verify that $v(0) = v''(0) = v^{(4)}(0) = \dots = v^{2n-4}(0) = 0$.
Now, we denote by $w_{\alpha,d}$ the function defined as follows:

$$w_{\alpha,d}(x) = \begin{cases} v(x), & x \in [0, \frac{1}{\alpha}], \\ d, & x \in [\frac{1}{\alpha}, L - \frac{1}{\alpha}], \\ v(L-x), & x \in [L - \frac{1}{\alpha}, L]. \end{cases} \tag{24}$$

Properties of function v ensure that $w_{\alpha,d} \in H(\Omega)$ and

$$\begin{aligned}
&\|w_{\alpha,d}\|^2_{H(\Omega)} \\
&\quad = \int_0^L \Big(\big(w^{(n)}_{\alpha,d}(x)\big)^2 - A(x)\big(w''_{\alpha,d}(x)\big)^2 + B(x)\big(w'_{\alpha,d}(x)\big)^2 + C(x)\big(w_{\alpha,d}(x)\big)^2 \Big)\,\mathrm{d}x \\
&\quad \leqslant \int_0^L \Big(\big(w^{(n)}_{\alpha,d}(x)\big)^2 - A^-\big(w''_{\alpha,d}(x)\big)^2 + B^+\big(w'_{\alpha,d}(x)\big)^2 + C^+\big(w_{\alpha,d}(x)\big)^2 \Big)\,\mathrm{d}x \\
&\quad \leqslant d^2\big(\mathcal{P} - A^-\mathcal{Q} + B^+\mathcal{R} + C^+\mathcal{S}\big),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{P} &:= \frac{2}{d^2}\int_0^{1/\alpha} \big(v^{(n)}(x)\big)^2\,\mathrm{d}x \\
&= \frac{2}{d^2}\int_0^{1/\alpha} \left(\frac{(2n-3)!}{(n-3)!}\bar{a}_{2n-3}\alpha^{2n-3}x^{n-3} + \frac{(2n-2)!}{(n-2)!}\bar{a}_{2n-2}\alpha^{2n-2}x^{n-2} \right)^2 \mathrm{d}x, \\
\mathcal{Q} &:= \frac{2}{d^2}\int_0^{1/\alpha} \big(v''(x)\big)^2\,\mathrm{d}x \\
&= \frac{2}{d^2}\int_0^{1/\alpha} \left(\sum_{k=2}^{n-1} \frac{(2k-1)!}{(2k-3)!}\bar{a}_{2k-1}\alpha^{2k-1}x^{2k-3} + \frac{(2n-2)!}{(2n-4)!}\bar{a}_{2n-2}\alpha^{2n-2}x^{2n-4} \right)^2 \mathrm{d}x, \\
\mathcal{R} &:= \frac{2}{d^2}\int_0^{1/\alpha} \big(v'(x)\big)^2\,\mathrm{d}x \\
&= \frac{2}{d^2}\int_0^{1/\alpha} \left(\sum_{k=1}^{n-1} \frac{(2k-1)!}{(2k-2)!}\bar{a}_{2k-1}\alpha^{2k-1}x^{2k-2} + \frac{(2n-2)!}{(2n-3)!}\bar{a}_{2n-2}\alpha^{2n-2}x^{2n-3} \right)^2 \mathrm{d}x, \\
\mathcal{S} &:= \frac{2}{d^2}\int_0^{1/\alpha} \big(v(x)\big)^2\,\mathrm{d}x + \left(L - \frac{2}{\alpha}\right) \\
&= \frac{2}{d^2}\int_0^{1/\alpha} \left(\sum_{k=1}^{n-1} \bar{a}_{2k-1}\alpha^{2k-1}x^{2k-1} + \bar{a}_{2n-2}\alpha^{2n-2}x^{2n-2} \right)^2 \mathrm{d}x + \left(L - \frac{2}{\alpha}\right).
\end{aligned}$$

Remark 7. For fixed n and L, the constants $\mathcal{P}, \mathcal{Q}, \mathcal{R}$, and $\mathcal{S}$ depend on $\alpha > 2/L$ and can be easily obtained. For instance,

- when $n = 3$, it results
$$v(x) = 2d\alpha x - 2d\alpha^3 x^3 + d\alpha^4 x^4$$
and
$$\mathcal{P} = 96\alpha^5, \qquad \mathcal{Q} = \frac{48}{5}\alpha^3, \qquad \mathcal{R} = \frac{104}{35}\alpha, \qquad \mathcal{S} = L - \frac{263}{315\alpha};$$
- when $n = 4$, it results
$$v(x) = \frac{9}{4}d\alpha x - \frac{5}{2}d\alpha^3 x^3 + \frac{9}{4}d\alpha^5 x^5 - d\alpha^6 x^6$$
and
$$\mathcal{P} = 3240\alpha^7, \qquad \mathcal{Q} = \frac{95}{7}\alpha^3, \qquad \mathcal{R} = \frac{513}{154}\alpha, \qquad \mathcal{S} = L - \frac{5983}{8008\alpha}.$$

The main tools in our approach of the superlinear case are critical point theorems. Now, for fixed $r > 0$, we define

$$\underline{\varphi}(r) := \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r}, \qquad \overline{\varphi}(r) := \sup_{u\in\Phi^{-1}(]0,r[)}\frac{\Psi(u)}{\Phi(u)}.$$

The following theorem, obtained in [3] as a consequence of [2, Thm. 4.1], guarantees the existence of at least one nonzero local minimum.

Theorem 3. (See [3, *Thm.* 2.3].) *Let X be a real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. *Assume that there exists $r > 0$ such that*

$$\underline{\varphi}(r) < \overline{\varphi}(r), \tag{25}$$

and for each $\lambda \in \Lambda_r :=]1/\overline{\varphi}(r),\, 1/\underline{\varphi}(r)[$, the function $J_\lambda = \Phi - \lambda\Psi$ satisfies $(\mathrm{PS})^{[r]}$ *condition. Then, for each $\lambda \in \Lambda_r$, there is $u_\lambda \in \Phi^{-1}(]0, r[)$ (hence, $u_\lambda \neq 0$) such that $J_\lambda(u_\lambda) \leqslant J_\lambda(u)$ for all $u_\lambda \in \Phi^{-1}(]0, r[)$ and $J'_\lambda(u_\lambda) = 0$.*

The following theorem established in [8] guarantees the existence of two nonzero critical points. It is a consequence of the previous nonzero local minimum theorem and the classical Ambrosetti–Rabinowitz theorem established in [1]; see also [9] and [19].

Theorem 4. *Let X be a real Banach space, and let Φ, $\Psi : X \to \mathbb{R}$ be two functions of class C^1 such that* $\inf_X \Phi(u) = \Phi(0) = \Psi(0) = 0$. *Assume that there is $r \in \mathbb{R}$ such that* (25) *holds and, for each $\lambda \in \Lambda_r :=]1/\overline{\varphi}(r),\, 1/\underline{\varphi}(r)[$, the function $J_\lambda = \Phi - \lambda\Psi$ satisfies* (PS) *condition, and it is unbounded from below. Then, for each $\lambda \in \Lambda_r$, the function J_λ admits at least two nonzero critical points $u_{\lambda,1}, u_{\lambda,2} \in X$ such that $J(u_{\lambda,1}) < 0 < J(u_{\lambda,2})$.*

For each $\alpha > 2/L$, the following technical constant will be useful:

$$\mathcal{K}_\alpha := \mathcal{P} - A^- \mathcal{Q} + B^+ \mathcal{R} + C^+ \mathcal{S},$$

where $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$ are the real numbers we have just introduced. Moreover, we define

$$\mathcal{A}(c) = \int_0^L \max_{|s| \leqslant c} F(t,s)\,\mathrm{d}t \quad \text{with } c > 0,$$

$$\mathcal{B}(d) = \int_{1/\alpha}^{L-1/\alpha} F(t,d)\,\mathrm{d}t + \inf_{s\in[0,d]} \left(\int_0^{1/\alpha} F(t,s)\,\mathrm{d}t + \int_{L-1/\alpha}^{L} F(t,s)\,\mathrm{d}t \right) \quad \text{with } d > 0.$$

4.1 Existence of at least one nonzero solution of (P)

Our first multiplicity result is the following.

Theorem 5. *Assume that there exist $\alpha > 2/L$ and two positive constants c, d with $d < c$ such that*

$$\frac{\mathcal{A}(c)}{c^2} < \frac{k}{2\eta^2} < \frac{\mathcal{K}_\alpha}{2} < \frac{\mathcal{B}(d)}{d^2}, \tag{26}$$

where k and η are introduced respectively in (13) *and Lemma* 1. *Then problem* (P) *admits at least one nonzero classical solution $\bar{u}$ such that $\|\bar{u}\|_\infty < c$.*

Proof. Let Φ and Ψ be defined as in (18) and $J = \Phi - \Psi$. Our aim is to apply Theorem 3 with $X = H(\Omega)$ and $\lambda = 1$. Clearly, Φ and Ψ satisfy all regularity assumptions (see in particular Remark 6). Now, we prove (25). To this end, put $r = c^2 k/(2\eta^2)$. On one hand, we have

$$\underline{\varphi}(r) = \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} \leqslant \frac{2\eta^2}{k}\frac{\mathcal{A}(c)}{c^2}. \tag{27}$$

Indeed, for each $u \in X$ with $\Phi(u) \leqslant r$, due to Lemma 1 and (13), it results

$$\|u\|_\infty \leqslant \|u\|_{C^{n-1}(\bar{\Omega})} \leqslant \eta\|u\|_{H^n(\Omega)} \leqslant \frac{\eta}{\sqrt{k}}\|u\| \leqslant \frac{\eta}{\sqrt{k}}\sqrt{2r} = c. \tag{28}$$

One has

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) \leqslant \sup_{\|u\|\leqslant\sqrt{2r}} \int_0^L F\big(t,u(t)\big)\,\mathrm{d}t \leqslant \int_0^L \max_{|s|\leqslant c} F(t,s)\,\mathrm{d}t = \mathcal{A}(c).$$

On the other hand, we claim that

$$\overline{\varphi}(r) = \sup_{u\in\Phi^{-1}(]0,r[)} \frac{\Psi(u)}{\Phi(u)} \geqslant \frac{2}{\mathcal{K}_\alpha}\frac{\mathcal{B}(d)}{d^2}. \tag{29}$$

For our goal, we consider the function $w_{\alpha,d}$ defined by (24). Clearly, $w_{\alpha,d} \in X$ and $\|w_{\alpha,d}\|^2 \leqslant \mathcal{K}_\alpha d^2$. Now, taking into account that from $d < c$ one has $A(c) \geqslant B(d)$, due to (26), it follows that $\Phi(w_{\alpha,d}) = \|w_{\alpha,d}\|^2/2 \leqslant d^2\mathcal{K}_\alpha/2 < r$. Moreover, since $w_{\alpha,d}(t) \in [0,d]$ for each $t \in [0,L]$, one has

$$\begin{aligned}\Psi(w_{\alpha,d}) &= \int\limits_{1/\alpha}^{L-1/\alpha} F(t,d)\,\mathrm{d}t + \int\limits_{0}^{1/\alpha} F\bigl(t, w_{\alpha,d}(t)\bigr)\,\mathrm{d}t + \int\limits_{L-1/\alpha}^{L} F\bigl(t, w_{\alpha,d}(t)\bigr)\,\mathrm{d}t \\ &\geqslant \int\limits_{1/\alpha}^{L-1/\alpha} F(t,d)\,\mathrm{d}t + \inf_{s\in[0,d]}\left(\int\limits_{0}^{1/\alpha} F(t,s)\,\mathrm{d}t + \int\limits_{L-1/\alpha}^{L} F(t,s)\,\mathrm{d}t\right) = \mathcal{B}(d),\end{aligned}$$

so that

$$\overline{\varphi}(r) \geqslant \frac{\Psi(w_{\alpha,d})}{\Phi(w_{\alpha,d})} \geqslant \frac{2}{\mathcal{K}_\alpha}\frac{\mathcal{B}(d)}{d^2},$$

and our claim is proved.

From (27), (29) one has

$$\underline{\varphi}(r) = \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} \leqslant 2\delta^2\frac{\mathcal{A}(c)}{c^2} < 1 < \frac{2}{\mathcal{K}_\alpha}\frac{\mathcal{B}(d)}{d^2} \leqslant \overline{\varphi}(r),$$

and (25) is achieved.

Hence, Theorem 3 ensures that J admits a local minimum $\bar{u}$, which, due to Remark 5, is a classical solution of problem (P). Finally, from $\Phi(\bar{u}) \leqslant r$ and (28) one has $\|\bar{u}\|_\infty < c$, and the proof is completed. □

Finally, we give two applications of Theorem 5

Example 2. Consider the function $f_2(s) = \ln(|s|+1) + |s|/(|s|+1) + 1$. Note that the function f_2 has no asymptotic sublinear or linear behaviour at zero, and hence, we cannot apply Theorem 1 or Theorem 2. Since $F_2(s) = s\ln(|s|+1) + s$, we get that for some $c > d > 0$, $L = 1$, and $\alpha > 2$,

$$\frac{\mathcal{A}(c)}{c^2} = \frac{\ln(c+1)+1}{c}, \qquad \frac{\mathcal{B}(d)}{d^2} = \left(1 - \frac{2}{\alpha}\right)\frac{\ln(d+1)+1}{d}.$$

Taking c sufficiently large and d sufficiently small, relation (26) holds if for some $\alpha > 2$,

$$\frac{k}{\eta^2} < \mathcal{K}_\alpha. \tag{30}$$

As a consequence, if (30) is fulfilled, then the boundary value problem (P) with f replaced by f_2 has at least one nonzero classical solution if $\Omega = (0,1)$.

Example 3. This example shows that the exponential function $f_3(s) = \mathrm{e}^s$ satisfies the requirements of Theorem 5 if $\Omega = (0,1)$. Indeed, since $F_3(s) = \mathrm{e}^s - 1$, we get that

$$\frac{\mathcal{A}(c)}{c^2} = \frac{\mathrm{e}^c - 1}{c^2} = g(c), \qquad \frac{\mathcal{B}(d)}{d^2} = \left(1 - \frac{2}{\alpha}\right)\left(\frac{\mathrm{e}^d - 1}{d^2}\right).$$

We can easily check that $\inf_{s>0} g(s) < g(1.6) < 1.55$. Taking d sufficiently small, relation (26) holds if for some $\alpha > 2$, $1.55 \leqslant k/(2\eta^2) < \mathcal{K}_\alpha/2$. As a consequence, the boundary value problem (P) with f replaced by f_3 has at least one nonzero solution.

4.2 Existence of at least two nonzero solutions of (P)

Our second result is the following.

Theorem 6. *Assume that there exist two positive constants c and d with $d < c$ such that* (26) *holds and there are $\sigma > 2$, $M > 0$ satisfying*

$$0 < \sigma F(x,s) \leqslant sf(x,s) \quad \forall x \in \overline{\Omega},\ \forall |s| > M. \tag{AR}$$

Then problem (P) *admits at least two nonzero classical solutions.*

Proof. Our aim is to apply Theorem 4. To this end, we define X, Φ, and Ψ as in the proof of Theorem 5. Therefore, the same computations of the previous proof ensure that (25) is verified. Further, by standard computations, condition (AR) implies that J satisfies (PS) condition and it is unbounded from below; see, for instance, [19]. Hence, Theorem 4 establishes the existence of two nonzero critical points of J, which, due to Remark 5, are classical solutions of problem (P) and the conclusion follows. □

Author contributions. All authors have read and approved the published version of the manuscript.

Conflicts of interest. The authors declare no conflicts of interest.

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