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Generalizations of Darbo's fixed point-theorem and its application to the solvability of a nonlinear fractional differential equation

Maha Belhadj[a], **Jamal Rezaei Roshan**[b, 1], **Stojan Radenović**[c]

[a] Department of Mathematics, Sousse University,
Higher School of Sciences and Technologies of H. Sousse,
MaPSFA Laboratory, LR11ES35, 4011, H. Sousse, Tunisia
belhadj.maha@essths.u-sousse.tn

[b] Department of Mathematics, Qaemshahr Branch,
Islamic Azad University, Qaemshahr, Iran
jmlroshan@gmail.com; jml.roshan@iau.ac.ir

[c] Faculty of Mechanical Engineering, University of Belgrade,
Kraljice Marije 16, 11120 Beograd 35, Sebia
sradenovic@mas.bg.ac.rs

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Abstract. This paper introduces novel fixed-point theorems for generalized Proinov contraction mappings utilizing the measure of noncompactness. These results significantly extend existing contraction principles and provide novel methods for analyzing nonlinear problems. We demonstrate the practical power of our theorems by establishing the existence of solutions to a broad class of nonlinear fractional differential equations with integral boundary conditions. An illustrative example underscores the effectiveness of our approach, promising impactful applications in fractional calculus and nonlinear analysis. Overall, these results enrich the theoretical framework and offer valuable insights for researchers working on complex dynamical systems and applied mathematical models.

Keywords: Proinov-type contraction, measure of noncompactness, fractional differential equation.

1 Introduction

The Banach contraction principle (BCP) [2] is a fundamental theorem that addresses several issues about the existence and uniqueness of fixed points in nonlinear analysis. It has been employed in a variety of fields such as physics, economics, biology, and different branches of science. In Banach spaces, Schauder [20] demonstrated in 1930 that

[1] Corresponding author.

all continuous compact mappings in closed convex subsets have at least one fixed point; this result generalizes the Brouwer fixed-point theorem to infinite-dimensional Banach spaces. BCP has many extensions that have appeared in several papers. In [9, 10], Ćirić investigated a new category of generalized contractions and derived the existence and uniqueness of fixed points, and in [7, 8], the authors established a generalization of BCP. Another generalization is due to Proinov; he proved new results for contractive form mappings, which was issued in 2020 [17]. The class of α-admissible mappings was introduced by Samet et al. [19] in 2012, and they established a general fixed-point result of contractive mapping named α-θ-contractive-type mapping in metric spaces. Furthermore, the authors in [13, 17] developed this type and obtained diverse fixed-point theorems. In [5], Alghamdi et al. defined an interesting form of contraction that was called α-ϑ-θ-contractions. They showed that if a mapping satisfies this new condition of contraction, then the existence and uniqueness of fixed point are verified.

Recently, the analysis of fractional differential equations and systems has attracted significant attention, especially, concerning the existence and uniqueness of solutions under various boundary conditions. Many studies have contributed valuable insights to this field. For example, Afshari et al. [1] investigated integral boundary value problems with impulsive conditions, employing a generalized contraction method to establish the existence of solutions involving the Atangana–Baleanu Caputo (ABC) fractional derivative in the Mittag-Lefler sense. Kassim et al. [14] examined conditions that lead to the nonexistence of solutions for a nonlinear system of fractional differential equations, utilizing properties of fractional derivatives, the test-function method, and integral inequalities. Roomi et al. [18] investigated the existence of solutions for specific classes of μ-Caputo fractional differential equations and inclusion problems with nonlocal μ-integral boundary conditions, applying F-contraction and convex F-contraction techniques supported by relevant examples. Furthermore, Zubair et al. [23] introduced the concept of fuzzy extended hexagonal b-metric spaces as a generalization of fuzzy rectangular b-metric spaces, proving various fixed point theorems, including Banach-type results, which serve as fundamental tools for solving nonlinear Caputo fractional differential equations. Lastly, Zubair et al. [24] investigated coupled systems of nonlinear fractional differential equations within complex-valued fuzzy rectangular b-metric spaces, establishing existence and uniqueness theorems supported by illustrative examples.

On the other hand, Kuratowski [16] described the idea of the measure of noncompactness (MNC). It has an important topic in the evolution of metric and topological fixed points in order to discuss the existence of solutions of fractional differential equations. Darbo [11] introduced a generalization of Schauder and BCP for contractions, concerning the measure of noncompactness, within nonempty closed convex subsets of Banach spaces. In [21], the authors presented new generalizations for α-θ-condensing mappings connected with a measure of noncompactness and proved various results that guaranteed the existence of fixed points. They also investigated a result for β-θ-condensing mappings via the class of β-admissible mappings. In [12, 15, 22], the authors examine the existence and stability of solutions to a category of nonlinear differential equations using the Caputo fractional derivative ${}^cD^\alpha$ under Riemann–Liouville fractional integral boundary conditions I^α. In the current work, we characterize the notion of α-ϑ-θ-contractions on nonempty subsets of Banach spaces and come up with new results about the existence of solutions for any measure of noncompactness that doesn't meet the maximum property. Moreover, we provide Krasnosel'skii-form fixed-point results. Our outcomes extend and develop the results of Proinov [17] and Darbo [11] in Banach spaces.

We examine the solvability of solutions for a category of nonlinear fractional differential equations (NFDE) characterized by ${}^{c}D^{\alpha}$ and involving I^{α}. Ultimately, we exemplify our findings with a case study.

2 Preliminaries

We start by stating Schauder's fixed-point theorem.

Theorem 1. (*See* [20].) *Consider D to be a closed, convex, bounded, and nonempty subset of Banach space $\mathbb{E}$. Assume that $\mathcal{T} : D \to D$ is a compact and continuous mapping. Then $\mathcal{T}$ possesses a fixed point.*

In this paper, $\mathbb{N}^* = \mathbb{N} \cup \{0\}$, $\mathbb{E}$ is a Banach space with norm $\|\cdot\|$, all nonempty and bounded subsets of $\mathbb{E}$ are represented by the set $\mathcal{B}(\mathbb{E})$, while all nonempty, bounded, and closed subsets of $\mathbb{E}$ are represented by the set $\mathcal{BC}(\mathbb{E})$. The family of all nonempty, bounded, closed, and convex subsets of $\mathbb{E}$ is represented by $\Omega(\mathbb{E})$, while the set of all relatively compact subsets of $\mathbb{E}$ is denoted by $\mathcal{K}(\mathbb{E})$.

We define the notion of an MNC.

Definition 1. (See [6].) A function $\Phi : \mathcal{B}(\mathbb{E}) \to \mathbb{R}_+$ is called an MNC on E if the following conditions are satisfied:

(i) $\Phi(\overline{\operatorname{conv}}(\mathcal{A})) = \Phi(\mathcal{A})$ for each $\mathcal{A} \in \mathcal{B}(\mathbb{E})$;
(ii) $\mathcal{A}_1 \subset \mathcal{A}_2 \Rightarrow \Phi(\mathcal{A}_1) \leqslant \Phi(\mathcal{A}_2)$ for each $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{B}(\mathbb{E})$;
(iii) $\operatorname{Ker}(\Phi) \neq \emptyset$, and if $\Phi(\mathcal{A}) = 0$, then $\mathcal{A} \subseteq \mathcal{K}(\mathbb{E})$;
(iv) $\Phi(\lambda\mathcal{A}_1 + (1-\lambda)\mathcal{A}_2) \leqslant \lambda\Phi(\mathcal{A}_1) + (1-\lambda)\Phi(\mathcal{A}_2)$ for each $\lambda \in [0,1]$, and $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{B}(\mathbb{E})$;
(v) If the sequence $\{\mathcal{A}_n\}$ of $\mathcal{BC}(\mathbb{E})$ is decreasing and $\lim_{n\to+\infty} \Phi(\mathcal{A}_n) = 0$, then $\mathcal{A}_{+\infty} = \bigcap_{n=1}^{+\infty} \mathcal{A}_n \neq \emptyset$.

Definition 2. (See [6].) We also mention the following conditions:

(i) If $\operatorname{Ker}(\Phi) = \mathcal{K}(\mathbb{E})$, then MNC Φ is said to be complete;
(ii) If $\Phi(\mathcal{A}_1 \cup \mathcal{A}_2) = \max\{\Phi(\mathcal{A}_1), \Phi(\mathcal{A}_2)\}$ for each $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{B}(\mathbb{E})$, then MNC Φ satisfies the maximum property.

Kuratowski MNC σ [16] can be characterized as follows:

$$\sigma(\mathcal{A}) = \inf\left\{ d > 0\colon \mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_i \text{ for some } \mathcal{A}_i \text{ with } \operatorname{diam}(\mathcal{A}_i) \leqslant d \right\}, \quad \mathcal{A} \subseteq \mathcal{B}(\mathbb{E}),$$

where $\operatorname{diam}(\mathcal{A})$ is the diameter of $\mathcal{A}$.

Theorem 2. (*See* [11].) *Let $\varPhi$ be an MNC and $D \in \varOmega(\mathbb{E})$. Assume that $\mathcal{T} : D \to D$ is continuous and*

$$\varPhi(\mathcal{T}\mathcal{A}) \leqslant k\varPhi(\mathcal{A}),$$

where $k \in [0, 1)$. Then $\mathcal{T}$ possesses a fixed point.

The following auxiliary definitions and theorems are used to support our main findings (see [5, 17, 19, 21] for more details).

Let $\vartheta, \theta : (0, +\infty) \to \mathbb{R}$ be two mappings under the following circumstances:

(a) ϑ is nondecreasing;
(b) $\theta(\iota) < \vartheta(\iota)$ for each $\iota > 0$;
(c) $\limsup_{\iota \to \iota_0^+} \theta(\iota) < \vartheta(\iota_0+)$ for each $\iota_0 > 0$;
(d) $\theta(\iota) < \vartheta(\iota-)$ for each $\iota > 0$.

Now we present two functions that satisfy the conditions of the auxiliary definition above.

Example 1. The mappings listed below meet requirements (a)–(d).

$$\vartheta(\iota) = \mathrm{e}^{\iota} \quad \text{and} \quad \theta(\iota) = 1 + \iota^2, \quad \iota \in (0, +\infty);$$

$$\vartheta(\iota) = \begin{cases} \frac{3}{2}, & \iota \in [0, 1], \\ \iota, & \iota \in (1, +\infty), \end{cases} \quad \text{and} \quad \theta(\iota) = \begin{cases} 2\iota^2, & \iota \in [0, \frac{1}{2}], \\ \frac{1}{2}, & \iota \in (\frac{1}{2}, 1], \\ \ln(\iota), & \iota \in (1, +\infty). \end{cases}$$

The class of α-admissible mappings was introduced by Samet et al. in 2012 [19].

Definition 3. (See [19].) Let $\alpha : \mathbb{E}\times\mathbb{E} \to [0, +\infty)$. If $\alpha(\nu, \omega) \geqslant 1$ implies $\alpha(\mathcal{T}\nu, \mathcal{T}\omega) \geqslant 1$ for any $\nu, \omega \in \mathbb{E}$, then $\mathcal{T} : \mathbb{E} \to \mathbb{E}$ is α-admissible.

Example 2. (See [19].) We define $\mathcal{T} : [0, +\infty) \to [0, +\infty)$ and $\alpha : [0, +\infty)\times[0, +\infty) \to [0, +\infty)$ by

$$\mathcal{T}\nu = \ln\bigl(1 + \mathrm{e}^{\nu}\bigr) \quad \text{and} \quad \alpha(\nu, \omega) = \begin{cases} 2, & \nu \geqslant \omega, \\ 0, & \nu < \omega, \end{cases}$$

respectively. Then $\mathcal{T}$ is α-admissible.

Theorem 3. (*See* [19].) *Assume that the metric space $(\mathbb{E}, d)$ is complete. Assume that $\mathcal{T} : \mathbb{E} \to \mathbb{E}$ is continuous and α-admissible, so that for any $\nu, \omega \in \mathbb{E}$,*

$$\alpha(\nu, \omega)d(\mathcal{T}\nu, \mathcal{T}\omega) \leqslant \phi\bigl(d(\nu, \omega)\bigr),$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function satisfying $\sum_{n=1}^{+\infty} \phi^n(\iota) < +\infty$. If there exists $\nu_0 \in \mathbb{E}$ such that

$$\alpha(\nu_0, \mathcal{T}\nu_0) \geqslant 1, \tag{1}$$

then $\mathcal{T}$ has a distinct fixed point.

In 2020, Proinov [17] presented a new form of contractions.

Definition 4. (See [17].) Consider the metric space $(\mathbb{E}, d)$. A Proinov-type contraction is $\mathcal{T} : \mathbb{E} \to \mathbb{E}$ if, for every $\nu, \omega \in \mathbb{E}$,

$$\vartheta\big(d(\mathcal{T}\nu, \mathcal{T}\omega)\big) \leqslant \theta\big(d(\nu, \omega)\big)$$

with $\theta(\iota) < \vartheta(\iota)$ for each $\iota > 0$.

Theorem 4. (*See* [5,17].) *Let* $(\mathbb{E}, d)$ *be a complete metric space. Assume that* $\mathcal{T} : \mathbb{E} \to \mathbb{E}$ *is a Proinov-type contraction such that* ϑ *and* θ *satisfy conditions* (a)–(c). $\mathcal{T}$ *then has a distinct fixed point.*

Definition 5. (See [17].) Consider the metric space $(\mathbb{E}, d)$. A generalized Proinov-type contraction is $\mathcal{T} : \mathbb{E} \to \mathbb{E}$ if, for every $\nu, \omega \in \mathbb{E}$,

$$\begin{aligned} &\vartheta\big(d(\mathcal{T}\nu, \mathcal{T}\omega)\big) \\ &\quad \leqslant \theta\left(\max\left\{d(\nu, \omega), d(\nu, \mathcal{T}\nu), d(\omega, \mathcal{T}\omega), \frac{1}{2}\big(d(\nu, \mathcal{T}\omega) + d(\omega, \mathcal{T}\nu)\big)\right\}\right) \end{aligned}$$

with $\theta(\iota) < \vartheta(\iota)$ for each $\iota > 0$.

Theorem 5. (*See* [17].) *Let* $(\mathbb{E}, d)$ *be a complete metric space. Assume that* $\mathcal{T} : \mathbb{E} \to \mathbb{E}$ *is a generalized Proinov-type contraction such that* ϑ *and* θ *satisfy conditions* (a)–(d). *Then* $\mathcal{T}$ *has a distinct fixed point.*

Definition 6. (See [5].) Consider the metric space $(\mathbb{E}, d)$. $\mathcal{T} : \mathbf{E} \to \mathbb{E}$ is an $\alpha, \vartheta, \theta$-contraction if for every $\nu, \omega \in \mathbb{E}$,

$$\begin{aligned} &\alpha(\nu, \omega)\vartheta\big(d(\mathcal{T}\nu, \mathcal{T}\omega)\big) \\ &\quad \leqslant \theta\left(\max\left\{d(\nu, \omega), d(\nu, \mathcal{T}\nu), d(\omega, \mathcal{T}\omega), \frac{1}{2}\big(d(\nu, \mathcal{T}\omega) + d(\omega, \mathcal{T}\nu)\big)\right\}\right) \end{aligned}$$

with $\theta(\iota) < \vartheta(\iota)$ for each $\iota > 0$.

Theorem 6. (*See* [5].) *Let* $(\mathbb{E}, d)$ *be a complete metric space. Assume that* $\mathcal{T} : \mathbb{E} \to \mathbb{E}$ *is an* $\alpha, \vartheta, \theta$*-contraction such that* ϑ *and* θ *satisfy conditions* (a)–(c). *Then* $\mathcal{T}$ *has a distinct fixed point.*

The authors defined a new kind of mapping known as β-admissible in [21]:

Definition 7. (See [21].) Let $\beta : \mathcal{B}(\mathbb{E}) \to [0, +\infty)$. If $\beta(\mathcal{A}) \geqslant 1$ implies $\beta(\overline{\mathrm{conv}}\mathcal{T}\mathcal{A}) \geqslant 1$ for any $\mathcal{A} \in \mathcal{B}(\mathbb{E})$, then $\mathcal{T} : \mathbb{E} \to \mathbb{E}$ is β-admissible.

Example 3. We define $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ and $\beta : 2^{\mathbb{R}} \to [0, +\infty)$ by

$$\mathcal{T}\nu = \begin{cases} \ln(1 - |\nu|), & |\nu| < 1, \\ \ln 2, & |\nu| \geqslant 1, \end{cases} \quad \text{and} \quad \beta(\mathcal{A}) = \begin{cases} 2, & \mathcal{A} \subseteq [-1, 1], \\ 0 & \text{otherwise}, \end{cases}$$

respectively. Then $\mathcal{T}$ is β-admissible.

3 Fixed-point results

In the sequel, by combinations of notions α-admissible and $\alpha, \vartheta, \theta$-contraction mappings, we present some new generalizations of Darbo's fixed-point theorem.

Theorem 7. *Let $D \in \Omega(\mathbb{E})$, Φ be an MNC, and $\alpha : \mathbb{E} \times \mathbb{E} \to [0, +\infty)$. Suppose that $\mathcal{T} : D \to D$ is continuous and α-admissible such that for each $\mathcal{A}_1, \mathcal{A}_2 \subset D$,*

$$\begin{aligned}&\alpha(\nu, \mathcal{T}\nu)\vartheta(\Phi(\mathcal{T}\mathcal{A}_1))\\&\quad \leqslant \theta\left(\max\left\{\Phi(\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\Phi(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\right\}\right),\end{aligned} \tag{2}$$

where ϑ and θ satisfy condition (c)*, and ϑ is lower semicontinuous. If there is a closed convex $\mathcal{A}_0 \subset D$ and $\nu_0 \in \mathcal{A}_0$ such that $\mathcal{T}$ satisfies* (1)*, then $\mathcal{T}$ possesses a fixed point.*

Proof. Let, for each $n \in \mathbb{N}$,

$$\mathcal{A}_n = \overline{\operatorname{conv}}(\mathcal{T}\mathcal{A}_{n-1}), \qquad \nu_n = \mathcal{T}\nu_{n-1}.$$

For some $m \in \mathbb{N}^*$, if $\Phi(\mathcal{A}_m) = 0$, then $\mathcal{A}_m \subseteq \mathcal{K}(\mathbb{E})$. Therefore, according to Theorem 1, $\mathcal{T}$ possesses a fixed point.

For every $n \in \mathbb{N}^*$, we now suppose that $\Phi(\mathcal{A}_n) > 0$. By Eq. (1), we have $\alpha(\nu_0, \nu_1) = \alpha(\nu_0, \mathcal{T}\nu_0) \geqslant 1$, and $\mathcal{T}$ is an α-admissible mapping. This signifies that $\alpha(\nu_1, \nu_2) \geqslant 1$, then $\alpha(\nu_n, \nu_{n+1}) \geqslant 1$ for each $n \in \mathbb{N}^*$. Next, by our contraction condition, we obtain

$$\begin{aligned}\vartheta\big(\Phi(\mathcal{A}_{n+1})\big) &\leqslant \alpha(\nu_n, \nu_{n+1})\vartheta\big(\Phi(\mathcal{A}_{n+1})\big)\\&= \alpha(\nu_n, \nu_{n+1})\vartheta\big(\Phi\big(\overline{\operatorname{conv}}(\mathcal{T}\mathcal{A}_n)\big)\big) = \alpha(\nu_n, \nu_{n+1})\vartheta\big(\Phi(\mathcal{T}\mathcal{A}_n)\big)\\&\leqslant \theta\left(\max\left\{\Phi(\mathcal{A}_n), \Phi(\mathcal{T}\mathcal{A}_n), \Phi(\mathcal{T}\mathcal{A}_{n+1}), \frac{1}{2}\Phi(\mathcal{T}\mathcal{A}_n \cup \mathcal{T}\mathcal{A}_{n+1})\right\}\right)\\&\leqslant \theta\left(\max\left\{\Phi(\mathcal{A}_n), \Phi(\mathcal{A}_{n+1}), \Phi(\mathcal{A}_{n+2}), \frac{1}{2}\Phi(\mathcal{A}_n \cup \mathcal{A}_{n+1})\right\}\right)\\&\leqslant \theta\left(\max\left\{\Phi(\mathcal{A}_n), \Phi(\mathcal{A}_n), \Phi(\mathcal{A}_n), \frac{1}{2}\Phi(\mathcal{A}_n)\right\}\right)\\&= \theta\big(\Phi(\mathcal{A}_n)\big).\end{aligned} \tag{3}$$

Since $\{\Phi(\mathcal{A}_n)\}$ is a decreasing sequence and $\Phi(\mathcal{A}_n) \geqslant 0$, there exists $\delta \geqslant 0$ so that $\lim_{n\to+\infty} \Phi(\mathcal{A}_n) = \delta$. We claim that $\delta = 0$. We presume the contrary, that is, $\delta > 0$ and by (3),

$$\begin{aligned}\vartheta(\delta) &\leqslant \liminf_{n\to+\infty} \vartheta\big(\Phi(\mathcal{A}_{n+1})\big) \leqslant \liminf_{n\to+\infty} \theta\big(\Phi(\mathcal{A}_n)\big)\\&\leqslant \limsup_{n\to+\infty} \theta\big(\Phi(\mathcal{A}_n)\big) \leqslant \limsup_{s\to\delta} \theta(s).\end{aligned}$$

This is absurd. Therefore, $\delta = 0$. According to Definition 1, $\mathcal{A}_{+\infty} = \bigcap_{n=1}^{+\infty} \mathcal{A}_n$ is a nonempty closed convex subset of $\mathbb{E}$ such that $\Phi(\mathcal{A}_{+\infty}) = 0$; i.e., $\mathcal{A}_{+\infty}$ is compact. As a result, by Theorem 1, $\mathcal{T} : \mathcal{A}_{+\infty} \to \mathcal{A}_{+\infty}$ has a fixed point. □

Corollary 1. *Let $D \in \Omega(\mathbb{E})$, Φ be an MNC, and $\alpha : \mathbb{E} \times \mathbb{E} \to [0, +\infty)$. Suppose that $\mathcal{T} : D \to D$ is continuous and α-admissible such that for each $\mathcal{A}_1, \mathcal{A}_2 \subset D$,*

$$\begin{aligned}&\alpha(\nu, \mathcal{T}\nu)\Phi(\mathcal{T}\mathcal{A}_1)\\&\quad \leqslant \theta\biggl(\max\biggl\{\Phi(\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\Phi(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\biggr\}\biggr),\end{aligned}$$

where $\theta : (0, +\infty) \to \mathbb{R}$ is a mapping such that:

(b′) $\theta(\iota) < \iota$ *for each* $\iota > 0$;
(c′) $\limsup_{\iota \to \iota_0^+} \theta(\iota) < \iota_0$ *for each* $\iota_0 > 0$.

Then $\mathcal{T}$ possesses a fixed point if there is a closed convex $\mathcal{A}_0 \subset D$ and $\nu_0 \in \mathcal{A}_0$ such that $\mathcal{T}$ satisfies (1).

Proof. The result is derived by selecting $\vartheta(\iota) = \iota$ in Theorem 7. □

Remark 1. Theorem 2 is obtained by selecting $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$, $\alpha(\nu, \omega) = 1$, and $\theta(\iota) = k\iota (0 \leqslant k < 1)$ for each $\iota \in (0, +\infty)$ in Corollary 1.

Theorem 8. *Let $D \in \Omega(\mathbb{E})$ and $\alpha : \mathbb{E} \times \mathbb{E} \to [0, +\infty)$. Suppose that $\mathcal{T} : D \to \mathbb{E}$ is continuous and α-admissible such that for each $\mathcal{A}_1, \mathcal{A}_2 \subset D$,*

$$\begin{aligned}&\alpha(\nu, \mathcal{T}\nu)\vartheta\bigl(\operatorname{diam}(\mathcal{T}\mathcal{A}_1)\bigr)\\&\quad \leqslant \theta\biggl(\max\biggl\{\operatorname{diam}(\mathcal{A}_1), \operatorname{diam}(\mathcal{T}\mathcal{A}_1), \operatorname{diam}(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\operatorname{diam}(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\biggr\}\biggr), \qquad (4)\end{aligned}$$

where ϑ and θ satisfy conditions (a)–(c)*, and ϑ is lower semicontinuous. If there is a closed convex $\mathcal{A}_0 \subset D$ and $\nu_0 \in \mathcal{A}_0$ such that $\mathcal{T}$ satisfies* (1)*, $\mathcal{T}$ then possesses a distinct fixed point.*

Proof. By choosing $\Phi(\mathcal{A}) = \operatorname{diam}(\mathcal{A})$ in Theorem 7, the existence of a fixed point can be seen immediately. Consequently, it is adequate to demonstrate the fixed point's uniqueness. Suppose that ν^* and ω^* are two fixed points such that $\nu^* \neq \omega^*$. Taking $\mathcal{A}_1 = \mathcal{A}_2 = \{\nu^*, \omega^*\}$, we can use the proof of Theorem 7 to get

$$\begin{aligned}&\vartheta\bigl(\|\nu^* - \omega^*\|\bigr)\\&\quad \leqslant \alpha(\nu, \mathcal{T}\nu)\vartheta\bigl(\|\mathcal{T}\nu^* - \mathcal{T}\omega^*\|\bigr) \leqslant \alpha(\nu, \mathcal{T}\nu)\vartheta\bigl(\operatorname{diam}(\mathcal{T}\mathcal{A}_1)\bigr)\\&\quad \leqslant \theta\biggl(\max\biggl\{\operatorname{diam}(\mathcal{A}_1), \operatorname{diam}(\mathcal{T}\mathcal{A}_1), \operatorname{diam}(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\operatorname{diam}(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\biggr\}\biggr)\\&\quad \leqslant \theta\bigl(\operatorname{diam}(\mathcal{A}_1)\bigr) < \vartheta\bigl(\operatorname{diam}(\mathcal{A}_1)\bigr) \leqslant \vartheta\bigl(\|\nu^* - \omega^*\|\bigr).\end{aligned}$$

This situation is illogical. Consequently, $\mathcal{T}$ possesses a distinct fixed point. □

Example 4. We define $\mathcal{T} : \mathcal{BC}(\mathbb{R}_+) \to \mathcal{BC}(\mathbb{R}_+)$ and $\alpha : \mathcal{BC}(\mathbb{R}_+) \times \mathcal{BC}(\mathbb{R}_+) \to [0, +\infty)$ by

$$\mathcal{T}\nu = \begin{cases} \frac{1-\nu}{4}, & \|\nu\| \leqslant 1, \\ 2\nu, & \|\nu\| > 1, \end{cases} \quad \text{and} \quad \alpha(\nu, \omega) = \begin{cases} 1, & \|\nu\| + \|\omega\| \leqslant 1, \\ 0, & \|\nu\| + \|\omega\| > 1, \end{cases}$$

respectively. Here $\mathcal{BC}(\mathbb{R}_+)$ denotes all compact intervals of $\mathbb{R}_+$.

Let $\Phi(\mathcal{A}) = \operatorname{diam}(\mathcal{A})$ be MNC in the sense of Definition 1.

Let $\|\nu\| + \|\omega\| \leqslant 1$, and let $\mathcal{A}_1 = [0, 1]$ and $\mathcal{A}_2 = [2, 4]$ be two subsets of $\mathcal{BC}(\mathbb{R}_+)$. We have

$$\Phi(\mathcal{A}_1) = 1, \qquad \Phi(\mathcal{T}\mathcal{A}_1) = \frac{1}{4}, \qquad \Phi(\mathcal{T}\mathcal{A}_2) = 4,$$

and

$$\begin{aligned} &\max\left\{\Phi(\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\Phi(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\right\} \\ &\quad = \max\left\{1, \frac{1}{4}, 4, \frac{\operatorname{diam}([0, \frac{1}{4}] \cup [4, 8])}{2}\right\} = \max\left\{1, \frac{1}{4}, 4, \frac{8}{2}\right\} \\ &\quad = 4, \end{aligned}$$

so

$$\begin{aligned} \vartheta(1) &= \alpha(\nu, \omega)\vartheta\big(\Phi(\mathcal{T}\mathcal{A}_1)\big) \\ &\leqslant \theta\left(\max\left\{\Phi(\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\Phi(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\right\}\right) \\ &= \theta(4) < \vartheta(4). \end{aligned}$$

Consequently, $\mathcal{T}$ satisfies Eq. (4). Hence, by Theorem 8, $\mathcal{T}$ possesses a fixed point $1/2$.

If $\|\nu\| + \|\omega\| > 1$, then Theorem 8 cannot be applied.

Theorem 9. *Let $D \in \Omega(\mathbb{E})$, Φ be a subadditive and complete MNC on* $\mathbf{E}$, *and $\alpha : \mathbb{E} \times \mathbb{E} \to [0, +\infty)$. Suppose that $\mathcal{T}, \mathcal{G} : D \to \mathbb{E}$ are continuous mappings such that*:

- (i) *$\mathcal{T}$ is α-admissible*;
- (ii) *$\mathcal{T}$ fulfil inequality* (2) *where ϑ and θ satisfy condition* (c) *and ϑ is lower semi-continuous*;
- (iii) *$\mathcal{G}$ is compact*;
- (iv) *$\mathcal{T}(\nu) + \mathcal{G}(\nu) \in D$, for each $\nu \in D$.*

If there is a closed convex $\mathcal{A}_0 \subset D$ and $\nu_0 \in \mathcal{A}_0$ such that $\mathcal{T}$ satisfies (1)*, then $\mathcal{T} + \mathcal{G}$ possesses a fixed point.*

Proof. It is obvious that $\mathcal{T} + \mathcal{G}$ is a well-defined self-mapping. Assume that $\mathcal{A}_1, \mathcal{A}_2 \subset D$, and since Φ is subadditive,

$$\begin{aligned} \alpha(\nu, \mathcal{T}\nu)\vartheta\big(\Phi\big((\mathcal{T} + \mathcal{G})(\mathcal{A}_1)\big)\big) &\leqslant \alpha(\nu, \mathcal{T}\nu)\vartheta\big(\Phi\big(\mathcal{T}(\mathcal{A}_1) + \mathcal{G}(\mathcal{A}_1)\big)\big) \\ &\leqslant \alpha(\nu, \mathcal{T}\nu)\vartheta\big(\Phi\big(\mathcal{T}(\mathcal{A}_1)\big) + \Phi\big(\mathcal{G}(\mathcal{A}_1)\big)\big). \end{aligned}$$

Consequently, given that $\mathcal{G}$ is compact and Φ represents a complete MNC in $\mathbb{E}$, we derive

$$\alpha(\nu, \mathcal{T}\nu)\vartheta\big(\Phi\big((\mathcal{T}+\mathcal{G})(\mathcal{A}_1)\big)\big) \leqslant \alpha(\nu, \mathcal{T}\nu)\vartheta\big(\Phi\big(\mathcal{T}(\mathcal{A}_1)\big)\big).$$

From Eq. (2)

$$\begin{aligned}&\alpha(\nu, \mathcal{T}\nu)\vartheta\big(\Phi\big((\mathcal{T}+\mathcal{G})(\mathcal{A}_1)\big)\big)\\ &\quad\leqslant \theta\left(\max\left\{\Phi(\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\Phi(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\right\}\right).\end{aligned}$$

Thus, by Theorem 7, $\mathcal{T}+\mathcal{G}$ possesses a fixed point. □

Corollary 2. *Let $D \in \Omega(\mathbb{E})$, Φ be a complete and subadditive MNC, and $\alpha : \mathbb{E}\times\mathbb{E} \to [0, +\infty)$. Suppose that $\mathcal{T}, \mathcal{G} : D \to \mathbb{E}$ are continuous such that*:

(i) *$\mathcal{T}$ is α-admissible, and $\vartheta : (0, +\infty) \to \mathbb{R}$ is a lower semicontinuous function such that for all $\iota_0 > 0$, $\vartheta(\iota_0+)$ exists, and*

$$\begin{aligned}&\alpha(\nu, \mathcal{T}\nu)\vartheta\big(\Phi(\mathcal{T}\mathcal{A}_1)\big)\\ &\quad\leqslant k\vartheta\left(\max\left\{\Phi(\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\Phi(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\right\}\right)\end{aligned}$$

for each $\mathcal{A}_1, \mathcal{A}_2 \subset D$ and $k \in [0, 1)$;

(ii) *$\mathcal{G}$ is compact*;

(iii) *$\mathcal{T}(\nu) + \mathcal{G}(\nu) \in D$, for each $\nu \in D$.*

If there is a closed convex $\mathcal{A}_0 \subset D$ and $\nu_0 \in \mathcal{A}_0$ such that $\mathcal{T}$ satisfies (1)*, then $\mathcal{T}+\mathcal{G}$ possesses a fixed point.*

Proof. By choosing $\theta(\iota) = k\vartheta(\iota)$ for $\iota \in (0, +\infty)$, we have

$$\limsup_{\iota\to\iota_0+}\theta(\iota) = k\limsup_{\iota\to\iota_0+}\vartheta(\iota) = k\lim_{\iota\to\iota_0+}\vartheta(\iota) = k\vartheta(\iota_0+) < \vartheta(\iota_0+).$$

So, by Theorem 9, $\mathcal{T}+\mathcal{G}$ possesses a fixed point. □

Corollary 3. *Let $D \in \Omega(\mathbb{E})$ and $\alpha : \mathbb{E}\times\mathbb{E} \to [0, +\infty)$. Suppose that $\mathcal{T}, \mathcal{G} : D \to \mathbb{E}$ are continuous such that*:

(i) *$\mathcal{T}$ is α-admissible, and $\vartheta : (0, +\infty) \to \mathbb{R}$ is a nondecreasing and lower semicontinuous function such that for all $\iota_0 > 0$, $\vartheta(\iota_0+)$ exists, and for all $u, \nu, \omega \in \mathcal{A}$ and $k \in [0, 1)$,*

$$\alpha(u, \mathcal{T}u)\vartheta\big(\|\mathcal{T}\nu - \mathcal{T}\omega\|\big) \leqslant k\vartheta\big(\|\nu-\omega\|\big); \tag{5}$$

(ii) *$\mathcal{G}$ is compact;*

(iii) *$\mathcal{T}(\nu) + \mathcal{G}(\nu) \in D$ for each $\nu \in D$.*

If there is a closed convex $\mathcal{A}_0 \subset D$ and $\nu_0 \in \mathcal{A}_0$ such that $\mathcal{T}$ satisfies (1)*, then $\mathcal{T}+\mathcal{G}$ possesses a fixed point.*

Proof. Let $\varPhi(\mathcal{A}) = \operatorname{diam}(\mathcal{A})$ be the MNC as a function of Definition 1. Let $\mathcal{A}_1, \mathcal{A}_2 \subset D$ and $\nu, \omega \in \mathcal{A}_1$.

From Eq. (5) and as ϑ is nondecreasing, we obtain

$$\begin{aligned}
&\alpha(u, \mathcal{T}u)\vartheta\bigl(\|\mathcal{T}\nu - \mathcal{T}\omega\|\bigr) \\
&\quad \leqslant k\vartheta\bigl(\|\nu - \omega\|\bigr) \leqslant k\vartheta\Bigl(\sup_{\nu,\omega\in\mathcal{A}_1} \|\nu - \omega\|\Bigr) = k\vartheta\bigl(\operatorname{diam}(\mathcal{A}_1)\bigr) \\
&\quad \leqslant k\vartheta\biggl(\max\biggl\{\operatorname{diam}(\mathcal{A}_1), \operatorname{diam}(\mathcal{T}\mathcal{A}_1), \operatorname{diam}(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\operatorname{diam}(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\biggr\}\biggr),
\end{aligned}$$

then

$$\begin{aligned}
&\alpha(u, \mathcal{T}u) \sup_{\nu,\omega\in\mathcal{A}_1} \vartheta\bigl(\|\mathcal{T}\nu - \mathcal{T}\omega\|\bigr) \\
&\quad \leqslant k\vartheta\biggl(\max\biggl\{Phi(\mathcal{A}_1), \varPhi(\mathcal{T}\mathcal{A}_1), \varPhi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\varPhi(\mathcal{T}\mathcal{A}_1) \cup \mathcal{T}\mathcal{A}_2\biggr\}\biggr).
\end{aligned}$$

Since ϑ is nondecreasing and lower semicontinuous, we obtain

$$\begin{aligned}
&\alpha(u, \mathcal{T}u)\vartheta\bigl(\varPhi(\mathcal{T}\mathcal{A}_1)\bigr) \\
&\quad \leqslant k\vartheta\biggl(\max\biggl\{\varPhi(\mathcal{A}_1), \varPhi(\mathcal{T}\mathcal{A}_1), \varPhi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\varPhi(\mathcal{T}\mathcal{A}_1) \cup \mathcal{T}\mathcal{A}_2\biggr\}\biggr).
\end{aligned}$$

Hence, from Corollary 2 $\mathcal{T} + \mathcal{G}$ possesses a fixed point. □

In the following, we show other results for β-admissible mappings.

Theorem 10. *Let $D \in \Omega(\mathbb{E})$, $\varPhi$ be an MNC, and $\beta : \mathcal{B}(\mathbb{E}) \to [0, +\infty)$. Suppose that $\mathcal{T} : D \to D$ is continuous and β-admissible such that for each $\mathcal{A}_1, \mathcal{A}_2 \subset D$,*

$$\begin{aligned}
&\beta(\mathcal{A}_1)\vartheta\bigl(\varPhi(\mathcal{T}\mathcal{A}_1)\bigr) \\
&\quad \leqslant \theta\biggl(\max\biggl\{\varPhi(\mathcal{A}_1), \varPhi(\mathcal{T}\mathcal{A}_1), \varPhi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\varPhi(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\biggr\}\biggr),
\end{aligned} \tag{6}$$

where ϑ and θ satisfy condition (c)*, and ϑ is lower semicontinuous. If there exists a closed convex $\mathcal{A}_0 \subset D$ such that*

$$\beta(\mathcal{A}_0) \geqslant 1, \tag{7}$$

then $\mathcal{T}$ possesses a fixed point.

Proof. We suppose

$$\mathcal{A}_n = \overline{\operatorname{conv}}(\mathcal{T}\mathcal{A}_{n-1}) \quad \text{for all } n \in \mathbb{N}.$$

If $\varPhi(\mathcal{A}_m) = 0$ for some $m \in \mathbb{N}^*$, then $\mathcal{A}_m \subseteq \mathcal{K}(\mathbb{E})$. Hence, by Theorem 1, $\mathcal{T}$ possesses a fixed point.

Now, we assume that $\varPhi(\mathcal{A}_n) > 0$ for each $n \in \mathbb{N}^*$. From Eq. (7) $\beta(\mathcal{A}_1) = \beta(\overline{\operatorname{conv}}(\mathcal{T}\mathcal{A}_0)) \geqslant 1$, then $\beta(\mathcal{A}_n) \geqslant 1$ for all $n \in \mathbb{N}^*$. Next, by our condition of contraction, we obtain

$$\begin{aligned}\vartheta\big(\Phi(\mathcal{A}_{n+1})\big) &\leqslant \beta(\mathcal{A}_n)\vartheta\big(\Phi(\mathcal{A}_{n+1})\big)\\ &= \beta(\mathcal{A}_n)\vartheta\big(\Phi\big(\overline{\operatorname{conv}}(\mathcal{T}\mathcal{A}_n)\big)\big) = \beta(\mathcal{A}_n)\vartheta\big(\Phi(\mathcal{T}\mathcal{A}_n)\big)\\ &\leqslant \theta\bigg(\max\bigg\{\Phi(\mathcal{A}_n), \Phi(\mathcal{T}\mathcal{A}_n), \Phi(\mathcal{T}\mathcal{A}_{n+1}), \frac{1}{2}\Phi(\mathcal{T}\mathcal{A}_n \cup \mathcal{T}\mathcal{A}_{n+1})\bigg\}\bigg)\\ &\leqslant \theta\bigg(\max\bigg\{\Phi(\mathcal{A}_n), \Phi(\mathcal{A}_{n+1}), \Phi(\mathcal{A}_{n+2}), \frac{1}{2}\Phi(\mathcal{A}_n \cup \mathcal{A}_{n+1})\bigg\}\bigg)\\ &\leqslant \theta\bigg(\max\bigg\{\Phi(\mathcal{A}_n), \Phi(\mathcal{A}_n), \Phi(\mathcal{A}_n), \frac{1}{2}\Phi(\mathcal{A}_n)\bigg\}\bigg)\\ &= \theta\big(\Phi(\mathcal{A}_n)\big).\end{aligned}$$

By the proof of Theorem 7, we obtain $\lim_{n\to+\infty}\Phi(\mathcal{A}_n) = 0$. Thus, by Definition 1, $\mathcal{A}_{+\infty} = \bigcap_{n=1}^{+\infty}\mathcal{A}_n$ is a nonempty closed convex subset of D such that $\Phi(\mathcal{A}_{+\infty}) = 0$; i.e., $\mathcal{A}_{+\infty}$ is compact. As a consequence, by Theorem 1, $\mathcal{T} : \mathcal{A}_{+\infty} \to \mathcal{A}_{+\infty}$ has a fixed point. □

Example 5. We define $\mathcal{T} : \mathcal{BC}(\mathbb{R}_+) \to \mathcal{BC}(\mathbb{R}_+)$ and $\beta : 2^{\mathcal{BC}(\mathbb{R}_+)} \to [0, +\infty)$ by

$$\mathcal{T}\nu = \begin{cases} 1 - \frac{\nu}{2}, & \|\nu\| \leqslant 1, \\ 3\nu - \frac{1}{2}, & \|\nu\| > 1, \end{cases} \quad \text{and} \quad \beta(\mathcal{A}) = \begin{cases} 1, & \mathcal{A} \subseteq [0,1], \\ 0 & \text{otherwise,} \end{cases}$$

respectively. Let $\Phi(\mathcal{A}) = \operatorname{diam}(\mathcal{A})$ be the MNC as a function of Definition 1.

If $\|\nu\| \leqslant 1$, let $\mathcal{A}_1 = [0,1]$ and $\mathcal{A}_2 = [2,3]$ be two subsets of $\mathcal{BC}(\mathbb{R}_+)$. Then we have $\Phi(\mathcal{A}_1) = 1$, $\Phi(\mathcal{T}\mathcal{A}_1) = 1/2$, $\Phi(\mathcal{T}\mathcal{A}_2) = 3$, and

$$\begin{aligned}&\max\bigg\{\Phi(\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\Phi(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\bigg\}\\ &\quad = \max\bigg\{1, \frac{1}{2}, 3, \frac{\operatorname{diam}([\frac{1}{2},1] \cup [\frac{11}{2}, \frac{17}{2}])}{2}\bigg\} = \max\bigg\{1, \frac{1}{2}, 3, \frac{8}{2}\bigg\}\\ &\quad = 4,\end{aligned}$$

so,

$$\begin{aligned}\vartheta\bigg(\frac{1}{2}\bigg) &= \beta\big(\mathcal{A}_1)\vartheta(\Phi(\mathcal{T}\mathcal{A}_1)\big)\\ &\leqslant \theta\bigg(\max\bigg\{\Phi(\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\Phi(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\bigg\}\bigg)\\ &= \theta(4) < \vartheta(4).\end{aligned}$$

Therefore, $\mathcal{T}$ satisfies Eq. (6), where $\vartheta : (0, +\infty) \to \mathbb{R}$ is nondecreasing. Thus, by Theorem 10, $\mathcal{T}$ possesses a fixed point $2/3$.

If $\|\nu\| > 1$, then Theorem 10 cannot be applied.

Remark 2. By choosing $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$, $\beta(\mathcal{A}_1) = 1$, $\vartheta(\iota) = \iota$, and $\theta(\iota) = k\iota (0 \leqslant k < 1)$, for each $\iota \in (0, +\infty)$ in Theorem 10, we obtain Theorem 2.

Theorem 11. *Let Φ be a complete and subadditive MNC, let $D \in \Omega(\mathbb{E})$, and let $\beta : \mathcal{B}(\mathbb{E}) \to [0, +\infty)$. Let $\mathcal{T}, \mathcal{G} : D \to \mathbb{E}$ be continuous in such a way that*:

(i) $\mathcal{T}$ *is β-admissible*;
(ii) $\mathcal{T}$ *fulfil* (6) *where ϑ and θ satisfy condition* (c) *and ϑ is lower semicontinuous*;
(iii) $\mathcal{G}$ *is compact*;
(iv) $\mathcal{T}(\nu) + \mathcal{G}(\nu) \in D$*, for each* $\nu \in D$.

If there is a closed convex $\mathcal{A}_0 \subset D$ such that $\mathcal{T}$ satisfies (7)*, then $\mathcal{T} + \mathcal{G}$ possesses a fixed point.*

Proof. It is obvious that $\mathcal{T} + \mathcal{G}$ is a well-defined self-mapping. Assume that $\mathcal{A}_1, \mathcal{A}_2 \subset D$, and since Φ is subadditive,

$$\begin{aligned}\beta(\mathcal{A}_1)\vartheta\big(\Phi\big((\mathcal{T} + \mathcal{G})(\mathcal{A}_1)\big)\big) &\leqslant \beta(\mathcal{A}_1)\vartheta\big(\Phi\big(\mathcal{T}(\mathcal{A}_1) + \mathcal{G}(\mathcal{A}_1)\big)\big) \\ &\leqslant \beta(\mathcal{A}_1)\vartheta\big(\Phi\big(\mathcal{T}(\mathcal{A}_1)\big) + \Phi\big(\mathcal{G}(\mathcal{A}_1)\big)\big).\end{aligned}$$

As a result, because $\mathcal{G}$ is compact and Φ is a complete MNC in $\mathbb{E}$, we obtain

$$\beta(\mathcal{A}_1)\vartheta\big(\Phi(mathcalT + \mathcal{G})(\mathcal{A}_1)\big) \leqslant \beta(\mathcal{A}_1)\vartheta\big(\Phi\big(\mathcal{T}(\mathcal{A}_1)\big)\big).$$

From Eq. (6)

$$\begin{aligned}&\beta(\mathcal{A}_1)\vartheta\big(\Phi\big((\mathcal{T} + \mathcal{G})(\mathcal{A}_1)\big)\big) \\ &\quad\leqslant \theta\left(\max\left\{\Phi(\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\Phi(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\right\}\right).\end{aligned}$$

Thus, by Theorem 10, $\mathcal{T} + \mathcal{G}$ possesses a fixed point. □

Corollary 4. *Let Φ be a complete and subadditive MNC, let $D \in \Omega(\mathbb{E})$, and let $\beta : \mathcal{B}(\mathbb{E}) \to [0, +\infty)$. Let $\mathcal{T}, \mathcal{G} : D \to \mathbb{E}$ be continuous in such a way that*:

(i) $\mathcal{T}$ *is β-admissible, and there exists a nondecreasing and lower semicontinuous function $\vartheta : (0, +\infty) \to \mathbb{R}$, provided that $\vartheta(\iota_0+)$ exists for each $\iota_0 > 0$ and such that for each $\mathcal{A} \subset D$,*

$$\begin{aligned}&\beta(\mathcal{A}_1)\vartheta\big(\Phi(\mathcal{T}\mathcal{A}_1)\big) \\ &\quad\leqslant k\vartheta\left(\max\left\{\Phi(\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\Phi(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\right\}\right)\end{aligned}$$

with $\mathcal{A}_1, \mathcal{A}_2 \subset D$ and $k \in [0, 1)$;
(ii) $\mathcal{G}$ *is compact*;
(iii) $\mathcal{T}(\nu) + \mathcal{G}(\nu) \in D$*, for each* $\nu \in D$.

If there is a closed convex $\mathcal{A}_0 \subset D$ such that $\mathcal{T}$ satisfies (7)*, then $\mathcal{T} + \mathcal{G}$ possesses a fixed point.*

Proof. For $\iota \in (0, +\infty)$, let $\theta(\iota) = k\vartheta(\iota)$ in Theorem 11. □

Corollary 5. *Let $D \in \Omega(\mathbb{E})$ and $\beta : \mathcal{B}(\mathbf{E}) \to [0, +\infty)$. Suppose that $\mathcal{T}, \mathcal{G} : D \to \mathbb{E}$ are continuous such that*:

(i) *$\mathcal{T}$ is β-admissible, and there exists a nondecreasing and lower semicontinuous function $\vartheta : (0, +\infty) \to \mathbb{R}$, provided that $\vartheta(\iota_0+)$ exists for each $\iota_0 > 0$ and such that for each $\nu, \omega \in \mathcal{A}_1$ and $k \in [0, 1)$,*

$$\beta(\mathcal{A}_1)\vartheta\big(\|\mathcal{T}\nu - \mathcal{T}\omega\|\big) \leqslant k\vartheta\big(\|\nu - \omega\|\big); \tag{8}$$

(ii) *$\mathcal{G}$ is compact*;

(iii) *$\mathcal{T}(\nu) + \mathcal{G}(\nu) \in D$, for each $\nu \in D$.*

If there is a closed convex $\mathcal{A}_0 \subset D$ such that $\mathcal{T}$ satisfies (7)*, then $\mathcal{T} + \mathcal{G}$ possesses a fixed point.*

Proof. Let $\Phi(\mathcal{A}) = \operatorname{diam}(\mathcal{A})$ be the MNC as a function of Definition 1. Let $\mathcal{A}_1, \mathcal{A}_2 \subset D$ and $\nu, \omega \in \mathcal{A}_1$. From Eq. (8) and as ϑ is nondecreasing, we obtain

$$\begin{aligned}
&\beta(\mathcal{A}_1)\vartheta\big(\|\mathcal{T}\nu - \mathcal{T}\omega\|\big) \\
&\quad \leqslant k\vartheta\big(\|\nu - \omega\|\big) \leqslant k\vartheta\Big(\sup_{\nu,\omega\in\mathcal{A}_1} \|\nu - \omega\|\Big) = k\vartheta\big(\operatorname{diam}(\mathcal{A}_1)\big) \\
&\quad \leqslant k\vartheta\bigg(\max\bigg\{\operatorname{diam}(\mathcal{A}_1), \operatorname{diam}(\mathcal{T}\mathcal{A}_1), \operatorname{diam}(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\operatorname{diam}(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\bigg\}\bigg),
\end{aligned}$$

then

$$\begin{aligned}
&\beta(\mathcal{A}_1) \sup_{\nu,\omega\in\mathcal{A}_1} \vartheta\big(\|\mathcal{T}\nu - \mathcal{T}\omega\|\big) \\
&\quad \leqslant k\vartheta\bigg(\max\bigg\{\Phi(\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\Phi(\mathcal{T}\mathcal{A}_1) \cup \mathcal{T}\mathcal{A}_2\bigg\}\bigg).
\end{aligned}$$

Since ϑ is nondecreasing and lower semicontinuous, we obtain

$$\begin{aligned}
&\beta(\mathcal{A}_1)\vartheta\big(\Phi(\mathcal{T}\mathcal{A}_1)\big) \\
&\quad \leqslant k\vartheta\bigg(\max\bigg\{\Phi(\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_1), \Phi(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\Phi(\mathcal{T}\mathcal{A}_1) \cup \mathcal{T}\mathcal{A}_2\bigg\}\bigg).
\end{aligned}$$

Hence, from Corollary 4 $\mathcal{T} + \mathcal{G}$ possesses a fixed point. □

Remark 3. If $\vartheta(\iota) = \iota$ for each $\iota \in [0, +\infty)$, then we obtain a version of Corollary 2.13 in [21] with $\psi(\iota) = k\iota$.

4 Fractional differential equations

Let $F := C(I = [0, T], \mathbb{R})$, $T \geqslant 1$, be the space of all continuous mappings, and let $D := \{\nu \in F\colon \|\nu\| \leqslant r\}$.

We denote by ${}^cD^\alpha$ and I^α the Caputo fractional derivative and Riemann–Liouville fractional of order α, respectively (see [3, 4]).

We show the existence of a solution of the following NFDE:

$$\begin{aligned} {}^cD^\alpha \nu(\iota) &= f\big(\iota, \nu(\iota)\big), \quad \iota \in I, \\ p_1\nu(0) + q_1\nu(T) &= I^\alpha g\big(T, \nu(T)\big), \\ p_2\nu'(0) + q_2\nu'(T) &= I^\alpha h\big(T, \nu(T)\big) \end{aligned} \tag{9}$$

with $f, g, h : I \times \mathbb{R} \to \mathbb{R}$, $\alpha \in]1, 2]$, and $p_1, p_2, q_1, q_2 \geqslant 0$.

A function $\nu \in F$ is a solution of system (9) if and only if ν satisfies [22]

$$\mathcal{T}(\nu)(\iota) = I^\alpha f_\iota - aI^\alpha f_T - bI^{\alpha-1} f_T + \frac{a}{q_1} I^\alpha g_T - \frac{b}{q_2} I^\alpha h_T, \quad \iota \in I,$$

with $f_\iota = f(\iota, \nu(\iota)$, $g_\iota = g(\iota, \nu(\iota))$, $h_\iota = h\big(\iota, \nu(\iota)\big)$, $a = q_1/(p_1 + q_1)$, and $b = q_2/(p_2 + q_2) \cdot (q_1T/(p_1 + q_1) - \iota)$.

We introduce the following definitions.

Definition 8. (See [3, 4].) The Caputo fractional derivative of order α for a mapping $f : [0, +\infty) \to \mathbb{R}$ is characterized as follows:

$${}^cD^\alpha f(\iota) = \frac{1}{\Gamma(n-\alpha)} \int_0^\iota f^{(n)}(s)(\iota - s)^{n-(1+\alpha)}\,\mathrm{d}s \quad \text{for } \alpha \in (n-1,\, n].$$

Definition 9. (See [3,4].) The Riemann–Liouville fractional integral of order α for a mapping $f : [0, +\infty) \to \mathbb{R}$ is characterized as follows:

$$I^\alpha f(\iota) = \frac{1}{\Gamma(\alpha)} \int_0^\iota f(s)(\iota - s)^{\alpha-1}\,\mathrm{d}s \quad \text{for } \alpha > 0.$$

Theorem 12. *Suppose that the following assumptions are satisfied*:

(a) *Mappings $f, g, h : I \times \mathbb{R} \to \mathbb{R}$ are continuous, and there are $k_1, k_2, k_3 \geqslant 0$ with $4k_1 + k_2 + 2k_3 < \Gamma(\alpha)/T^{\alpha+1}$ such that for all $\nu, \omega \in \mathbb{R}$ and $\iota \in I$,*

$$\begin{aligned} \big|f(\iota, \nu) - f(\iota, \omega)\big| &\leqslant \frac{4k_1T^{\alpha+1}}{\Gamma(\alpha)} \ln\big(1 + |\nu - \omega|\big), \\ \big|g(\iota, \nu) - g(\iota, \omega)\big| &\leqslant \frac{k_2T^{\alpha+1}}{\Gamma(\alpha)} \ln\big(1 + |\nu - \omega|\big), \\ \big|h(\iota, \nu) - h(\iota, \omega)\big| &\leqslant \frac{2k_3T^{\alpha+1}}{\Gamma(\alpha)} \ln\big(1 + |\nu - \omega|\big); \end{aligned}$$

(b) $L_1 := \sup_{\iota \in I} |f(\iota, \nu)| < +\infty$, $L_2 := \sup_{\iota \in I} |g(\iota, \nu)| < +\infty$, *and* $L_3 := \sup_{\iota \in I} |h(\iota, \nu)| < +\infty$;

(c) *There exists $r > 0$ such that*

$$\frac{T^{\alpha+1}}{\Gamma(\alpha)}(4L_1 + L_2 + 2L_3) \leqslant r;$$

(d) *For each* $\nu, \omega \in F$ *and* $\iota \in I$,

$$\gamma\big(\nu(\iota), \omega(\iota)\big) \geqslant 1 \to \gamma\big(\mathcal{T}\nu(\iota), \mathcal{T}\omega(\iota)\big) \geqslant 1,$$

where $\mathcal{T} : D \to D$ *is defined by*

$$\mathcal{T}(\nu)(\iota) = I^\alpha f_\iota - aI^\alpha f_T - bI^{\alpha-1} f_T + \frac{a}{q_1} I^\alpha g_T - \frac{b}{q_2} I^\alpha h_T, \quad \iota \in I.$$

If there is an $\nu_0 \in D$ *such that* $\mathcal{T}$ *satisfies* $\gamma(\nu_0(\iota), \mathcal{T}\nu_0(\iota)) \geqslant 1$, *then a solution to problem* (9) *is in* D.

Proof. First, we show that $\mathcal{T}$ is well-defined. Let $\nu \in D$. Then

$$\begin{aligned}
&\big|I^\alpha f\big(\iota, \nu(\iota)\big) - aI^\alpha f\big(\iota, \nu(\iota)\big)\big| \\
&\quad\leqslant \frac{1}{\Gamma(\alpha)}\left(\int_0^\iota \big|f\big(s, \nu(s)\big)\big|(\iota - s)^{\alpha-1}\,\mathrm{d}s + a\int_0^T \big|f\big(s, \nu(s)\big)\big|(T - s)^{\alpha-1}\,\mathrm{d}s\right) \\
&\quad\leqslant \frac{2\|f(s, \nu(s))\|}{\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1}\,\mathrm{d}s \leqslant \frac{2L_1 T^\alpha}{\alpha\Gamma(\alpha)},
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\big|bI^{\alpha-1} f\big(T, \nu(T)\big)\big| \\
&\quad\leqslant \frac{b}{\Gamma(\alpha-1)} \int_0^\iota \big|f\big(s, \nu(s)\big)\big|(\iota - s)^{\alpha-2}\,\mathrm{d}s \leqslant \frac{2T\|f(s, \nu(s))\|}{\Gamma(\alpha-1)} \int_0^T (T - s)^{\alpha-2}\,\mathrm{d}s \\
&\quad\leqslant \frac{2L_1 T^\alpha}{(\alpha-1)\Gamma(\alpha-1)},
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\left|\frac{a}{q_1} I^\alpha g\big(T, \nu(T)\big) - \frac{b}{q_2} I^\alpha h\big(\iota, \nu(\iota)\big)\right| \\
&\quad\leqslant \frac{1}{\Gamma(\alpha)}\big(\big\|g\big(s, \nu(s)\big)\big\| + 2T\big\|h\big(s, \nu(s)\big)\big\|\big) \int_0^T (T - s)^{\alpha-1}\,\mathrm{d}s \\
&\quad\leqslant \frac{1}{\alpha\Gamma(\alpha)}\big(L_2 T^\alpha + 2L_3 T^{\alpha+1}\big).
\end{aligned} \tag{12}$$

Combining Eqs. (10), (11), and (12), we obtain

$$\begin{aligned}
\big|\mathcal{T}(\nu)(\iota)\big| &\leqslant \frac{2L_1 T^\alpha}{\alpha\Gamma(\alpha)} + \frac{2L_1 T^\alpha}{(\alpha-1)\Gamma(\alpha-1)} + \frac{1}{\alpha\Gamma(\alpha)}\big(L_2 T^\alpha + 2L_3 T^{\alpha+1}\big) \\
&\leqslant T^{\alpha+1}\left(\frac{2L_1}{\Gamma(\alpha+1)} + \frac{2L_1}{\Gamma(\alpha)} + \frac{L_2 + 2L_3}{\Gamma(\alpha+1)}\right) \\
&\leqslant \frac{T^{\alpha+1}}{\Gamma(\alpha)}\big(4L_1 + L_2 + 2L_3\big) \leqslant r.
\end{aligned}$$

Accordingly, $\|\mathcal{T}(\nu)\| \leqslant r$ for each $\nu \in D$, that is, $\mathcal{T}(\nu) \in D$. Hence, $\mathcal{T}$ is well-defined.

We will now demonstrate the continuity of $\mathcal{T}$. In order to observe this, let $\nu, \omega \in D$ and $\varepsilon > 0$ such that $|\nu - \omega| < \varepsilon$. We have

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\alpha)}\int_0^\iota \left|f\bigl(s,\nu(s)\bigr) - f\bigl(s,\omega(s)\bigr)\right|(\iota - s)^{\alpha-1}\,\mathrm{d}s\right| \\
&\quad\leqslant \frac{1}{\Gamma(\alpha)}\int_0^\iota \left|f\bigl(s,\nu(s)\bigr) - f\bigl(s,\omega(s)\bigr)\right|(\iota - s)^{\alpha-1}\,\mathrm{d}s \\
&\quad\leqslant \frac{T^\alpha}{\alpha\Gamma(\alpha)}\left\|f\bigl(s,\nu(s)\bigr) - f\bigl(s,\omega(s)\bigr)\right\| \leqslant \frac{T^{\alpha+1}}{\Gamma(\alpha)}k_1 \ln\bigl(1 + |\nu - \omega|\bigr), \qquad (13)
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{a}{\Gamma(\alpha)}\int_0^T \left|f\bigl(s,\nu(s)\bigr) - f\bigl(s,\omega(s)\bigr)\right|(T - s)^{\alpha-1}\,\mathrm{d}s\right| \\
&\quad\leqslant \frac{a}{\Gamma(\alpha)}\int_0^T \left|f\bigl(s,\nu(s)\bigr) - f\bigl(s,\omega(s)\bigr)\right|(T - s)^{\alpha-1}\,\mathrm{d}s \\
&\quad\leqslant \frac{T^\alpha}{\alpha\Gamma(\alpha)}\left\|f\bigl(s,\nu(s)\bigr) - f\bigl(s,\omega(s)\bigr)\right\| \leqslant \frac{T^{\alpha+1}}{\Gamma(\alpha)}k_1 \ln\bigl(1 + |\nu - \omega|\bigr), \qquad (14)
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{b}{\Gamma(\alpha-1)}\int_0^T \left|f\bigl(s,\nu(s)\bigr) - f\bigl(s,\omega(s)\bigr)\right|(T - s)^{\alpha-2}\,\mathrm{d}s\right| \\
&\quad\leqslant \frac{a}{\Gamma(\alpha)}\int_0^T \left|f\bigl(s,\nu(s)\bigr) - f\bigl(s,\omega(s)\bigr)\right|(T - s)^{\alpha-2}\,\mathrm{d}s \\
&\quad\leqslant \frac{2T^\alpha}{\Gamma(\alpha)}\left\|f\bigl(s,\nu(s)\bigr) - f\bigl(s,\omega(s)\bigr)\right\| \leqslant \frac{2T^{\alpha+1}}{\Gamma(\alpha)}k_1 \ln\bigl(1 + |\nu - \omega|\bigr), \qquad (15)
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{a}{q_1\Gamma(\alpha)}\int_0^T \left|g\bigl(s,\nu(s)\bigr) - g\bigl(s,\omega(s)\bigr)\right|(T - s)^{\alpha-1}\,\mathrm{d}s\right| \\
&\quad\leqslant \frac{a}{q_1\Gamma(\alpha)}\int_0^T \left|g\bigl(s,\nu(s)\bigr) - g\bigl(s,\omega(s)\bigr)\right|(T - s)^{\alpha-1}\,\mathrm{d}s \\
&\quad\leqslant \frac{T^\alpha}{\alpha\Gamma(\alpha)}\left\|g\bigl(s,\nu(s)\bigr) - g\bigl(s,\omega(s)\bigr)\right\| \leqslant \frac{T^{\alpha+1}}{\Gamma(\alpha)}k_2 \ln\bigl(1 + |\nu - \omega|\bigr), \qquad (16)
\end{aligned}$$

$$\left|\frac{b}{q_2\Gamma(\alpha)}\int_0^T \left|h\bigl(s,\nu(s)\bigr) - h\bigl(s,\omega(s)\bigr)\right|(T - s)^{\alpha-1}\,\mathrm{d}s\right|$$

$$\leqslant \frac{b}{q_2 \Gamma(\alpha)} \int_0^T \left|h\big(s, \nu(s)\big) - h\big(s, \omega(s)\big)\right| (T-s)^{\alpha-1} \, \mathrm{d}s$$
$$\leqslant \frac{2T^{\alpha+1}}{\alpha \Gamma(\alpha)} \left\|h\big(s, \nu(s)\big) - h\big(s, \omega(s)\big)\right\| \leqslant \frac{2T^{\alpha+1}}{\Gamma(\alpha)} k_3 \ln\big(1 + |\nu - \omega|\big). \tag{17}$$

Combining Eqs. (13)–(17), we obtain

$$\begin{aligned}
&\left|\mathcal{T}(\nu)(\iota) - \mathcal{T}(\omega)(\iota)\right| \\
&\quad \leqslant 4\frac{T^{\alpha+1}}{\Gamma(\alpha)} k_1 \ln\big(1 + |\nu - \omega|\big) + \frac{T^{\alpha+1}}{\Gamma(\alpha)} k_2 \ln\big(1 + |\nu - \omega|\big) \\
&\qquad + \frac{2T^{\alpha+1}}{\Gamma(\alpha)} k_3 \ln(1 + |\nu - \omega|) \\
&\quad \leqslant \frac{T^{\alpha+1}}{\Gamma(\alpha)} (4k_1 + k_2 + 2k_3) \ln\big(1 + |\nu - \omega|\big) \\
&\quad \leqslant \ln\big(1 + |\nu - \omega|\big) < \varepsilon.
\end{aligned}$$

Now, we claim that $\mathcal{T}$ satisfies Eq. (4). Let $\mathcal{A}_1, \mathcal{A}_2 \subset D$ and $\nu, \omega \in \mathcal{A}_1$, and we obtain

$$\left|\mathcal{T}(\nu)(\iota) - \mathcal{T}(\omega)(\iota)\right| \leqslant \ln\big(1 + |\nu - \omega|\big),$$

so,

$$\begin{aligned}
&\mathrm{e}^{|\mathcal{T}(\nu)(\iota) - \mathcal{T}(\omega)(\iota)|} \\
&\quad \leqslant 1 + |\nu - \omega| \leqslant 1 + \operatorname{diam}(\mathcal{A}_1) \\
&\quad \leqslant 1 + \max\left\{\operatorname{diam}(\mathcal{A}_1), \operatorname{diam}(\mathcal{T}\mathcal{A}_1), \operatorname{diam}(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\operatorname{diam}(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\right\},
\end{aligned}$$

then

$$\begin{aligned}
&\vartheta(\operatorname{diam}(\mathcal{T}\mathcal{A}_1) \\
&\quad = \mathrm{e}^{\operatorname{diam}(\mathcal{T}\mathcal{A}_1)} \\
&\quad \leqslant \theta\left(\max\left\{\operatorname{diam}(\mathcal{A}_1), \operatorname{diam}(\mathcal{T}\mathcal{A}_1), \operatorname{diam}(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\operatorname{diam}(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\right\}\right)
\end{aligned}$$

with $\vartheta(\iota) = \mathrm{e}^{\iota}$ and $\theta(\iota) = 1 + \iota$.

Moreover, we define $\alpha : D \times D \to [0, +\infty)$ by

$$\alpha(\nu, \omega) = \begin{cases} 1, & \gamma(\nu(\iota), \omega(\iota)) \geqslant 1,\ \iota \in I, \\ 0 & \text{else.} \end{cases}$$

Thus,

$$\begin{aligned}
&\alpha(\nu, \mathcal{T}\nu)\vartheta(\operatorname{diam}(\mathcal{T}\mathcal{A}_1) \\
&\quad \leqslant \theta\left(\max\left\{\operatorname{diam}(\mathcal{A}_1), \operatorname{diam}(\mathcal{T}\mathcal{A}_1), \operatorname{diam}(\mathcal{T}\mathcal{A}_2), \frac{1}{2}\operatorname{diam}(\mathcal{T}\mathcal{A}_1 \cup \mathcal{T}\mathcal{A}_2)\right\}\right).
\end{aligned}$$

Hence, $\mathcal{T}$ satisfies Eq. (2).

If $\alpha(\nu,\omega) \geqslant 1$ for each $\nu,\omega \in D$, then $\gamma(\nu(\iota),\omega(\iota)) \geqslant 1$. By condition (c), we get $\gamma(\mathcal{T}\nu(\iota),\mathcal{T}\omega(\iota)) \geqslant 1$, which implies that $\alpha(\mathcal{T}\nu,\mathcal{T}\omega) \geqslant 1$. So, $\mathcal{T}: D \to D$ is α-admissible for each $\nu,\omega \in D$.

Finally, since there exists $\nu_0 \in D$ such that $\alpha(\nu_0,\mathcal{T}\nu_0) \geqslant 1$, hence, by Theorem 7, $\mathcal{T}$ possesses a fixed point in D. □

Example 6. We examine the following nonlinear fractional differential equation:

$$\begin{aligned}
&{}^{c}D^{3/2}\nu(\iota) = \frac{\iota^2+1}{64}\mathrm{e}^{-\iota^2}\ln\bigl(1+\bigl|\nu(\iota)\bigr|\bigr), \quad \iota \in I = [0,1],\\
&\frac{1}{2}\nu(0)+\frac{1}{2}\nu(1) = \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 \frac{(1-s)^{1/2}\ln(1+|\cos\nu(s)|)}{s^2+32}\,\mathrm{d}s,\\
&\frac{1}{2}\nu'(0)+\frac{1}{2}\nu'(1) = \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 \frac{(1-s)^{1/2}\ln(1+|\sin\nu(s)|)}{s^4+16}\,\mathrm{d}s,
\end{aligned} \tag{18}$$

where $\nu \in F_1 = \{\nu \in C([0,1],\mathbb{R}):\ \|\nu\| \leqslant 1\}$.

(a) Mappings $f,g,h : I\times\mathbb{R} \to \mathbb{R}$ are continuous and are defined by

$$f(\iota,\nu) = \frac{\iota^2+1}{64}\mathrm{e}^{-\iota^2}\ln(1+|\nu|),$$
$$g(\iota,\nu) = \frac{\ln(1+|\cos\nu|)}{\iota^2+32}, \qquad h(\iota,\nu) = \frac{\ln(1+|\sin\nu|)}{\iota^4+16}.$$

For each $\nu,\omega \in F_1$ and $\iota \in I$,

$$\begin{aligned}
&\bigl|f\bigl(\iota,\nu(\iota)\bigr)-f\bigl(\iota,\omega(\iota)\bigr)\bigr|\\
&\quad= \frac{\iota^2+1}{64}\mathrm{e}^{-\iota^2}\bigl|\ln\bigl(1+\bigl|\nu(\iota)\bigr|\bigr)-\ln\bigl(1+\bigl|\omega(\iota)\bigr|\bigr)\bigr|\\
&\quad\leqslant \frac{1}{32}\ln\frac{1+|\nu(\iota)|}{1+|\omega(\iota)|} \leqslant 4k_1\frac{T^{\alpha+1}}{\Gamma(\alpha)}\ln(1+|\nu(\iota)-\omega(\iota)|),\\
&\bigl|g\bigl(\iota,\nu(\iota)\bigr)-g\bigl(\iota,\omega(\iota)\bigr)\bigr|\\
&\quad= \frac{1}{\iota^2+32}\bigl|\ln\bigl(1+\bigl|\cos\nu(\iota)\bigr|\bigr)-\ln\bigl(1+\bigl|\cos\omega(\iota)\bigr|\bigr)\bigr|\\
&\quad\leqslant \frac{1}{32}\ln\bigl(1+\bigl|\nu(\iota)-\omega(\iota)\bigr|\bigr) \leqslant k_2\frac{T^{\alpha+1}}{\Gamma(\alpha)}\ln\bigl(1+\bigl|\nu(\iota)-\omega(\iota)\bigr|\bigr),\\
&\bigl|h\bigl(\iota,\nu(\iota)\bigr)-h\bigl(\iota,\omega(\iota)\bigr)\bigr|\\
&\quad= \frac{1}{\iota^4+16}|\ln(1+|\sin\nu(\iota)|)-\ln(1+|\sin\omega(\iota)|)|\\
&\quad\leqslant \frac{1}{16}\ln\bigl(1+\bigl|\nu(\iota)-\omega(\iota)\bigr|\bigr) \leqslant 2k_3\frac{T^{\alpha+1}}{\Gamma(\alpha)}\ln\bigl(1+\bigl|\nu(\iota)-\omega(\iota)\bigr|\bigr)
\end{aligned}$$

with

$$k_1 = \frac{\sqrt{\pi}}{256}, \qquad k_2 = \frac{\sqrt{\pi}}{64}, \qquad k_3 = \frac{\sqrt{\pi}}{64},$$

and

$$4k_1 + k_2 + 2k_3 = \frac{20\sqrt{\pi}}{256} < \frac{\sqrt{\pi}}{2} = \frac{\Gamma(\alpha)}{T^{\alpha+1}}.$$

(b)

$$L_1 := \sup_{\iota\in I} \bigl|f\bigl(\iota, \nu(\iota)\bigr)\bigr| < \frac{\|\nu\|}{32} \leqslant \frac{1}{32} < +\infty,$$

$$L_2 := \sup_{\iota\in I} \bigl|g\bigl(\iota, \nu(\iota)\bigr)\bigr| < \frac{\|\nu\|}{32} \leqslant \frac{1}{32} < +\infty,$$

$$L_3 := \sup_{\iota\in I} \bigl|h\bigl(\iota, \nu(\iota)\bigr)\bigr| < \frac{\|\nu\|}{16} \leqslant \frac{1}{16} < +\infty.$$

(c) There exists $r > 0$ such that

$$\begin{aligned}\frac{T^{\alpha+1}}{\Gamma(\alpha)}(4L_1 + L_2 + 2L_3) &\leqslant \frac{2}{\sqrt{\pi}}\left(4\frac{\|\nu\|}{32} + \frac{\|\nu\|}{32} + 3\frac{\|\nu\|}{16}\right)\\ &\leqslant \frac{22}{32\sqrt{\pi}}\|\nu\| \leqslant \frac{22}{32\sqrt{\pi}} = r.\end{aligned}$$

(d) Mapping $\gamma : F_1 \times F_1 \to [0, +\infty)$ is defined by

$$\gamma(\nu, \omega) = \begin{cases} 1, & \|\nu\| + \|\omega\| \leqslant 1,\\ 0, & \|\nu\| + \|\omega\| > 1.\end{cases}$$

For each $(\nu, \omega) \in F_1 \times F_1$ and $\iota \in I$. If $\gamma(\nu, \omega) \geqslant 1$, then $\|\nu\| + \|\omega\| \leqslant 1$, and we obtain

$$\begin{aligned}&\bigl|I^\alpha f\bigl(\iota, \nu(\iota)\bigr) - aI^\alpha f\bigl(\iota, \nu(\iota)\bigr)\bigr|\\ &\quad\leqslant \left(\frac{1}{64\Gamma(\frac{3}{2})} + \frac{1}{128\Gamma(\frac{3}{2})}\right)\int_0^\iota (\iota - s)^{1/2}\bigl(s^2 + 1\bigr)\mathrm{e}^{-s^2}\bigl|\ln\bigl(1 + \bigl|\nu(s)\bigr|\bigr)\bigr|\,\mathrm{d}s\\ &\quad\leqslant \left(\frac{1}{32\Gamma(\frac{3}{2})} + \frac{1}{64\Gamma(\frac{3}{2})}\right)\int_0^1 (1 - s)^{1/2}\bigl|\ln\bigl(1 + \bigl|\nu(s)\bigr|\bigr)\bigr|\,\mathrm{d}s, \end{aligned} \tag{19}$$

$$\begin{aligned}&\bigl|bI^{\alpha-1} f\bigl(T, \nu(T)\bigr)\bigr|\\ &\quad\leqslant \frac{1}{128\Gamma(\frac{1}{2})}\left|\frac{1}{2} - \iota\right|\int_0^1 (1 - s)^{-1/2}\bigl(s^2 + 1\bigr)\mathrm{e}^{-s^2}\bigl|\ln\bigl(1 + \bigl|\nu(s)\bigr|\bigr)\bigr|\,\mathrm{d}s\\ &\quad\leqslant \frac{1}{128\Gamma(\frac{1}{2})}\int_0^1 (1 - s)^{-1/2}\bigl|\ln\bigl(1 + \bigl|\nu(s)\bigr|\bigr)\bigr|\,\mathrm{d}s, \end{aligned} \tag{20}$$

$$\left|\frac{a}{q_1}I^\alpha g\bigl(T,\nu(T)\bigr)\right| \leqslant \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 (1-s)^{1/2}\frac{1}{s^2+32}\bigl|\ln\bigl(1+\bigl|\cos\nu(s)\bigr|\bigr)\bigr|\,\mathrm{d}s$$

$$\leqslant \frac{1}{32\Gamma(\frac{3}{2})}\int_0^1 (1-s)^{1/2}\bigl|\ln\bigl(1+\bigl|\cos\nu(s)\bigr|\bigr)\bigr|\,\mathrm{d}s \tag{21}$$

$$\left|\frac{b}{q_2}I^\alpha h\bigl(\iota,\nu(\iota)\bigr)\right| \leqslant \frac{1}{\Gamma(\frac{3}{2})}\left|\iota-\frac{1}{2}\right|\int_0^1 (1-s)^{1/2}\frac{1}{s^4+16}\bigl|\ln\bigl(1+\bigl|\sin\nu(s)\bigr|\bigr)\bigr|\,\mathrm{d}s$$

$$\leqslant \frac{1}{32\Gamma(\frac{3}{2})}\int_0^1 (1-s)^{1/2}\bigl|\ln\bigl(1+\bigl|\sin\nu(s)\bigr|\bigr)\bigr|\,\mathrm{d}s. \tag{22}$$

We combine Eqs. (19), (20), (21), and (22), and we obtain

$$\bigl|\mathcal{T}(\nu)(\iota)\bigr| \leqslant \left(\frac{7}{64\Gamma(\frac{3}{2})}+\frac{1}{128\Gamma(\frac{1}{2})}\right)\int_0^1 \ln\bigl(1+\bigl|\nu(s)\bigr|\bigr)\,\mathrm{d}s\frac{29}{128\sqrt{\pi}}\int_0^1 \bigl|\nu(s)\bigr|\,\mathrm{d}s$$

$$\leqslant \frac{29}{128\sqrt{\pi}}\|\nu\| \leqslant \frac{29}{128\sqrt{\pi}} \leqslant \frac{22}{32\sqrt{\pi}}.$$

Hence, $\|\mathcal{T}\nu\| \leqslant 29/(128\sqrt{\pi})$. Similarly, we can show that $\|\mathcal{T}\omega\| \leqslant 29/(128\sqrt{\pi})$.

Consequently, $\|\mathcal{T}\nu\| + \|\mathcal{T}\omega\| \leqslant 58/(128\sqrt{\pi}) = 0.25565 < 1$. This necessitates that $\gamma(\mathcal{T}\nu(\iota), \mathcal{T}\omega(\iota) \geqslant 1$ for all $(\nu,\omega) \in F_1 \times F_1$ and $\iota \in I$.

Finally, we define $\nu_0 : [0,1] \to \mathbb{R}$ by $\nu_0(\iota) = 29\iota/(128\sqrt{\pi})$, and we have $\|\nu_0\| + \|\mathcal{T}\nu_0\| \leqslant 58/(128\sqrt{\pi}) < 1$.

Hence, there exists $\nu_0 \in D = \{\nu \in F\colon \|\nu\| \leqslant 22/(32\sqrt{\pi})\}$ such that $\gamma(\nu_0, \mathcal{T}\nu_0) \geqslant 1$. Consequently, by Theorem 12, Eq. (18) has a solution in space F_1.

5 Conclusion

Novel theorems of the existence of fixed point for α-ϑ-θ-contractions by considering an arbitrary MNC have been shown. From these results we establish new Krasnosel'skii-form fixed-point theorems. Our study is encouraged by the possible application of nonlinear fractional differential equations of order α. Finally, an illustrative example has been presented to demonstrate our results.

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