Nonlinear Analysis: Modelling and Control, Vol. 30, No. 6, 1145–1162
https://doi.org/10.15388/namc.2025.30.43775

Boundary value problems for third-order differential equations involving singular $\varPhi$-Laplacian operators*

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Received: January 2, 2024 / **Revised:** July 16, 2025 / **Published online:** October 10, 2025

Abstract. We study strongly nonlinear, third-order differential equations of type

$$\big(\varPhi\big(k(t)u''(t)\big)\big)' = f\big(t, u(t), u'(t), u''(t)\big), \quad \text{a.e. } t \in J,$$

where $\varPhi$ is the singular $\varPhi$-Laplacian operator. That is, $\varPhi : (-r, r) \to \mathbb{R}$, $r > 0$, is a generic strictly increasing homeomorphism with bounded domain, which generalizes the relativistic operator $\varPhi(u) := u/\sqrt{r^2 - u^2}$. Moreover, k is a nonnegative continuous function, which can vanish on a set of zero measure, so such equations can be singular, and f is a general Carathédory function. For these equations, we investigate boundary value problems both in compact intervals (when $J = [a, b]$) and in a half-line (with $J = [a, +\infty)$), and we prove existence results under mild assumptions. Our approach is based on fixed point techniques.

Keywords: boundary value problems on unbounded domains, heteroclinic solutions, nonlinear differential operators, singular $\varPhi$-Laplacian operator, singular equation.

1 Introduction

In this paper, we investigate the existence of solutions for a class of boundary value problems (BVPs for short) associated to a strongly nonlinear, third-order differential equation of the type

$$\big(\varPhi\big(k(t)u''(t)\big)\big)' = f\big(t, u(t), u'(t), u''(t)\big), \tag{1}$$

where $\varPhi : (-r, r) \to \mathbb{R}$, $r > 0$, is the so-called *singular $\varPhi$-Laplacian*, that is, a generic strictly increasing homeomorphism with bounded domain (not necessarily satisfying

*This article was supported by the Research project of MUR - Prin 2022 "Nonlinear differential problems with applications to real phenomena" [grant No. 2022ZXZTN2].

$\Phi(0) = 0$). Moreover, k is a nonnegative measurable function that can vanish in a set of zero measure; consequently, Eq. (1) can be *singular*. Finally, the term f appearing in the right-hand side is a Carathédory function.

The nonlinear differential operator that appears in (1), that is, the singular Φ-Laplacian, is a general homeomorphism whose main prototype is the one-dimensional relativistic operator

$$\Phi(u) := \frac{u}{\sqrt{1-u^2}}, \quad u \in (-1, 1). \tag{2}$$

Differential equations involving (2), as well as the N-dimensional p-relativistic operator ($p \geqslant 2$), have been recently studied by many authors, using different techniques. We mention here, for instance, [8, 19, 24, 29–31]; see also the survey [33] and references therein.

Equations of this kind fit into a general context of differential equations governed by nonlinear differential operators. Among them, one of the most investigated is the classical p-Laplacian operator

$$\Phi_p(u) := u|u|^{p-2}, \quad p > 1.$$

Equations involving a more general operator, the Φ-Laplacian – that is, a strictly increasing homeomorphisms Φ from $\mathbb{R}$ to $\mathbb{R}$ – are also well-studied. In fact, these equations arise in many applicative models: for instance, in non-Newtonian fluid theory, nonlinear elasticity, diffusion of flows in porous media, theory of capillary surfaces, and, more recently, the modeling of glaciology [14,25,34]. In this framework, the theory concerning second-order differential problems has been considered by many authors (see, e.g., [10–13, 17, 18, 20, 23, 32] and references therein). Moreover, many papers are also devoted to other types of differential operators instead of the usual Φ-Laplacian: the case in which the increasing homeomorphism Φ is defined on the whole real line but is *not surjective*, whose main prototype is the mean curvature operator

$$\Phi(u) := \frac{u}{\sqrt{1+u^2}}, \quad u \in \mathbb{R},$$

(see e.g. [7, 9]), and the already mentioned *singular* Φ-Laplacian as in (2).

In addition, boundary value problems associated with equations of higher order have been considered as well in view of applications. For instance, third-order equations occur in some models in fluid mechanics or electromagnetic frameworks; see, e.g., [21, 26], see also the more recent papers [16, 28] for results on Blasius equation, and [5] about Falkner–Skan equation. However, as far as we know, for these equations, a general theory has not been fully developed yet. Some results have been obtained, e.g., in [1–4, 6, 22, 27, 35, 36].

Due to the wide class of their applications, here we focus on BVPs associated to Eq. (1). The fact that here we consider a very general equation allows us to say that our results generalize, in some sense, many results present in literature and mentioned above. Moreover, for similar reasons, motivated by many applications in various biological, physical and chemical models, such as front propagation in reaction–diffusion equations, we consider also existence of solutions on noncompact intervals.

In order to obtain our existence results, we use a fixed point approach. The crucial remark is that the existence of a solution of the BVP is equivalent to the existence of a fixed point of a suitable operator. Then we prove that the abstract operator verifies the assumptions of Schauder fixed point theorem.

We stress that our main results require mild assumptions. In fact, we do not ask for a Wintner–Nagumo-type condition on the right-hand side of (1), and we do not assume the existence of lower and upper solutions. On the other hand, these hypotheses are quite usual and are assumed in most of the quoted papers: for example, in the recent [4] in which we studied a similar problem but with a usual Φ-Laplacian $\Phi : \mathbb{R} \to \mathbb{R}$.

As it was pointed out by Bereanu and Mawhin in [7], if the differential equation is governed by a singular Φ-Laplacian, the search of the a priori bounds for possible solutions can be simpler. Therefore, as in the present paper, an approach based on fixed point theory usually gives existence results under milder conditions with respect to corresponding problems with a classical Φ-Laplacian. On the other hand, as far as we know, the case of a *bounded* Φ-Laplacian $\Phi : \mathbb{R} \to (-s, s)$ is open for third-order equations.

The paper is divided in two parts. In the first one, we study Eq. (1) on a bounded interval $I := [a, b]$; namely, we consider the following boundary value problem:

$$\begin{aligned} &\bigl(\Phi\bigl(k(t)u''(t)\bigr)\bigr)' = f\bigl(t, u(t), u'(t), u''(t)\bigr) \quad \text{a.e. on } I, \\ &u(a) = \gamma, \qquad u'(a) = \alpha, \qquad u'(b) = \beta, \end{aligned} \tag{3}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. Assuming $1/k$ to be in $L^1(I)$, it is quite natural to look for solutions of (3) in the space $W^{2,1}(I)$. A similar framework can be found in the recent [2,4,12,20]. Under mild conditions, we obtain a very general existence result for the solutions of the BVP (3); see Theorem 2 below. In particular, we require the following inequality on the boundary data:

$$|\beta - \alpha| < r \cdot \left\| \frac{1}{k} \right\|_{L^1},$$

and we show that such inequality is also necessary for the solvability of (3); see Remark 1.

In the second part of the paper, by a limit argument, we use Theorem 2 to investigate the existence of solutions of Eq. (1) on the half-line $\Lambda := [a, +\infty)$. In other words, we turn our attention to the BVP

$$\begin{aligned} &\bigl(\Phi\bigl(k(t)u''(t)\bigr)\bigr)' = f\bigl(t, u(t), u'(t), u''(t)\bigr) \quad \text{a.e. on } \Lambda, \\ &u(a) = \gamma, \qquad u'(a) = \alpha, \qquad u'(+\infty) = \beta. \end{aligned} \tag{4}$$

Under the assumption that $1/k$ is in $L^1(\Lambda)$, our approach allows us to obtain the existence of solutions $u \in W^{2,1}_{\mathrm{loc}}(\Lambda)$ of (4) that are, in some sense, heteroclinic. Again, in our result (see Theorem 3) we do not require a Wintner–Nagumo-type condition nor the existence of lower and upper solutions, differently as in the recent [2] in which the author studies the case of the usual Φ-Laplacian $\Phi : \mathbb{R} \to \mathbb{R}$. Furthermore, it would be interesting to discuss the existence of solutions of (4) when $1/k$ is just in L^1_{loc} (including, for example, the case in which k is a positive constant), possibly asking for some more conditions on the right-hand side f. We leave this as an open question.

The paper contains also specific examples illustrating both Theorems 2 and 3. We mention here Example 2 in which we consider the following BVP on $\Lambda := [0, +\infty)$:

$$\begin{aligned} &\big(\Phi\big(k(t)u''(t)\big)\big)' = \mu(t)g_1\big(u(t), u'(t), u''(t)\big) + u''(t)g_2\big(t, u(t), u'(t)\big), \\ &u(0) = \gamma, \qquad u'(0) = \alpha, \qquad u'(+\infty) = \beta. \end{aligned} \tag{5}$$

Here $\Phi: (-1, 1) \to \mathbb{R}$, so that $r = 1$; for example, one may consider the relativistic operator (2). In addition, $k : \Lambda \to \mathbb{R}$ is a.e. positive with $1/k \in L^1(\Lambda)$, $k(0) = 0$, and $\|1/k\|_{L^1(\Lambda)} = 1$, the function $\mu : \Lambda \to \mathbb{R}$ belongs to $L^1(\Lambda)$ with $\mu(t) \geqslant 0$ for a.a. $t \in \Lambda$, and the maps $g_1 : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $g_2 : \Lambda\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous and bounded. Then we show that Theorem 3 applies, so problem (5) admits solutions, provided that $|\beta - \alpha| < 1$.

2 Preliminary results

In this section, we study Eq. (1) on a bounded interval $I := [a, b]$. Namely, we consider the following BVP:

$$\begin{aligned} &\big(\Phi\big(k(t)u''(t)\big)\big)' = f\big(t, u(t), u'(t), u''(t)\big) \quad \text{a.e. on } I, \\ &u(a) = \gamma, \qquad u'(a) = \alpha, \qquad u'(b) = \beta, \end{aligned} \tag{6}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are given constants, the operator $\Phi : (-r, r) \to \mathbb{R}$, $r > 0$, is a generic strictly increasing homeomorphism, $f : I\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a Carathédory function, and $k : I \to \mathbb{R}$ is a measurable function satisfying $k(t) > 0$ for a.e. $t \in I$ and $1/k \in L^1(I)$.

We introduce the closed, convex subspace of $W^{2,1}(I)$ defined as

$$\mathcal{W}_\gamma(I) = \big\{u \in W^{2,1}(I)\colon\ u(a) = \gamma\big\}.$$

Note that looking for a solution $u \in W^{2,1}(I)$ of (6) is equivalent to find a solution $u \in \mathcal{W}_\gamma(I)$ of the Neumann-type problem

$$\begin{aligned} &\big(\Phi\big(k(t)u''(t)\big)\big)' = f\big(t, u(t), u'(t), u''(t)\big) \quad \text{a.e. on } I, \\ &u'(a) = \alpha, \qquad u'(b) = \beta. \end{aligned} \tag{7}$$

We study problem (7) by means of the following auxiliary functional BVP:

$$\begin{aligned} &\big(\Phi\big(k(t)u''(t)\big)\big)' = F_u(t) \quad \text{a.e. on } I, \\ &u'(a) = \alpha, \qquad u'(b) = \beta, \end{aligned} \tag{8}$$

where $F : \mathcal{W}_\gamma(I) \to L^1(I)$, $u \mapsto F_u$, is a functional operator.

By a solution of the BVP (8) we mean a function $u \in \mathcal{W}_\gamma(I)$, with $u'(a) = \alpha$, $u'(b) = \beta$, such that $|k(t)u''(t)| < r$ for all $t \in I$, moreover

$$\Phi \circ (k \cdot u'') \in W^{1,1}(I) \quad \text{and} \quad \big(\Phi\big(k(t)u''(t)\big)\big)' = F_u(t) \quad \text{a.e. on } I.$$

We remark that, if $u \in W^{2,1}(I)$ is such that $\Phi \circ (k \cdot u'') \in W^{1,1}(I)$, the continuity of Φ^{-1} implies $k \cdot u'' \in C(I)$, meaning that $k \cdot u''$ has a continuous extension on the whole interval I.

From now on we assume that $F : \mathcal{W}_\gamma(I) \to L^1(I)$ is continuous and there exists a function $\eta \in L^1(I)$ such that

$$\bigl|F_u(t)\bigr| \leqslant \eta(t) \quad \text{a.e. on } I, \text{ for every } u \in \mathcal{W}_\gamma(I). \tag{9}$$

Moreover, we introduce the integral operator $\mathcal{H} : \mathcal{W}_\gamma(I) \to C(I)$ defined by

$$\mathcal{H}_u(t) = \int_a^t F_u(s)\,\mathrm{d}s, \quad t \in I.$$

Notice that, by assumption (9), the operator $\mathcal{H}$ is continuous as well, and we have

$$\bigl|\mathcal{H}_u(t)\bigr| \leqslant \|\eta\|_{L^1(I)} \quad \text{for every } u \in \mathcal{W}_\gamma(I),\ t \in I. \tag{10}$$

For the sake of brevity, in the sequel, we denote

$$k_1 := \left\|\frac{1}{k}\right\|_{L^1}.$$

We start by proving the following lemma.

Lemma 1. *Assume that condition* (9) *holds. Assume further that*

$$|\beta - \alpha| < r k_1. \tag{11}$$

Then, for every $u \in \mathcal{W}_\gamma(I)$, there exists a unique constant $I_u \in \mathbb{R}$ such that

$$\int_a^b \frac{1}{k(t)} \Phi^{-1}\bigl(I_u + \mathcal{H}_u(t)\bigr)\,\mathrm{d}t = \beta - \alpha.$$

Moreover, the following estimate holds:

$$|I_u| \leqslant \left|\Phi\left(\frac{\beta - \alpha}{k_1}\right)\right| + \|\eta\|_{L^1(I)} \quad \textit{for every } u \in \mathcal{W}_\gamma(I). \tag{12}$$

Proof. By (10), for every $\xi \in \mathbb{R}$, $u \in \mathcal{W}_\gamma(I)$ and $t \in I$, we have

$$\xi - \|\eta\|_{L^1(I)} \leqslant \xi + \mathcal{H}_u(t) \leqslant \xi + \|\eta\|_{L^1(I)}.$$

Then, since $\Phi^{-1} : \mathbb{R} \to (-r, r)$ is strictly increasing and k is positive, we get

$$\begin{aligned} &\Phi^{-1}\bigl(\xi - \|\eta\|_{L^1(I)}\bigr) \int_a^b \frac{1}{k(t)}\,\mathrm{d}t \\ &\quad \leqslant \int_a^b \frac{1}{k(t)} \Phi^{-1}\bigl(\xi + \mathcal{H}_u(t)\bigr)\,\mathrm{d}t \leqslant \Phi^{-1}\bigl(\xi + \|\eta\|_{L^1(I)}\bigr) \int_a^b \frac{1}{k(t)}\,\mathrm{d}t. \end{aligned}$$

So, the function $\varphi_u : \mathbb{R} \to (-rk_1, rk_1)$ given by

$$\varphi_u(\xi) := \int_a^b \frac{1}{k(t)} \Phi^{-1}\bigl(\xi + \mathcal{H}_u(t)\bigr)\,\mathrm{d}t$$

is well-defined and continuous by Lebesgue's dominated convergence theorem. Moreover, since Φ^{-1} is strictly increasing and surjective and k is positive, then φ_u is also strictly increasing and surjective; in particular,

$$\lim_{\xi \to -\infty} \varphi_u(\xi) = -rk_1, \qquad \lim_{\xi \to +\infty} \varphi_u(\xi) = rk_1,$$

implying that φ_u is a homeomorphism. Therefore, if $|\beta - \alpha| < rk_1$, for every $u \in \mathcal{W}_a(I)$, there exists a unique $I_u \in \mathbb{R}$ such that

$$\int_a^b \frac{1}{k(t)} \Phi^{-1}\bigl(I_u + \mathcal{H}_u(t)\bigr)\,\mathrm{d}t = \beta - \alpha.$$

Now, by the mean value theorem, we get that for every $u \in \mathcal{W}_\gamma(I)$, there exists a value $t_u \in I$ such that

$$\beta - \alpha = \int_a^b \frac{1}{k(t)} \Phi^{-1}\bigl(I_u + \mathcal{H}_u(t)\bigr)\,\mathrm{d}t = \Phi^{-1}\bigl(I_u + \mathcal{H}_u(t_u)\bigr) \int_a^b \frac{1}{k(t)}\,\mathrm{d}t.$$

Thus, $\Phi^{-1}(I_u + \mathcal{H}_u(t_u)) = (\beta - \alpha)/k_1$, so that

$$I_u + \mathcal{H}_u(t_u) = \Phi\biggl(\frac{\beta - \alpha}{k_1}\biggr).$$

Hence, estimate (12) directly follows from (10). □

Now we state the following existence result for the functional problem (8). The proof can be carried out applying the Schauder fixed point theorem as in [4, Thm. 2.2], and therefore, is omitted.

Theorem 1. *Suppose conditions* (9) *and* (11) *hold. Then problem* (8) *admits a solution.*

The following remark shows that condition (11) is also necessary for the existence of solutions of (8).

Remark 1. Let $u \in \mathcal{W}_\gamma(I)$ be a solution of (8). Then $|\beta - \alpha| < rk_1$. Indeed, $u' \in W^{1,1}(I)$ and

$$\beta - \alpha = u'(b) - u'(a) = \int_a^b u''(t)\,\mathrm{d}t,$$

so that, by definition of solution,

$$|\beta-\alpha| = \left|\int_a^b u''(t)\,\mathrm{d}t\right| \leqslant \int_a^b \left|u''(t)\right|\mathrm{d}t \leqslant \int_a^b \frac{r}{k(t)}\,\mathrm{d}t = rk_1.$$

Notice that, taking into account Remark 1, the necessary condition (11) implies that all solutions of (8) are a priori bounded. In fact, the following remark holds.

Remark 2. Let $u \in \mathcal{W}_\gamma(I)$ be a solution of (8), and assume that (11). Then, since $|k(t)u''(t)| < r$ for all $t \in I$, by direct integration, we obtain the following estimates:

(i) $|u'(t)| < |\alpha| + rk_1$ for all $t \in I$;
(ii) $|u(t)| < |\gamma| + (b-a)(|\alpha| + rk_1)$ for all $t \in I$.

3 Existence results on bounded intervals

In this section, we prove an existence result for problem (7); see Theorem 2. Let us first clarify the meaning of solution.

Definition 1. A function $u \in \mathcal{W}_\gamma(I)$ is said to be a *solution* of problem (7) if it satisfies $|k(t)u''(t)| < r$ for all $t \in I$; $\Phi \circ (k \cdot u'') \in W^{1,1}(I)$; and $(\Phi(k(t)u''(t)))' = f(t, u(t), u'(t), u''(t))$ for a.e. $t \in I$, $u'(a) = \alpha$ and $u'(b) = \beta$.

Thanks to a priori estimates as in Remark 2, we obtain the following existence result.

Theorem 2. *Assume that condition* (11) *holds, and let* $M := |\alpha| + rk_1$. *Moreover, suppose*

(H) *for every* $R > 0$ *and for every nonnegative function* $\ell \in L^1(I)$, *there exists a nonnegative function* $h = h_{R,\ell} \in L^1(I)$ *such that*

$$\left|f\bigl(t, x, y, z(t)\bigr)\right| \leqslant h(t) \quad \textit{for a.e. } t \in I,$$

for every $x, y \in \mathbb{R}$ *such that* $|x|, |y| \leqslant R$ *and every* $z \in L^1(I)$ *such that* $|z(t)| \leqslant \ell(t)$ *for a.e.* $t \in I$.

Then problem (7) *admits a solution* $u \in \mathcal{W}_\gamma(I)$ *such that*

$$-|\gamma| - M(t-a) \leqslant u(t) \leqslant |\gamma| + M(t-a) \quad \textit{for all } t \in I.$$

Proof. For simplicity, we let $\psi \in L^1(I)$ be defined by

$$\psi(t) := \frac{r}{k(t)} \quad \text{for a.e. } t \in I. \tag{13}$$

Now, following [11, App. A], we introduce a truncation operator. Let $\xi, \zeta \in L^1(I)$ be such that $\xi(t) \leqslant \zeta(t)$ for a.e. $t \in I$. Define $\mathcal{T}^{\xi,\zeta} : L^1(I) \to L^1(I)$ by

$$\mathcal{T}_x^{\xi,\zeta}(t) = \max\bigl\{\xi(t), \min\bigl\{x(t), \zeta(t)\bigr\}\bigr\} \quad \text{for a.e. } t \in I. \tag{14}$$

Furthermore, given $u \in W^{2,1}(I)$, we denote

$$\mathcal{D}_{u'}(t) := \mathcal{T}^{-\psi,\psi}_{(\mathcal{T}^{-M,M}_{u'})'}(t) \quad \text{for a.e. } t \in I, \tag{15}$$

where ψ is the function defined in (13). Note that definition (15) is well-posed since $\mathcal{T}^{-M,M}_{u'}$ is in $W^{1,1}(I)$; see, e.g., [11, Lemma A.1] for details.

Let us now define the operator $F : \mathcal{W}_\gamma(I) \to L^1(I)$, $u \mapsto F_u$, by

$$F_u(t) := f\bigl(t, \mathcal{T}^{\sigma,\tau}_u(t), \mathcal{T}^{-M,M}_{u'}(t), \mathcal{D}_{u'}(t)\bigr) \quad \text{for a.e. } t \in I,$$

where σ, τ are the linear functions $\sigma(t) := -|\gamma| - M(t-a)$ and $\tau(t) := |\gamma| + M(t-a)$, respectively.

Consider the auxiliary problem

$$\begin{aligned} &\bigl(\Phi\bigl(k(t)u''(t)\bigr)\bigr)' = F_u(t) \quad \text{a.e. on } I, \\ &u'(a) = \alpha, \qquad u'(b) = \beta. \end{aligned} \tag{16}$$

Following the steps of the proof of [4, Thm. 3.3] and using assumption (H), one can prove that the definition of F is well-posed. Moreover, one can check that condition (9) in the statement of Theorem 1 above holds, so that the auxiliary problem (16) admits a solution.

To complete the proof, we have to show that any solution u of (16) is also a solution of (7). This follows from suitable a priori estimates as in Remark 2.

Indeed, let u be a solution of (16). Then, in particular, $|k(t)u''(t)| < r$ for all $t \in I$. Hence, by direct integration, $|u'(t)| < |\alpha| + rk_1 := M$ for all $t \in I$. Consequently,

$$\mathcal{T}^{-M,M}_{u'}(t) = u'(t) \quad \text{for all } t \in I.$$

Moreover,

$$\bigl|u(t)\bigr| \leqslant |\gamma| + M(t-a) \quad \text{for all } t \in I,$$

and this implies, with the notation introduced above,

$$\mathcal{T}^{\sigma,\tau}_u(t) = u(t) \quad \text{for all } t \in I.$$

Furthermore, note that $|k(t)u''(t)| < r$ for all $t \in I$ and thus

$$\mathcal{D}_{u'}(t) := \mathcal{T}^{-\psi,\psi}_{(\mathcal{T}^{-M,M}_{u'})'}(t) = \mathcal{T}^{-\psi,\psi}_{u''}(t) = u''(t) \quad \text{for a.e. } t \in I.$$

From the above discussion it follows

$$F_u(t) := f\bigl(t, u(t), u'(t), u''(t)\bigr) \quad \text{a.e. on } I,$$

and consequently, u is a solution of (7).

Hence, problem (7) admits a solution $u \in \mathcal{W}_\gamma(I)$ such that

$$-|\gamma| - M(t-a) \leqslant u(t) \leqslant |\gamma| + M(t-a) \quad \text{for all } t \in I.$$

This completes the proof. □

Let us now provide a first example illustrating Theorem 2. We consider a quite general class of nonlinearities f such that $f(t,x,y,\cdot)$ is sublinear.

Example 1. Consider the following BVP on $I := [0,1]$:

$$\begin{aligned} &\big(\Phi\big(k(t)u''(t)\big)\big)' = \mu(t)g\big(u(t),u'(t)\big) + \big(u''(t)\big)^{\alpha} \quad \text{a.e. on } [0,1],\\ &u(0)=\gamma, \qquad u'(0)=\alpha, \qquad u'(1)=\beta, \end{aligned} \tag{17}$$

where $\Phi : (-1,1) \to \mathbb{R}$ is a strictly increasing homeomorphism (so that $r=1$); $k : I \to \mathbb{R}$ is a.e. positive with $1/k \in L^1(I)$ and (to fix ideas) $\|1/k\|_{L^1} = 1$; for example, one can choose $k(t) = 2\sqrt{t}$. Moreover, $\mu : I \to \mathbb{R}$ belongs to $L^1(I)$ with $\mu(t) \geqslant 0$ for a.a. $t \in I$; $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous; and $0 < \alpha \leqslant 1$.

Let us prove that, by Theorem 2, problem (17) admits solutions, provided that

$$|\beta - \alpha| < 1.$$

Let

$$f(t,x,y,z) := \mu(t)g(x,y) + z^{\alpha}.$$

Note that f is a Carathéodory function. Let us prove that assumption (H) is satisfied. Indeed, we fix $R > 0$ and $\ell \in L^1(I)$, and we let

$$M_R := \max_{(x,y)\in[-R,R]^2} \big|g(x,y)\big|$$

and, for a.e. $t \in I$,

$$\widehat{\ell}(t) := \max\big\{1, \ell(t)\big\}.$$

Note that $\widehat{\ell} \in L^1(I)$. Then

$$\big|f\big(t,x,y,z(t)\big)\big| \leqslant \mu(t)\big|g(x,y)\big| + \big|z(t)\big|^{\alpha} \leqslant \mu(t)M_R + \widehat{\ell}(t) =: h_{R,\ell}(t)$$

whenever $|x| \leqslant R$, $|y| \leqslant R$, and $z \in L^1(I)$ with $|z(t)| \leqslant \ell(t)$ for a.e. $t \in I$, where $h_{R,\ell} \in L^1_+(I)$. Thus, (H) is verified.

If $|\beta - \alpha| < 1$, then condition (11) is satisfied as well. Consequently, for any $\gamma \in \mathbb{R}$, Theorem 2 applies, and therefore, problem (17) admits a solution $u \in \mathcal{W}_\gamma(I)$ such that $-|\gamma| - Mt \leqslant u(t) \leqslant |\gamma| + Mt$ for all $t \in [0,1]$, where $M := |\alpha| + 1$.

4 Existence results on unbounded intervals

We now turn our attention to Eq. (1) on a half-line $\Lambda := [a,+\infty)$. That is,

$$\big(\Phi\big(k(t)u''(t)\big)\big)' = f\big(t,u(t),u'(t),u''(t)\big) \quad \text{a.e. on } \Lambda. \tag{18}$$

More precisely, from now on we assume that $f : \Lambda\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a Carathédory function, $k : \Lambda \to \mathbb{R}$ is a measurable function satisfying $k(t) > 0$ for a.e. $t \in \Lambda$ and $1/k \in L^1(\Lambda)$.

In addition to the above requirements, we suppose that k vanishes at least at one point, namely, we assume that

(C) k is continuous, and there exists $t_0 \in \Lambda$ such that $k(t_0) = 0$.

Under these assumptions, we study the solvability of the following BVP:

$$\begin{aligned} &\big(\Phi\big(k(t)u''(t)\big)\big)' = f\big(t, u(t), u'(t), u''(t)\big) \quad \text{a.e. on } \Lambda, \\ &u(a) = \gamma, \qquad u'(a) = \alpha, \qquad u'(+\infty) = \beta. \end{aligned} \tag{19}$$

Definition 2. A function $u \in W^{2,1}_{\mathrm{loc}}(\Lambda)$ is said to be a *solution* of (19) if it satisfies $|k(t)u''(t)| < r$ for all $t \in \Lambda$; $\Phi \circ (k \cdot u'') \in W^{1,1}_{\mathrm{loc}}(\Lambda)$; moreover,

$$\big(\Phi\big(k(t)u''(t)\big)\big)' = f\big(t, u(t), u'(t), u''(t)\big) \quad \text{for a.e. } t \in \Lambda,$$

$u(a) = \gamma$, $u'(a) = \alpha$, and $\lim_{t\to+\infty} u'(t) = \beta$.

Remark 3. We stress that (C) is a technical condition needed by the singularity of the homeomorphism $\Phi : (-r, r) \to \mathbb{R}$. It would be interesting to discuss the solvability of the BVP (19) without assuming (C).

As in the previous section, when u is a solution of (19), the function $k \cdot u''$ has a continuous extension; in fact, the next remark holds.

Remark 4. If $u \in W^{2,1}_{\mathrm{loc}}(\Lambda)$ is any function such that $\Phi \circ (k \cdot u'') \in W^{1,1}_{\mathrm{loc}}(\Lambda)$, the continuity of Φ^{-1} implies $k \cdot u'' \in C(\Lambda)$, meaning that $k \cdot u''$ has a continuous extension on the whole of Λ.

The following is our main result. For the sake of brevity, in the sequel, we denote

$$k_{1,\Lambda} := \left\| \frac{1}{k} \right\|_{L^1(\Lambda)}.$$

Theorem 3. *Assume that condition* (C) *holds. Assume further that*

$$|\beta - \alpha| < r k_{1,\Lambda}, \tag{20}$$

and let $M := |\alpha| + r k_{1,\Lambda}$. *Moreover, suppose that*

(H′) *for every* $R > 0$ *and for every nonnegative function* $\ell \in L^1(\Lambda)$*, there exists a nonnegative function* $h = h_{R,\ell} \in L^1(\Lambda)$ *such that*

$$\big|f\big(t, x(t), y, z(t)\big)\big| \leqslant h(t) \quad \textit{for a.e. } t \in \Lambda,$$

for every $x, z \in L^1(\Lambda)$*, and* $y \in \mathbb{R}$ *such that* $|x(t)| \leqslant |\gamma| + R(t - a)$ *for a.e.* $t \in \Lambda$, $|y| \leqslant R$*, and* $|z(t)| \leqslant \ell(t)$ *for a.e.* $t \in \Lambda$.

Then problem (19) *admits a solution* $u \in W^{2,1}_{\mathrm{loc}}(\Lambda)$ *such that*

$$-|\gamma| - M(t - a) \leqslant u(t) \leqslant |\gamma| + M(t - a) \quad \textit{for all } t \in \Lambda.$$

In the proof of Theorem 3, we apply a sequential approach. Roughly speaking, we restrict Eq. (18) on a sequence of compact intervals exhausting $[a, +\infty)$, and we apply the results of the previous section to get the existence of a solution in any compact interval.

Lemma 2 below is a convergence result.

Lemma 2. *Assume that conditions* (C)*,* (20)*, and* (H′) *hold. For all* $n \in \mathbb{N}$*,* $n > a$*, let* $I_n := [a, n]$*, and let* $\{u_n\}_n$ *be a sequence* $u_n \in W^{2,1}(I_n)$*,* $n \in \mathbb{N}$*, which verifies the following conditions*:

(i) $|k(t)u_n''(t)| < r$ *for all* $t \in I_n$;
(ii) $\Phi \circ (k \cdot u_n'') \in W^{1,1}(I_n)$;
(iii) $(\Phi(k(t)u_n''(t)))' = f(t, u_n(t), u_n'(t), u_n''(t))$ *for a.e.* $t \in I_n$;
(iv) $u_n(a) = \gamma$*,* $u_n'(a) = \alpha$*, and* $u_n'(n) = \beta$*.*

Then the sequence $\{x_n\}_n$ *in* $W^{2,1}_{\mathrm{loc}}(\Lambda)$ *defined by*

$$x_n(t) := \begin{cases} u_n(t) & \text{for } t \in I_n, \\ u_n(n) + \beta(t - n) & \text{for } t > n, \end{cases}$$

admits a subsequence uniformly convergent in Λ *to a solution* $x_0 \in W^{2,1}_{\mathrm{loc}}(\Lambda)$ *of* (19).

Proof. Let us consider the following sequences. Given $n > a$, for $t \in \Lambda$, we define

$$y_n(t) := \begin{cases} u_n'(t) & \text{if } t \in I_n, \\ \beta & \text{for } t > n; \end{cases} \qquad z_n(t) := \begin{cases} u_n''(t) & \text{if } t \in I_n \text{ and } \exists u_n''(t), \\ 0 & \text{otherwise}; \end{cases}$$

and

$$w_n(t) := \begin{cases} (\Phi(k(t)u_n''(t)))' & \text{if } t \in I_n \text{ and } \exists (\Phi(k(t)u_n''(t)))', \\ 0 & \text{otherwise.} \end{cases}$$

Now, for every $n > a$, from the assumption $|k(t)u_n''(t)| < r$ for all $t \in I_n$, since $k(t) > 0$ for a.e. $t \in \Lambda$, it follows

$$\left|u_n''(t)\right| < \frac{r}{k(t)} \quad \text{for a.e. } t \in I_n. \tag{21}$$

Therefore, denoting

$$\psi(t) := \frac{r}{k(t)} \in L^1(\Lambda),$$

we obtain $|z_n(t)| \leqslant \psi(t)$ for a.e. $t \in \Lambda$, and so $\{z_n\}_n$ is a sequence of uniformly integrable functions in $L^1(\Lambda)$.

Moreover, for every $n > a$, by direct integration we get $|u_n'(t)| < |\alpha| + rk_{1,\Lambda} = M$ for all $t \in I_n$ and $|u_n(t)| \leqslant |\gamma| + M(t - a)$ for all $t \in I_n$. Therefore, from assumption (H′) it follows that there exists a nonnegative function $h = h_{M,\psi} \in L^1(\Lambda)$ such that

$$\left|w_n(t)\right| \leqslant h(t) \quad \text{for a.e. } t \in \Lambda. \tag{22}$$

So, $\{w_n\}_n$ is a sequence of uniformly integrable functions in $L^1(\Lambda)$ as well.

Hence, we can apply the Dunford–Pettis theorem (see, e.g., [15]) obtaining the existence of two functions $\xi, \zeta \in L^1(\Lambda)$ such that (up to a subsequence)

$$z_n \rightharpoonup \zeta \quad \text{and} \quad w_n \rightharpoonup \xi \quad \text{in } L^1(\Lambda) \text{ as } n \to \infty.$$

Now, observe that $y_n(a) = u'_n(a) = \alpha$ for every $n > a$, so that

$$y_n(t) = y_n(a) + \int_a^t y'_n(s)\,\mathrm{d}s = y_n(a) + \int_a^t z_n(s)\,\mathrm{d}s$$
$$\underset{n\to\infty}{\longrightarrow} \alpha + \int_a^t \zeta(s)\,\mathrm{d}s =: y_0(t) \quad \text{for all } t \in \Lambda.$$

By definition, y_0 is absolutely continuous on Λ, and $y_0(a) = \alpha$. Moreover, as already pointed out, $|y_n(t)| \leqslant M$ for all $t \in \Lambda$. Consequently, by applying Lebesgue's dominated convergence theorem on the compact subintervals of Λ, since $x_n(a) = u_n(a) = \gamma$ for every $n > a$, we get

$$x_n(t) = x_n(a) + \int_a^t x'_n(s)\,\mathrm{d}s = x_n(a) + \int_a^t y_n(s)\,\mathrm{d}s$$
$$\underset{n\to\infty}{\longrightarrow} \gamma + \int_a^t y_0(s)\,\mathrm{d}s =: x_0(t) \quad \text{for all } t \in \Lambda. \tag{23}$$

Observe that by construction $x''_0 = y'_0 = \zeta \in L^1(\Lambda)$, so that $x_0 \in W^{2,1}(\Lambda) \subseteq W^{2,1}_{\mathrm{loc}}(\Lambda)$. Therefore, to complete the proof of the lemma, we have to show that x_0 is a solution of (19).

According to Remark 4, we denote by $\mathcal{K}_n$ the unique continuous function defined on I_n such that

$$\mathcal{K}_n(t) = k(t)u''_n(t) \quad \text{for a.e. } t \in I_n.$$

Recall that $|k(t)u''_n(t)| < r$ for all $t \in I_n$ and note that, by assumption (C), there is $n_0 > a$ such that $|\mathcal{K}_n(t_0)| = 0$ for every $n > n_0$.

Moreover, taking into account that

$$u''_n \equiv x''_n \quad \text{and} \quad k \cdot u''_n \equiv \mathcal{K}_n \quad \text{a.e. on } I_n,$$

it is possible to find a set $\Lambda_1 \subseteq \Lambda$, independent of n and with vanishing Lebesgue measure, with the following property: for every $t \in \Lambda \setminus \Lambda_1$, there is $\bar{n}_1 > a$ such that for every $n > \bar{n}_1$, one has

$$\Phi\bigl(k(t)x''_n(t)\bigr) = \Phi\bigl(\mathcal{K}_n(t)\bigr) = \int_{t_0}^t w_n(s)\,\mathrm{d}s.$$

Thus, since $w_n \rightharpoonup \xi$ in $L^1(\Lambda)$ and $\Phi^{-1} : \mathbb{R} \to (-r, r)$ is continuous, we obtain

$$k(t)x''_n(t) \underset{n\to\infty}{\longrightarrow} \Phi^{-1}\Biggl(\int_{t_0}^t \xi(s)\,\mathrm{d}s\Biggr) =: \mathcal{X}(t) \quad \text{for every } t \in \Lambda \setminus \Lambda_1. \tag{24}$$

Note that, by its definition, the function $\mathcal{X}$ is in fact continuous on Λ. Indeed, in view of the pointwise convergence (23), $\mathcal{X}$ is the unique continuous function defined on Λ such that $k(t)x_0''(t) = \mathcal{X}(t)$ for a.e. $t \in \Lambda$. Moreover, the following properties hold: $-r < \mathcal{X}(t) < r$ for every $t \in \Lambda$, $\Phi \circ \mathcal{X}$ is absolutely continuous on Λ, and $(\Phi \circ \mathcal{X})' = \xi \in L^1(\Lambda)$.

Now, by (24) and the fact that $k > 0$ a.e. on Λ, we get

$$z_n(t) = x_n''(t) \underset{n\to\infty}{\longrightarrow} \frac{1}{k(t)}\mathcal{X}(t) \quad \text{for a.e. } t \in \Lambda; \tag{25}$$

on the other hand, by (21), we can apply a standard dominated convergence argument to show that z_n converges to $\mathcal{X}/k$ also in $L^1(\Lambda)$. Thus, since $z_n \rightharpoonup \zeta$ in $L^1(\Lambda)$ as $n \to \infty$, we get

$$\zeta(t) = \frac{1}{k(t)}\,\mathcal{X}(t) \quad \text{for a.e. } t \in \Lambda. \tag{26}$$

So, by (26) and the fact that $x_n'' \to \mathcal{X}/k$ in $L^1(\Lambda)$, we have

$$x_n'' \to \zeta = x_0'' \quad \text{in } L^1(\Lambda).$$

Summing up, the function x_0 defined in (23) has the following properties: $x_0 \in W^{2,1}(\Lambda)$ and $x_0'' = \zeta = \mathcal{X}/k \in L^1(\Lambda)$. Moreover, $k(t)x_0''(t) = \mathcal{X}(t)$ for a.e. $t \in \Lambda$, and thus $|k(t)x_0''(t)| < r$ for a.e. $t \in \Lambda$. In addition, $\Phi \circ (k \cdot x_0'') = \Phi \circ \mathcal{X} \in W^{1,1}(\Lambda)$ and $(\Phi \circ (k \cdot x_0''))' = \xi$.

We claim that x_0 solves the differential equation (18).

For this purpose, we observe that, by (25) and (26), we have $x_n''(t) \to \zeta(t) = x_0''(t)$, $n \to \infty$, for a.e. $t \in \Lambda$. Moreover, taking into account that

$$x_n'' \equiv u_n'' \quad \text{and} \quad \bigl(\Phi\bigl(k(t)u_n''(t)\bigr)\bigr)' = f\bigl(t, u_n(t), u_n'(t), u_n''(t)\bigr) \quad \text{a.e. on } I_n,$$

it is possible to find a set $\Lambda_2 \subseteq \Lambda$, independent of n and with vanishing Lebesgue measure, with the following property: for every $t \in \Lambda \setminus \Lambda_2$, there is $\bar{n}_2 > a$ such that for every $n > \bar{n}_2$, one has

$$w_n(t) = \bigl(\Phi\bigl(k(t)u_n''(t)\bigr)\bigr)' = f\bigl(t, u_n(t), u_n'(t), u_n''(t)\bigr) = f(t, x_n(t), x_n'(t), x_n''(t)).$$

Thus, since $x_n(t) \to x_0(t)$ for all $t \in \Lambda$ and f is a Carathéodory function, we get

$$\lim_{n\to\infty} w_n(t) = f\bigl(t, x_0(t), x_0'(t), x_0''(t)\bigr) \quad \text{for every } t \in \Lambda \setminus \Lambda_2.$$

On the other hand, by (22), we can apply Lebesgue's dominated convergence theorem, obtaining that $w_n \to f(t, x_0, x_0', x_0'')$ in $L^1(\Lambda)$. Therefore, since $w_n \rightharpoonup \xi$ in $L^1(\Lambda)$, we get

$$\bigl(\Phi\bigl(k(t)x_0''(t)\bigr)\bigr)' = \xi(t) = f\bigl(t, x_0(t), x_0'(t), x_0''(t)\bigr) \quad \text{for a.e. } t \in \Lambda,$$

and this proves that x_0 is a solution of (18), as claimed.

Finally, we show that $\lim_{t\to\infty} x_0'(t) = \lim_{t\to\infty} y_0(t) = \beta$. In fact, since $y_n' \to y_0'$ in $L^1(\Lambda)$ and, for $n > a$, the estimate

$$\sup_{\Lambda} |y_n - y_0| \leqslant \|y_n' - y_0'\|_{L^1(\Lambda)}$$

holds (being $y_n(a) = u_n'(a) = \alpha$ for every n), we get that $y_n \to y_0$ uniformly on Λ. Consequently,

$$\lim_{t\to\infty} y_0(t) = \lim_{n\to\infty}\Bigl(\lim_{t\to\infty} y_n(t)\Bigr) = \beta.$$

Therefore, x_0 is a solution of (19), and this completes the proof. □

We now prove Theorem 3.

Proof of Theorem 3. Let $n > a$ be fixed. As a first step, we prove that there exists a solution u_n of the BVP on I_n

$$\begin{aligned} &\bigl(\Phi\bigl(k(t)u''(t)\bigr)\bigr)' = f\bigl(t, u(t), u'(t), u''(t)\bigr) \quad \text{a.e. on } I_n,\\ &u(a) = \gamma, \qquad u'(a) = \alpha, \qquad u'(n) = \beta. \end{aligned} \tag{27}$$

This can be proved following the same approach of Theorem 2 above, namely, by applying Theorem 1 to a suitable truncated auxiliary problem.

More precisely, we define the operator

$$F : \mathcal{W}_\gamma(I_n) \to L^1(I_n), \quad u \mapsto F_u$$

by

$$F_u(t) := f\bigl(t, \mathcal{T}_u^{\sigma,\tau}(t), \mathcal{T}_{u'}^{-M,M}(t), \mathcal{D}_{u'}(t)\bigr) \quad \text{for a.e. } t \in I_n,$$

where the truncation operator $\mathcal{T}$ is as in (14); $\sigma(t) := -|\gamma| - M(t-a)$ and $\tau(t) := |\gamma| + M(t-a)$; finally, given $u \in W^{2,1}(I_n)$, we denote

$$\mathcal{D}_{u'}(t) := \mathcal{T}^{-\psi,\psi}_{(\mathcal{T}_{u'}^{-M,M})'}(t) \quad \text{for a.e. } t \in I_n,$$

where $\psi \in L^1(I_n)$ is defined by

$$\psi(t) := \frac{r}{k(t)} \quad \text{for a.e. } t \in I_n.$$

Observe that, by assumption (20), condition (11) holds as well. So, arguing as in Theorem 2 above, we prove that for any $n > a$, there exists a solution $u_n \in \mathcal{W}_\gamma(I_n)$ of (27). That is, $|k(t)u_n''(t)| < r$ for all $t \in I_n$; $\Phi\circ(k\cdot u_n'') \in W^{1,1}(I_n)$; $(\Phi(k(t)u_n''(t)))' = f(t, u_n(t), u_n'(t), u_n''(t))$ for a.e. $t \in I_n$, moreover, $u_n'(a) = \alpha$ and $u_n'(n) = \beta$.

To complete the proof of Theorem 3, we follow a limit argument. Namely, we apply Lemma 2 above to the sequence $\{u_n\}_n$ obtaining the existence of a solution $x_0 \in W^{2,1}_{\mathrm{loc}}(\Lambda)$ of (19). This concludes the proof. □

We conclude with the following example illustrating Theorem 3.

Example 2. Consider the following BVP on $\Lambda := [0, +\infty)$:

$$\begin{aligned} &\big(\Phi(k(t)u''(t))\big)' = \mu(t)g_1\big(u(t), u'(t), u''(t)\big) \\ &\qquad\qquad\qquad\qquad + u''(t)g_2\big(t, u(t), u'(t)\big) \quad \text{a.e. on } [0, +\infty), \\ &u(0) = \gamma, \qquad u'(0) = \alpha, \qquad u'(+\infty) = \beta. \end{aligned} \tag{28}$$

Here the map $\Phi : (-1, 1) \to \mathbb{R}$ is a strictly increasing homeomorphism (so that $r = 1$); $\Phi(0) = 0$; $k : [0, +\infty) \to \mathbb{R}$ is a.e. positive with $1/k \in L^1(\Lambda)$, $k(0) = 0$, and (to fix ideas) $\|1/k\|_{L^1([0,+\infty))} = 1$; for example, one can choose $k(t) = \pi\sqrt{t}\,(t + 1)$. Moreover, $\mu : \Lambda \to \mathbb{R}$ belongs to $L^1(\Lambda)$ with $\mu(t) \geqslant 0$ for a.a. $t \in \Lambda$, and the maps $g_1 : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g_2 : \Lambda \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and bounded.

Let us prove that, by Theorem 3, problem (28) admits solutions, provided that

$$|\beta - \alpha| < 1.$$

Let

$$f(t, x, y, z) := \mu(t)g_1(x, y, z) + zg_2(t, x, y).$$

Note that f is a Carathéodory function. Let us prove that assumption (H$'$) is satisfied. Indeed, we fix $R > 0$ and $\ell \in L^1(\Lambda)$, and we let

$$M_1 := \max_{(x,y,z)} \big|g_1(x, y, z)\big| \quad \text{and} \quad M_2 := \max_{(t,x,y)} \big|g_2(t, x, y)\big|.$$

Then

$$\big|f\big(t, x(t), y, z(t)\big)\big| \leqslant \mu(t)M_1 + \ell(t)M_2 =: h_{R,\ell}(t)$$

whenever $|y| \leqslant R$ and $x, z \in L^1(\Lambda)$ are such that $|z(t)| \leqslant \ell(t)$ for a.e. $t \in \Lambda$, where $h_{R,\ell} \in L^1_+(\Lambda)$. Thus, (H$'$) is verified.

Note that condition (C) is satisfied, and if $|\beta - \alpha| < 1$, then also (20) holds. Consequently, for any $\gamma \in \mathbb{R}$, we can apply Theorem 3 and obtain that problem (28) admits a solution $u \in W^{2,1}_{\mathrm{loc}}(\Lambda)$ with

$$-|\gamma| - Mt \leqslant u(t) \leqslant |\gamma| + Mt \quad \text{for all } t \geqslant 0,$$

where $M := |\alpha| + 1$.

Author contributions. The authors contributed equally and significantly in writing this article. All authors have read and approved the published version of the manuscript.

Conflicts of interest. The authors declare no conflicts of interest.

Acknowledgment. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and gratefully acknowledge the support provided by GNAMPA-INdAM during the preparation of this work.

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