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# Trajectory controllability of semilinear dynamic systems on time scales

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**Abstract.** This paper explores the trajectory controllability of semilinear dynamic systems defined over time scales, which is an important aspect in understanding and manipulating the behavior of such systems across discrete and continuous domains. We address the controllability of these systems under the assumption that the nonlinearities satisfy a Lipschitz-type condition. Our approach involves a detailed analysis of how these conditions impact the ability to steer the system's state along a desired trajectory within a finite-time horizon. We establish sufficient conditions for trajectory controllability ($T$-controllability), providing a theoretical framework that extends classical results from differential and difference equations to the broader context of time-scale calculus. To illustrate the practical implications of our theoretical findings, we include several numerical examples that demonstrate the application of our results to specific semilinear dynamic systems, highlighting the versatility and effectiveness of our approach.

**Keywords:** $T$-controllability, Lipschitz condition, regressive, rd-continuous, $\Delta$-differentiable, $\Delta$-Carathéodory condition.

## 1 Introduction

Time-scale calculus is a mathematical framework that offers a unified framework for analyzing the behavior of systems across various time domains. It is used to investigate systems mixed with discrete and continuous dynamics [5, 10, 11]. Time-scale calculus has inspiring applications in a wide range of fields, including physics, engineering, economics, and biological studies [3, 4]. In recent years, the theory of dynamic systems on

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time scales, including a range of qualitative features, has made notable advances [29, 30]. We refer to the books (see [10, 11]) and recent papers (see [1, 2]) for more details on time-scale calculus.

In the field of control theory, the concept of "controllability" is used to describe the theoretical possibility of achieving controlled trajectories between the specified initial and final states of a system. Introduced by Kalman in 1960, the notion of controllability holds significant relevance in numerous fields of applied mathematics. The notion of controllability of linear dynamic systems defined on nonuniform time scales was developed by Davis et al. [18]. Since then, there has been a rapid evolution in the study of time-scale control systems, including nonlinear systems. It offers a rich theoretical framework for analyzing and designing control systems that operate in various time domains. For dynamic systems on time scales, stability, controllability, and observability have been the main areas of focus in recent years [12].

A stronger version of controllability is the concept of $T$-controllability introduced by Chalishajar et al. [16]. This notion of controllability attracted many researchers because the control system is not only transferred from a given initial state to a desired final target influenced by a control function, but also follows a desired path or trajectory. It is an important problem due to its widespread applications in domains like robotics, autonomous vehicles, aeronautical engineering, space exploration, and industrial automation, where the ability to regulate the trajectory of a system improves safety and performance with optimal cost. Some important applications of $T$-controllabillity have been listed below:

(i) $T$-controllability in aircraft design and control ensures that the aircraft can follow desired flight trajectories, avoid obstructions, and maintain stability during maneuvers. For both human and unmanned aerial vehicles (UAVs), this is essential [28].

(ii) Robotic systems rely on $T$-controllability to perform tasks with accuracy and efficiency. Robots must adhere to predefined paths or trajectories when performing tasks such as assembly, welding, or material handling in manufacturing environments [33].

(iii) $T$-controllability is essential for self-driving automobiles and other autonomous vehicles to avoid collisions, navigate challenging areas, and obey traffic laws. Control algorithms regulate vehicle trajectory according to sensor inputs and predefined paths [27].

(iv) For satellites and spacecraft to eventually land at their intended locations inside or outside the solar system, $T$-controllability is important. During missions, it allows for accurate trajectory corrections, orbit insertion, and navigation [17].

(v) $T$-controllability is often used in industrial processes to control automated machinery such as robotic arms and conveyor systems [32].

(vi) In traffic control systems, $T$-controllability is important for managing intersections, optimizing traffic flow, and coordinating vehicle movement to reduce congestion and increase safety [27].

(vii) $T$-controllability can also be applied in health care systems, for example, in controlling the spread of infectious diseases [7].

These are just a few examples, and the applications of $T$-controllability continue to expand as technology advances and new challenges arise in various fields. There are very few researchers who have studied the $T$-controllability problem for dynamical systems. $T$-controllability of semilinear systems is examined by Chalishajar [14, 15]. Bin and Liu [8] and Klamka et al. [25] have examined the $T$-controllability of certain semilinear dynamical systems with delays. The concept of $T$-controllability has recently been expanded to include fractional dynamical systems [22, 23, 31]. In general, the study of the $T$-controllability of dynamical systems on time scales is a challenging problem, as it requires the analysis of both continuous and discrete-time dynamics and their interactions as well. Therefore, from an application standpoint, the study is quite interesting, and the concept of $T$-controllability for dynamic systems on time scales is still an unexplored topic.

The structure of this paper is as follows: The basic notation and primary concepts are discussed in Section 2. In Section 3, we state the problem under consideration with the required definition. In Section 4, we first prove the $T$-controllability of the 1-dimensional dynamic equation, and we extend the results to an $n$-dimensional dynamic system in Section 5. We provide examples in the final section to illustrate the obtained theoretical results.

## 2 Preliminaries

We will first recall certain standard notations and terminologies to be used for further discussion [10].

A time scale $\mathbb{T}$ is an arbitrary closed nonempty subset of $\mathbb{R}$. $\mathbb{T}$ can be a continuum, such as the real line, or a discrete set, such as integers. A time scale interval is defined by $[c,d]_{\mathbb{T}} = \{t \in \mathbb{T}\colon\ c \leqslant t \leqslant d\}$ or $[c,d]_{\mathbb{T}} = [c,d] \cap \mathbb{T}$. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as

$$\sigma(t) = \begin{cases} \inf\{s \in \mathbb{T}\colon\ t < s\} & \text{if } t < d, \\ d & \text{if } t = d. \end{cases}$$

Also, $t$ is called right dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$. Let $c = \min \mathbb{T}$ and $d = \max \mathbb{T}$. We denote

$$\mathbb{T}^k = \mathbb{T} \setminus \big(\rho(d), d\big], \qquad \mathbb{T}_0 = \mathbb{T} \setminus \{d\}.$$

So, $\mathbb{T}^k = \mathbb{T}$ if $d$ is left dense, otherwise, $\mathbb{T}^k = \mathbb{T}_0$.

An rd-continuous function is defined as $f : \mathbb{T} \to \mathbb{R}$, where $f$ is continuous at every right-dense point and has a left-sided limit at every left-dense point. If, for every $t \in \mathbb{T}$, $1 + \mu(t)f(t) \neq 0$, then $f : \mathbb{T} \to \mathbb{R}$ is regressive.

Let $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^n)$ denote the set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}^n$. Equipped with the norm, $\|f\| := \sup_{t\in\mathbb{T}} \|f(t)\|$, this space turns into a Banach space [21]. We denote by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R}^n)$ the set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}^n$.

We need the following definitions that are borrowed from [21]. We recall some notions and results related to the theory of $\Delta$-measure and $\Delta$-Lebesgue integration introduced by Bohner and Guseinov [9].

**Definition 1.** A set $A \subset \mathbb{T}$ is said to be $\Delta$-measurable if for every set $E \subset \mathbb{T}$,

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*\big(E \cap (\mathbb{T} \setminus A)\big),$$

where

$$m_1^*(E) = \begin{cases} \inf\{\sum_{k=1}^m (d_k - c_k)\colon\ E \subset \bigcup_{k=1}^m [c_k, d_k),\ c_k, d_k \in \mathbb{T}\} & \text{if } d \notin E, \\ \infty & \text{if } d \in E. \end{cases}$$

The $\Delta$-measure on $\mathcal{M}(m_1^*) := \{A \subset \mathbb{T}\colon\ A \text{ is } \Delta\text{-measurable}\}$, denoted by $\mu_\Delta$, is the restriction of $m_1^*$ to $\mathcal{M}(m_1^*)$. So, $(\mathbb{T}, \mathcal{M}(m_1^*), \mu_\Delta)$ is a complete measurable space.

**Definition 2.** Let $E \subset \mathbb{T}$ be a $\Delta$-measurable set, and let $f : \mathbb{T} \to \mathbb{R}^n$ be a $\Delta$-measurable function. We say that $f \in \mathcal{L}_\Delta^1(E, \mathbb{R}^n)$, provided $\int_E \|f(s)\| \Delta s < \infty$.

According to [21], the set $\mathcal{L}_\Delta^1(\mathbb{T}_0, \mathbb{R}^n)$ is a Banach space endowed with the norm

$$\|f\|_{\mathcal{L}_\Delta^1} := \int_{\mathbb{T}_0} \big\|f(s)\big\| \Delta s.$$

Similarly, we can say that a function $f \in \mathcal{L}_\Delta^2(\mathbb{T}_0, \mathbb{R}^n)$ if it is $\Delta$-measurable on $\mathbb{T}$ and if $\int_{\mathbb{T}_0} \|f(s)\|^2 \Delta s < \infty$.

For the case $n = 1$, we use $\mathcal{L}_\Delta^1(\mathbb{T}_0)$ and $\mathcal{L}_\Delta^2(\mathbb{T}_0)$.

**Definition 3.** A function $f : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be $\Delta$-Carathéodory if the following conditions hold:

(i) The map $t \to f(t, x)$ is $\Delta$-measurable for every $x \in \mathbb{R}^n$.
(ii) The map $x \to f(t, x)$ is continuous for $\Delta$-a.e. $t \in \mathbb{T}_0$.
(iii) For every $r > 0$, there exists a function $h_r \in \mathcal{L}_\Delta^1(\mathbb{T}_0, [0, \infty))$ such that $\|f(t, x)\| \leqslant h_r(t)$ for $\Delta$-a.e. $t \in \mathbb{T}_0$ and for every $x \in \mathbb{R}^n$ such that $\|x\| \leqslant r$.

**Definition 4.** (See [1,2].) If $p \in \mathcal{R}$, then the exponential function is defined by

$$e_p(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}\big(p(\tau)\big) \Delta\tau \right\}$$

for $s, t \in \mathbb{T}$ with the cylindrical transformation

$$\xi_h(z) = \begin{cases} \frac{1}{h} \log(1 + hz) & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

**Theorem 1.** (*See* [11].) *Assume that $f, g : \mathbb{T} \to R$ are both $\Delta$-differentiable at $t \in \mathbb{T}^k$. Then we have the following*:

(i) *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is $\Delta$-differentiable at t, and*

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

(ii) *For $\alpha \in R$, the function $\alpha f : \mathbb{T} \to \mathbb{R}$ is $\Delta$-differentiable at $t$, and*

$$(\alpha f)^{\Delta}(t) = \alpha f(t).$$

(iii) *The product $fg : \mathbb{T} \to \mathbb{R}$ is $\Delta$-differentiable at $t$, and*

$$(fg)^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g\big(\sigma(t)\big) = f^{\Delta}(t)g(t) + f\big(\sigma(t)\big)g^{\Delta}(t).$$

(iv) *If $g(t)g(\sigma(t)) \neq 0$, then the quotient $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is $\Delta$-differentiable at $t$, and*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

**Theorem 2.** (*See* [11].) *Let $t_0 \in \mathbb{T}^{\kappa}$ and $k : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}$ be continuous at $(t, t)$, where $t > t_0$ and $t \in \mathbb{T}^{\kappa}$. Assume that $k^{\Delta}(t, \cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. Suppose that for any $\varepsilon > 0$, there exists a neighborhood $U$ of $t$, independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$\big|\big[k\big(\sigma(t), \tau\big) - k(s, \tau)\big] - k^{\Delta}(t, \tau)\big[\sigma(t) - s\big]\big| \leqslant \varepsilon\big|\sigma(t) - s\big| \quad \forall s \in U.$$

*If $k^{\Delta}$ denotes the derivative of $k$ with respect to the first variable, then*

$$F(t) = \int_{t_0}^{t} k(t, \tau)\,\Delta\tau$$

*yields*

$$F^{\Delta}(t) = \int_{t_0}^{t} k^{\Delta}(t, \tau)\,\Delta\tau + k\big(\sigma(t), t\big).$$

**Theorem 3.** (*See* [10].) *Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$.*

(i) *If $\mathbb{T} = \mathbb{R}$, then*

$$\int_{a}^{b} f(t)\,\Delta t = \int_{a}^{b} f(t)\,\mathrm{d}t,$$

*where the integral on the right is the usual Riemann integral.*

(ii) *If $\mathbb{T} = \mathbb{Z}$, then*

$$\int_{a}^{b} f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum_{t=b}^{a-1} f(t) & \text{if } a > b. \end{cases}$$

(iii) *If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, then*

$$\int_{a}^{b} f(t)\Delta t = \begin{cases} \sum_{k=a/h}^{b/h-1} f(kh)h & \text{if} a < b, \\ 0 & \text{if } a = b, \\ -\sum_{k=b/h}^{a/h-1} f(kh)h & \text{if } a > b. \end{cases}$$

## 3 Statement of the problem

We consider the nonlinear regressive dynamic equation of the form

$$\begin{aligned} x^{\Delta}(t) &= a(t)x(t) + b\big(t, u(t)\big) + f\big(t, x(t)\big), \\ x(0) &= x_0 \end{aligned} \tag{1}$$

defined for all $t \in [0, T]_{\mathbb{T}} = J$. Here $x^{\Delta}(t)$ is the Hilger derivative of $x$ at $t$, $a \in \mathcal{L}^1_{\Delta}(J)$, and $b : J \times \mathbb{R} \to \mathbb{R}$. For $t \in J$, the state $x(t)$ and the control $u(t)$ are real-valued. Further, $f : J \times \mathbb{R} \to \mathbb{R}$ is a nonlinear function satisfying the $\Delta$-Carathéodory conditions.

**Definition 5.** The dynamic system (1) is said to be completely controllable on $J$ if for any $x_0, x_1 \in \mathbb{R}$ and fixed $T > 0$, there exists a control input $u \in \mathcal{L}^2_{\Delta}(J)$ such that the corresponding solution $x(\cdot)$ of (1) satisfies $x(T) = x_1$.

Let $\mathscr{T}$ be the set of all rd-continuous functions $z(\cdot)$ defined on $J$ such that $z(0) = x_0$, $z(T) = x_1$, and $z^{\Delta}$ exists a.e. and is rd-continuous. We denote by $\mathscr{T}$ the set of all trajectories of system (1).

**Definition 6.** The dynamic system (1) is said to be $T$-controllable on $J$ if for any given $z \in \mathscr{T}$, there exists a control input $u \in \mathcal{L}^2_{\Delta}(J)$ such that the corresponding solution $x(\cdot)$ of (1) satisfies $x(t) = z(t)$ a.e. on $J$.

It is important to note that $T$-controllability implies the complete controllability of system (1).

## 4 $T$-controllability of dynamic equation on time scales

Note that system (1) is nonlinear in both state $x(t)$ and control $u(t)$. First, let us consider the nonlinear regressive dynamic equation with $b(t, u) = b(t)u(t)$, i.e., we assume that the control term appears linearly:

$$\begin{aligned} x^{\Delta}(t) &= a(t)x(t) + b(t)u(t) + f\big(t, x(t)\big), \\ x(0) &= x_0, \quad t \in J, \end{aligned} \tag{2}$$

where the functions $a(t)$ and $f(t, x(t))$ are rd-continuous and regressive on $J$. We need to make the following assumptions to prove the $T$-controllability of system (2):

(A1) $b(t)$ is rd-continuous and does not vanish for any $t \in J$.

(A2) $f(\cdot, \cdot)$ is Lipschitz continuous with respect to the second argument, i.e., there exists a constant $\alpha > 0$ such that

$$\big|f(t, x) - f(t, z)\big| \leqslant \alpha |x - z| \quad \forall x, z \in \mathbb{R},\ t \in J.$$

**Theorem 4.** *Suppose that* (A1)–(A2) *are satisfied. Then system* (2) *is $T$-controllable.*

*Proof.* For each control $u \in \mathcal{L}^2_\Delta(J)$, the standard arguments and assumptions (A1)–(A2) guarantee the existence and uniqueness of the solution to system (2) [10].

Under the above assumptions, it is easy to formulate the control explicitly to prove $T$-controllability of the nonlinear system (2). To see this, let us proceed as follows.

Let $z(t)$ be a given trajectory in $\mathscr{T}$. We define a control function $u(t)$ by

$$u(t) = \frac{z^\Delta(t) - a(t)z(t) - f(t, z(t))}{b(t)}.$$

The above-defined control function is rd-continuous. With this control, (2) becomes

$$\begin{aligned} x^\Delta(t) &= a(t)x(t) + z^\Delta(t) - a(t)z(t) - f\big(t, z(t)\big) + f\big(t, x(t)\big), \\ x(0) &= x_0, \quad t \in J. \end{aligned}$$

Setting $w(t) = x(t) - z(t)$, we have

$$\begin{aligned} w^\Delta(t) &= a(t)w(t) + f\big(t, x(t)\big) - f\big(t, z(t)\big), \\ w(0) &= 0, \quad t \in J. \end{aligned} \tag{3}$$

Using the transition function $e_a(t,s) = \exp(\int_s^t \xi_{h(\tau)}(a(\tau))\Delta\tau)$, where

$$\xi_h(z) = \begin{cases} \frac{1}{h}\log(1 + hz) & \text{if } h \neq 0, \\ z & \text{if } h = 0, \end{cases}$$

and the dynamic equation $y^\Delta(t) = a(t)y(t)$, (3) can be rewritten as

$$w(t) = \int_0^t e_a\big(t, \sigma(s)\big)\big(f\big(s, x(s)\big) - f\big(s, z(s)\big)\big)\,\Delta s.$$

Thus, by (A2), we have

$$\big|w(t)\big| \leqslant \alpha \int_0^t \big|e_a\big(t, \sigma(s)\big)\big|\big|x(s) - z(s)\big|\,\Delta s,$$

that is,

$$\big|x(t) - z(t)\big| \leqslant \alpha \int_0^t \big|e_a\big(t, \sigma(s)\big)\big|\big|x(s) - z(s)\big|\,\Delta s.$$

Hence, it follows from Gronwall's inequality that $|x(t) - z(t)| = 0$, proving the $T$-controllability of system (2). □

Let us now examine the case where the control term appears in a nonlinear fashion. To prove the $T$-controllability, we require the following assumptions:

(B1) $b(t, u)$ is rd-continuous on $J \times \mathbb{R}$.

(B2) $b(t, u)$ is coercive in the second variable, i.e., $b(t, u) \to \pm\infty$ as $u \to \pm\infty$ for each $t \in J$.

Based on the above assumptions, we prove the following theorem.

**Theorem 5.** *Suppose that conditions* (A2)*,* (B1)*, and* (B2) *are satisfied. Then the nonlinear system* (1) *is $T$-controllable.*

*Proof.* For each fixed $u(t)$, the existence and uniqueness of the solution of Eq. (1) is guaranteed by the Lipschitz continuity of $f$. Moreover, this solution satisfies the integral equation

$$x(t) = e_a(t,0)x_0 + \int_0^t e_a\big(t,\sigma(s)\big)b\big(s,u(s)\big)\,\Delta s + \int_0^t e_a\big(t,\sigma(s)\big)f\big(s,x(s)\big)\,\Delta s.$$

Let $z(t)$ be a prescribed trajectory in $\mathscr{T}$. We want to construct a control $u(t)$ satisfying

$$z(t) = e_a(t,0)z_0 + \int_0^t e_a\big(t,\sigma(s)\big)b\big(s,u(s)\big)\,\Delta s + \int_0^t e_a\big(t,\sigma(s)\big)f\big(s,z(s)\big)\,\Delta s.$$

The above equation can be rewritten as

$$z(t) - e_a(t,0)z_0 - \int_0^t e_a\big(t,\sigma(s)\big)f\big(s,z(s)\big)\,\Delta s = \int_0^t e_a\big(t,\sigma(s)\big)b\big(s,u(s)\big)\,\Delta s.$$

Differentiating both sides with respect to $t$ and applying the differentiation rules (see Theorem 2) to the above integral terms, we have

$$\begin{aligned} &z^{\Delta}(t) - a(t)e_a(t,0)z_0 - \int_0^t a(t)e_a\big(t,\sigma(s)\big)f\big(s,z(s)\big)\,\Delta s - e_a\big(t,\sigma(t)\big)f\big(t,z(t)\big) \\ &\quad = \int_0^t a(t)e_a\big(t,\sigma(s)\big)b\big(s,u(s)\big)\,\Delta s + e_a\big(t,\sigma(t)\big)b\big(t,u(t)\big). \end{aligned} \tag{4}$$

Equation (4) can be rewritten as

$$w(t) = \int_0^t k(t,s)w(s)\,\Delta s + w_0(t), \tag{5}$$

where $w(t) = b(t,u(t))$, $k(t,s) = -a(t)e_a(t,\sigma(s))$, and the left-hand side of (4) is denoted by $w_0(t)$. It is to be noted that (5) is a linear Volterra integral equation of the second kind, and it has a unique solution $w(t)$ for each given $w_0(t)$ (refer [26]). Hence, it is sufficient to extract $u(t)$ from the solution $w(t)$. To extract $u(t)$, we use the technique developed by Deimling [20] (for more details, see Appendix B).

Let us consider the multivalued function $G : J \to 2^{\mathbb{R}}$ defined by $G(t) = \{u \in \mathbb{R}: b(t,u(t)) = w(t)\}$. Since $b(\cdot,\cdot)$ and $w(\cdot)$ are rd-continuous, by hypothesis (B2), $G(t)$ is nonempty for all $t$ and upper semicontinuous. That is, $t_n \to 0$ implies $G(t_n) \subset G(0) + \bar{B}_\varepsilon(0)$ for all $n \geqslant n(\varepsilon,0)$. Further, $G$ has compact values. Hence, $G$ is Lebesgue measurable and therefore has a measurable selection $u(\cdot)$. This function $u$ is the required control that steers the nonlinear system along the prescribed trajectory $z(t)$. □

## 5 $T$-controllability of nonlinear dynamic systems

Now let us extend the results to an $n$-dimensional system represented by the nonlinear regressive dynamic system of the form

$$\begin{aligned} x^{\Delta}(t) &= A(t)x(t) + B\big(t, u(t)\big) + F\big(t, x(t)\big), \\ x(0) &= x_0 \end{aligned} \tag{6}$$

for all $t \in J$. For $t \in J$, the state vector $x(t) \in \mathbb{R}^n$, $n > 1$, and the control vector $u \in \mathcal{L}^2_{\Delta}(J, \mathbb{R}^n) := U$. Here the matrix $A(t) \in \mathbb{R}^{n\times n}$ and $B : J \times U \to \mathbb{R}^n$. In addition, $F : J \times \mathbb{R}^n \to \mathbb{R}^n$ is nonlinear and rd-continuous.

We make the following assumptions on (6):

(C1) $B(t, u)$ is a measurable function of $t$ for all $u \in U$ and rd-continuous with respect to $u$ for almost all $t \in J$, and it satisfies the linear growth condition

$$\big\|B(t, u)\big\|_{\mathbb{R}^n} \leqslant b_0(t) + b_1\|u\|_U \quad \forall u \in U,\ t \in J.$$

(C2) $F(t, x)$ is measurable with respect to $t$ for all $x \in \mathbb{R}^n$ and rd-continuous with respect to $x$ for almost all $t \in J$, and it satisfies the growth condition

$$\big\|F(t, x)\big\|_{\mathbb{R}^n} \leqslant f_0(t) + f_1(t)\|x\|_{\mathbb{R}^n} \quad \forall x \in \mathbb{R}^n,\ t \in J.$$

Under assumptions (C1) and (C2), the solution $x(t)$ of system (6) satisfies the Volterra integral equation

$$\begin{aligned} x(t) &= e_A(t, 0)x_0 \\ &\quad + \int_0^t e_A\big(t, \sigma(t)\big)B\big(t, u(\tau)\big)\,\Delta\tau + \int_0^t e_A\big(t, \sigma(t)\big)F\big(t, x(\tau)\big)\,\Delta\tau. \end{aligned}$$

Let $\mathscr{G}$ be the set of all functions $z(\cdot)$ defined on $J = [0, T]_{\mathbb{T}}$ such that $z(0) = x_0$, $z(T) = x_1$, and $z^{\Delta}$ exists a.e. and is rd-continuous. We denote by $\mathscr{G}$ the set of all trajectories of system (6).

**Definition 7.** The dynamic system (6) is said to be $T$-controllable if for any $z \in \mathscr{G}$, there exists a $\mathcal{L}^2_{\Delta}$-function $u : J \to \mathbb{R}^n$ such that the corresponding solution $x(\cdot)$ of (6) satisfies $x(t) = z(t)$ a.e. on $J$.

Now we are in a position to prove the $T$-controllability of system (6).

**Theorem 6.** *Suppose that*

(i) *$F(t, x)$ is Lipschitz continuous with respect to $x$, i.e., there exists a constant $\beta > 0$ such that*

$$\big|F(t, x_1) - F(t, x_2)\big| \leqslant \beta|x_1 - x_2| \quad \forall x_1, x_2 \in \mathbb{R}^n,\ t \in J.$$

(ii) $B(t,u)$ *satisfies monotonicity and coercivity conditions, i.e.,*

$$\langle B(t,u) - B(t,v),\, u - v\rangle \geqslant 0 \quad \forall u,v \in U,\ t \in J,$$

*and*

$$\lim_{\|u\|\to\infty} \frac{\langle B(t,u), u\rangle}{\|u\|} = \infty.$$

*Then the nonlinear system* (6) *is $T$-controllable by a measurable control function $u: J \to \mathbb{R}^n$.*

*Proof.* Let $z \in \mathscr{G}$ be any prescribed trajectory. We want to find a control $u(\cdot)$ satisfying

$$z(t) = e_A(t,0)z_0 + \int_0^t e_A\bigl(t,\sigma(s)\bigr)B\bigl(s,u(s)\bigr)\,\Delta s + \int_0^t e_A\bigl(t,\sigma(s)\bigr)F\bigl(s,z(s)\bigr)\,\Delta s.$$

The above equation can be rewritten as

$$z(t) - e_A(t,0)z_0 - \int_0^t e_A\bigl(t,\sigma(s)\bigr)F\bigl(s,z(s)\bigr)\,\Delta s = \int_0^t e_A\bigl(t,\sigma(s)\bigr)B\bigl(s,u(s)\bigr)\,\Delta s.$$

Differentiating with respect to $t$, we get

$$z^{\Delta}(t) - A(t)e_A(t,0)z_0 - \int_0^t A(t)e_A\bigl(t,\sigma(s)\bigr)F\bigl(s,z(s)\bigr)\,\Delta s - e_A\bigl(t,\sigma(t)\bigr)F\bigl(t,z(t)\bigr)$$
$$= \int_0^t A(t)e_A\bigl(t,\sigma(s)\bigr)B\bigl(s,u(s)\bigr)\,\Delta s + e_A\bigl(t,\sigma(t)\bigr)B\bigl(t,u(t)\bigr). \tag{7}$$

The above equation becomes

$$y(t) = \int_0^t k(t,s)y(s)\,\Delta s + y_0(t),$$

where $y(t) = B(t,u(t))$, $k(t,s) = -A(t)e_A(t,\sigma(s))$, and $y_0(t)$ is the left-hand side of (7). Define an operator $P : \mathcal{L}^2_{\Delta}(J,\mathbb{R}^n) \to \mathcal{L}^2_{\Delta}(J,\mathbb{R}^n)$ by

$$(Py)(t) = \int_0^t k(t,s)y(s)\,\Delta s, \tag{8}$$

and $P$ is a bounded linear operator. Furthermore, for sufficiently large $n$, it can be shown that $P^n$ is a contraction [20]. Hence, for given $y_0 \in \mathrm{C_{rd}}(J,\mathbb{R}^n)$, there exists a unique solution $y$ for (8) by the generalized Banach contraction principle. Therefore, $T$-controllability follows if we can extract $u(t)$ from the relation

$$B\big(t, u(t)\big) = y(t), \quad t \in J. \tag{9}$$

To see this, let us define an operator $\mathcal{N} : \mathcal{L}^2_{\Delta}(J, \mathbb{R}^n) \to \mathcal{L}^2_{\Delta}(J, \mathbb{R}^n)$ by

$$(\mathcal{N}u)(t) = B\big(t, u(t)\big).$$

Assumptions (C1) and (C2) imply that the operator $\mathcal{N}$ is well-defined, continuous, and bounded. Hypothesis (ii) shows that $\mathcal{N}$ is monotone and coercive. A hemicontinuous monotone mapping is of type M (for more details; see the Appendix A). Therefore, by Theorem A1 in Appendix A, the nonlinear map $\mathcal{N}$ is onto. Hence, there exists a control $u(\cdot)$ that satisfies (9). The measurability of $u(t)$ follows as $u \in \mathcal{L}^2_{\Delta}(J, \mathbb{R}^n)$, which proves the $T$-controllability of system (6). □

## 6 Numerical examples

The purpose of this section is to demonstrate, using illustrative examples, the efficiency of the results developed that were presented in Sections 4 and 5.

The numerical validation of the derived theoretical results was carried out using MATLAB. For the first example involving linear systems, solutions were obtained analytically. However, the subsequent examples involve nonlinear systems, which necessitate the decomposition of the nonlinear operator. This is achieved using an iterative technique known as the method of successive approximations, which incrementally refines the solution until convergence is achieved. Iterative methods are particularly effective for handling the complexities in nonlinear equations. Such numerical techniques have previously been employed to validate theoretical results for controllability, as demonstrated in [6].

Here we shall consider two different dynamic systems, namely a linear system and a nonlinear system defined on different time scales:

$$\mathbb{T} = \mathbb{R}, \quad \mathbb{T} = \mathbb{Z}, \quad \mathbb{T} = h\mathbb{Z}, \quad \mathbb{T} = [0, 2]_{h\mathbb{Z}} \cup [2, 4]_{\mathbb{R}},$$

and

$$\mathbb{T} = [0, 3]_{h\mathbb{Z}} \cup [3, 6]_{\mathbb{R}}.$$

*Example 1.* Consider the linear dynamic system

$$\begin{aligned} x^{\Delta}(t) &= -4x(t) + \sin\big(2t^2 + 1\big)u(t), \quad t \in [0, 4]_{\mathbb{T}}, \\ x(0) &= 1. \end{aligned} \tag{10}$$

In this case, $a(t) = -4$, $b(t) = \sin(2t^2 + 1)$, $x_0 = 1$, and $x(4) = 17$. Let $z(t) = t^2 + 1 \in \mathscr{T}$ be a prescribed trajectory. Clearly, $z(0) = 1$ and $z(4) = 17$.

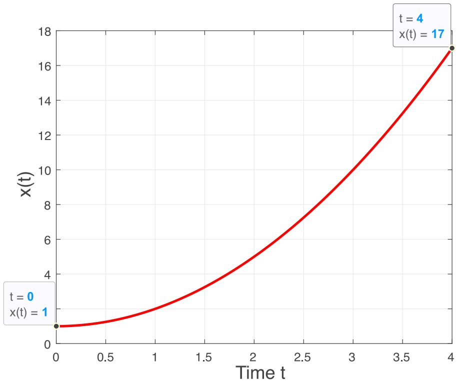


**Figure 1.** Trajectory of the state $x(t)$ during the interval $[0,4]_{\mathbb{R}}$.

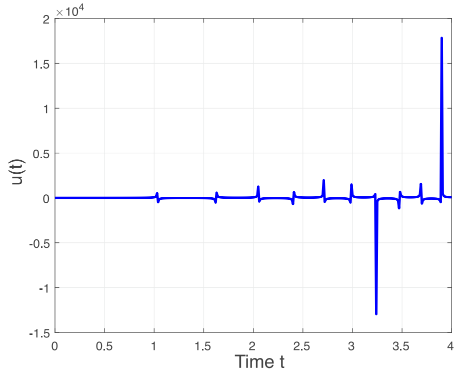


**Figure 2.** Control input $u(t)$ of the system during the interval $[0,4]_{\mathbb{R}}$.

Define the control function

$$u(t) = \frac{z^{\Delta}(t) - a(t)z(t)}{b(t)} = \frac{(t^2+1)^{\Delta} + 4(t^2+1)}{\sin(2t^2+1)} = \frac{t + \sigma(t) + 4(t^2+1)}{\sin(2t^2+1)}. \tag{11}$$

The solution of system (10) is given by

$$x(t) = e_{\ominus 4}(t,0)x_0 + \int_0^t e_{\ominus 4}(t,\sigma(s))\big(s + \sigma(s) + 4(s^2+1)\big)\,\Delta s.$$

*Case 1*. For the case $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$ and $e_a(t,t_0) := \mathrm{e}^{a(t-t_0)} = \mathrm{e}^{-4t}$. The control function (11) is defined by

$$u(t) = \frac{t + \sigma(t) + 4(t^2+1)}{\sin(2t^2+1)} = \frac{2t + 4(t^2+1)}{\sin(2t^2+1)},$$

and hence, the solution is given by

$$x(t) = \mathrm{e}^{-4t} + \int_0^t \mathrm{e}^{-4(t-s)}\big(2s + 4s^2 + 4\big)\,\Delta s = t^2 + 1 = z(t) \quad \forall t \in [0,4]_{\mathbb{T}}.$$

The controlled state trajectory of system (10) and the corresponding input function are shown in Figs. 1 and 2.

*Case 2*. For the case $\mathbb{T} = \mathbb{Z}$, we have

$$\sigma(t) = t + 1 \quad \text{and} \quad e_a(t,t_0) := (1+a)^{t-t_0} = (-3)^t.$$

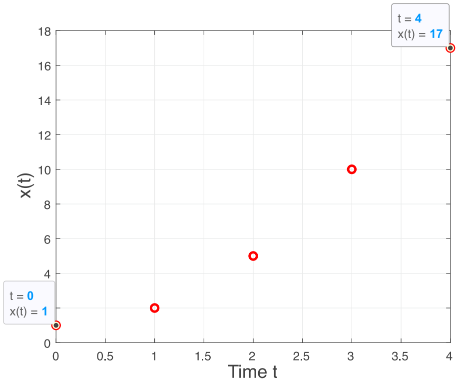


**Figure 3.** Trajectory of the system $x(t)$ during the interval $[0,4]_{\mathbb{Z}}$.

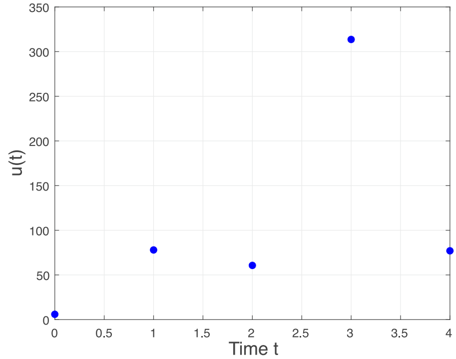


**Figure 4.** Steering control $u(t)$ of the system during the interval $[0,4]_{\mathbb{Z}}$.

The control function defined by (11) takes the form

$$u(t) = \frac{2t + 4t^2 + 5}{\sin(2t^2 + 1)}.$$

Hence, the solution is given by

$$\begin{aligned} x(t) &= e_{\ominus 4}(t,0) + \int_0^t e_{\ominus 4}\bigl(t,\sigma(s)\bigr)\bigl(s + \sigma(s) + 4\bigl(s^2+1\bigr)\bigr)\,\Delta s \\ &= e_{\ominus 4}(t,0) + \int_0^t e_{\ominus 4}(t,s+1)\bigl(2s + 4s^2 + 5\bigr)\,\Delta s \\ &= (-3)^t + (-3)^{t-1}\sum_{s=0}^{t-1}(-3)^{-s}\bigl(2s+4s^2+5\bigr) \\ &= t^2 + 1 = z(t) \quad \forall t \in [0,4]_{\mathbb{T}}. \end{aligned}$$

Figures 3 and 4 depict the controlled state trajectory of system (10) and the corresponding input function.

*Case 3*. For the case $\mathbb{T} = h\mathbb{Z}$ and $h > 0$, we have

$$\sigma(t) = t + h \quad \text{and} \quad e_a(t,t_0) := (1 + ah)^{(t-t_0)/h} = (1-4h)^{t/h}.$$

The control function is

$$u(t) = \frac{2t + 4t^2 + 4 + h}{\sin(2t^2 + 1)}.$$

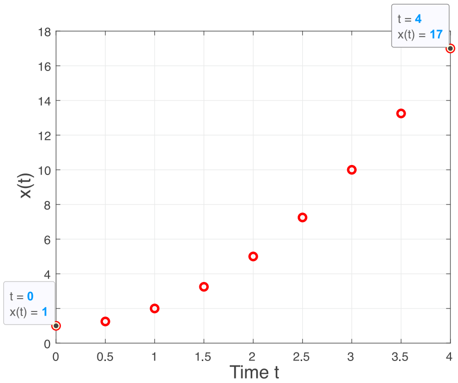


**Figure 5.** Trajectory of the system state $x(t)$ during the interval $[0, 4]_{(1/2)\mathbb{Z}}$.

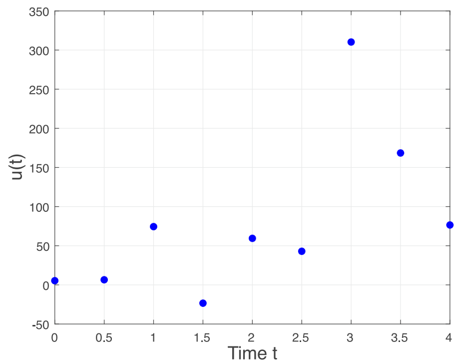


**Figure 6.** Control function $u(t)$ of the system during the interval $[0, 4]_{(1/2)\mathbb{Z}}$.

Hence, the solution is given by

$$\begin{aligned} x(t) &= (1-4h)^{t/h} + \int_0^t (1-4h)^{(t-s-h)/h}(2s+h+4s^2+4)\,\Delta s \\ &= (1-4h)^{t/h} + (1-4h)^{t/h-1}h\Biggl\{2h\sum_{s=0}^{t/h-1}(1-4h)^{-s}s \\ &\quad + (h+4)\sum_{s=0}^{t/h-1}(1-4h)^{-s} + 4h^2\sum_{s=0}^{t/h-1}(1-4h)^{-s}s^2\Biggr\}. \end{aligned}$$

Let us choose $h = 1/2$. Hence, the solution becomes

$$\begin{aligned} x(t) &= \left(1-4\left(\frac{1}{2}\right)\right)^{2t} x_0 \\ &\quad + \left(1-4\left(\frac{1}{2}\right)\right)^{2t-1}\frac{1}{2}\left\{\sum_{s=0}^{2t-1}\left(1-4\left(\frac{1}{2}\right)\right)^{-s}\left(s+\frac{1}{2}+4\left(\frac{1}{4}\right)s^2\right)\right\} \\ &= t^2+1 = z(t) \quad \forall t \in [0,4]_{\mathbb{T}}. \end{aligned}$$

Figures 5 and 6 present the controlled state trajectory and the input function of system (10).

*Case 4*. For the case $\mathbb{T} = [0, 2]_{(1/2)\mathbb{Z}} \cup [2, 4]_{\mathbb{R}}$, where $h = 1/2$, we have:

(i) If $\mathbb{T} = h\mathbb{Z}$, then $e_a(t, t_0) := (1 + ah)^{(t-t_0)/h}$ and $\sigma(t) := t + h$.
(ii) If $\mathbb{T} = \mathbb{R}$, then $e_a(t, t_0) := \mathrm{e}^{a(t-t_0)}$ and $\sigma(t) := t$.

Consider the linear dynamic system (10), the initial points $x(0) = 1$ and $x(2) = 5$, and the final points $x(2) = 5$ and $x(4) = 17$. Let $z(t) = t^2 + 1 \in \mathscr{T}$. Clearly, $z(2) = 5$ and $z(4) = 17$.

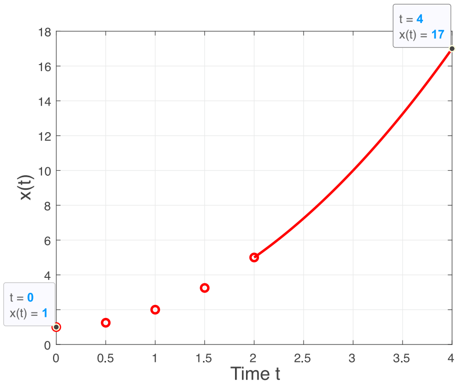


**Figure 7.** Trajectory of the system state $x(t)$ is $[0,2]_{(1/2)\mathbb{Z}} \cup [2,4]_{\mathbb{R}}$.

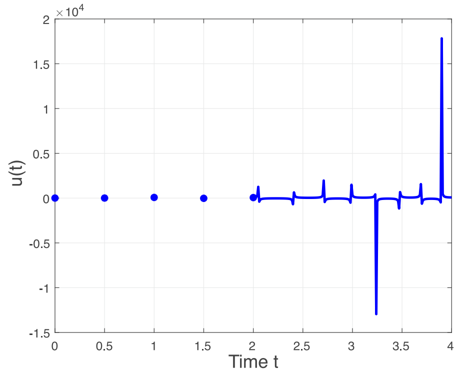


**Figure 8.** Control input $u(t)$ of the system $[0,2]_{(1/2)\mathbb{Z}} \cup [2,4]_{\mathbb{R}}$.

As in Case 3 of Example 1, for $t \in [0,2]_{(1/2)\mathbb{Z}}$, the solution follows and is ignored. The control function $u(t)$ is given by

$$u(t) = \begin{cases} \frac{2t+h+4t^2+4}{\sin(2t^2+1)}, & t \in [0,2]_{h\mathbb{Z}}, \\ \frac{2t+4(t^2+1)}{\sin(2t^2+1)} & t \in [2,4]_{\mathbb{R}}. \end{cases}$$

For $t \in [2,4]_{\mathbb{R}}$, the solution is given by

$$x(t) = e_{\ominus 4}(t,2)x(2) + \int_2^t e_{\ominus 4}\bigl(t,\sigma(s)\bigr)\bigl(s+\sigma(s)+4\bigl(s^2+1\bigr)\bigr)\,\Delta s.$$

Using (ii), we get

$$\begin{aligned} x(t) &= \mathrm{e}^{-4(t-2)}x(2) + \int_2^t \mathrm{e}^{-4(t-s)}\bigl(2s+4s^2+4\bigr)\,\Delta s \\ &= 5\mathrm{e}^{-4(t-2)} + \mathrm{e}^{-4t}\int_2^t \mathrm{e}^{4s}\bigl(2s+4s^2+4\bigr)\,\Delta s \\ &= t^2+1 = z(t) \quad \forall t \in [2,4]_{\mathbb{R}}. \end{aligned}$$

Figures 7 and 8 depict the controlled state trajectory and input function of system (10).

*Example 2.* Consider the nonlinear dynamic system

$$\begin{aligned} &x^{\Delta}(t) = -2x(t) + (t+1)u(t) - 10\sin x(t), \quad t \in [0,6]_{\mathbb{T}}, \\ &x(0) = 1. \end{aligned} \tag{12}$$

In this case, $a(t) = -2$, $b(t) = t+1$, $x_0 = 1$ and $x(6) = 73$. Let $z(t) = 2t^2+1 \in \mathscr{T}$ be a prescribed trajectory. Clearly, $z(0) = 1$ and $z(4) = 33$.

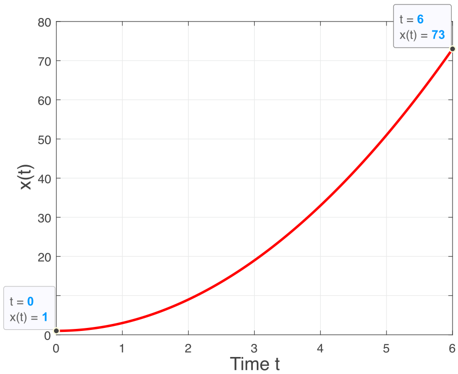


**Figure 9.** Trajectory of the state $x(t)$ during the interval $[0, 6]_{\mathbb{R}}$.

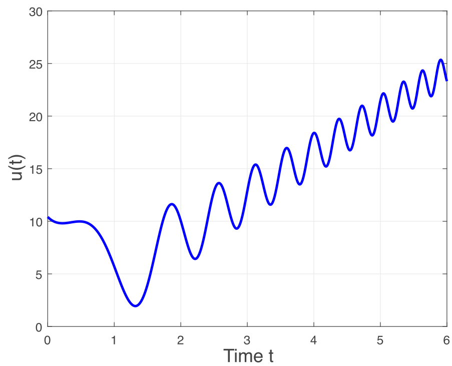


**Figure 10.** Control input $u(t)$ of the system during the interval $[0, 6]_{\mathbb{R}}$.

Define the control function

$$u(t) = \frac{z^{\Delta}(t) + 2z(t) + 10 \sin z(t)}{t+1} = \frac{2t + 2\sigma(t) + 4t^2 + 2 + 10 \sin(2t^2 + 1)}{t+1}. \quad (13)$$

The solution of system (12) and (13) is

$$x(t) = e_{\ominus 2}(t, 0)x_0 + \int_0^t e_{\ominus 2}\bigl(t, \sigma(s)\bigr)b(s)u(s)\,\Delta s + \int_0^t e_{\ominus 2}\bigl(t, \sigma(s)\bigr) f\bigl(s, x(s)\bigr)\,\Delta s$$
$$= e_{\ominus 2}(t, 0) + \int_0^t e_{\ominus 2}\bigl(t, \sigma(s)\bigr)\bigl[2s + 2\sigma(s) + 4s^2 + 2\bigr]\,\Delta s.$$

*Case 1*. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $e_\alpha(t, t_0) := \mathrm{e}^{\alpha(t-t_0)} = \mathrm{e}^{-2t}$.
The control function is

$$u(t) = \frac{4t + 2 + 4t^2 + 10 \sin(2t^2 + 1)}{t+1},$$

and hence, the solution is given by

$$x(t) = \mathrm{e}^{-2t} + \int_0^t \mathrm{e}^{-2(t-s)}\bigl(4s + 4s^2 + 2\bigr)\,\mathrm{d}s = 2t^2 + 1 = z(t) \quad \forall\, t \in [0, 6]_{\mathbb{T}}.$$

The controlled state trajectory of system (12) and the corresponding input function are plotted in Figs. 9 and 10.

*Case 2*. If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$ and $e_\alpha(t, t_0) := (1 + \alpha)^{t-t_0} = (-1)^t$.
The control function is

$$u(t) = \frac{4t + 4 + 4t^2 + 10 \sin(2t^2 + 1)}{t+1}.$$

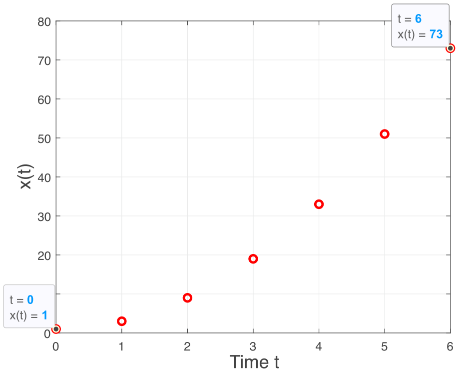


**Figure 11.** Trajectory of the system $x(t)$ during the interval $[0, 6]_{\mathbb{Z}}$.

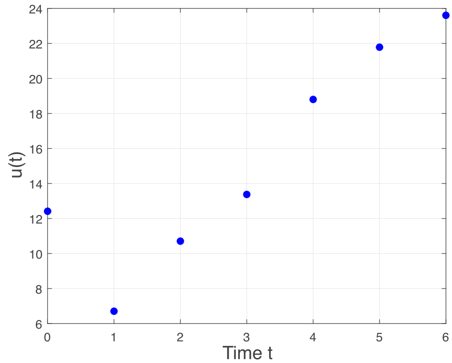


**Figure 12.** Steering control $u(t)$ of the system during the interval $[0, 6]_{\mathbb{Z}}$.

Hence, the solution is given by

$$x(t) = (-1)^t + \int_0^t (-1)^{t-s-1}\big(2s + 2(s+1) + 4s^2 + 2\big)\,\Delta s$$
$$= 2t^2 + 1 = z(t) \quad \forall t \in [0, 6]_{\mathbb{T}}.$$

The controlled state trajectory of system (12) and the corresponding input function are shown in Figs. 11 and 12.

*Case 3*. If $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, then $\sigma(t) = t + h$ and $e_\alpha(t, t_0) := (1 + \alpha h)^{(t-t_0)/h} = (1 - 2h)^{t/h}$. The control function is

$$u(t) = \frac{4t + 2 + 2h + 4t^2 + 10\sin(2t^2 + 1)}{t + 1}.$$

Hence, the solution is given by

$$x(t) = (1 - 2h)^{t/h} + \int_0^t (1 - 2h)^{(t-s-h)/h}\big(2s + 2(s+h) + 4s^2 + 2\big)\,\Delta s$$
$$= (1 - 2h)^{t/h} + (1 - 2h)^{t/h-1}\int_0^t (1 - 2h)^{-sh}\big(4s + 2h + 4s^2 + 2\big)\,\Delta s$$
$$= (1 - 2h)^{t/h} + (1 - 2h)^{t/h-1} h\Bigg\{4h \sum_{s=0}^{t/h-1} (1 - 2h)^{-s} s$$
$$+ 2(h+1) \sum_{s=0}^{t/h-1} (1 - 2h)^{-s} + 4h^2 \sum_{s=0}^{t/h-1} (1 - 2h)^{-s} s^2\Bigg\}.$$

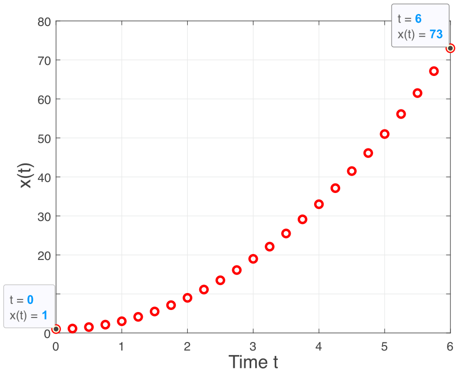


**Figure 13.** Trajectory of the system state $x(t)$ during the interval $[0, 6]_{(1/4)\mathbb{Z}}$.

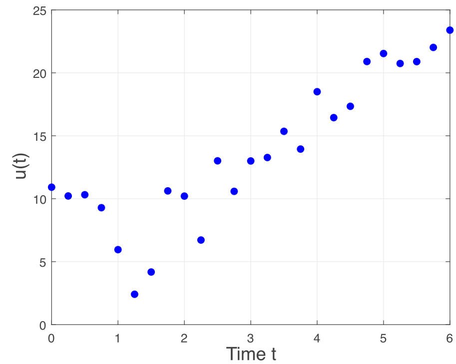


**Figure 14.** Control input $u(t)$ of the system during the interval $[0, 6]_{(1/4)\mathbb{Z}}$.

Taking $h = 1/4$, we have

$$\begin{aligned} x(t) &= \left(\frac{1}{2}\right)^{4t} + \left(\frac{1}{2}\right)^{4t-1} \frac{1}{4}\left[\sum_{s=0}^{4t-1}\left(\frac{1}{2}\right)^{-s} s + \left(\frac{5}{2}\right)\sum_{s=0}^{4t-1}\left(\frac{1}{2}\right)^{-s} \right. \\ &\quad \left. + \left(\frac{1}{4}\right)\sum_{s=0}^{4t-1}\left(\frac{1}{2}\right)^{-s} s^2\right] \\ &= \left(\frac{1}{2}\right)^{4t} + \left(\frac{1}{2}\right)^{4t-1} \frac{1}{4}\left[\sum_{s=0}^{4t-1}\left(\frac{1}{2}\right)^{-s}\left(s + \frac{5}{2} + \frac{s^2}{4}\right)\right] \\ &= \left(\frac{1}{2}\right)^{4t} + \left(\frac{1}{2}\right)^{4t-1} \frac{1}{4}\left(2\cdot 16^t + 4\cdot 2^{4t}t^2 - 2\right) \\ &= 2t^2 + 1 = z(t) \quad \forall t \in [0, 6]_{\mathbb{T}}. \end{aligned}$$

Figures 13 and 14 present the controlled state trajectory and the input function of system (12).

*Case 4*. For the case $\mathbb{T} = [0, 3]_{h\mathbb{Z}} \cup [3, 6]_{\mathbb{R}}$ and $h > 0$, we consider system (12) with initial state $x(0) = 1$ and $x(3) = 19$ and the final state $x(3) = 19$ and $x(6) = 73$. Let $z(t) = 2t^2 + 1 \in \mathscr{T}$. Clearly, $z(3) = 19$ and $z(6) = 73$.

(i) If $\mathbb{T} = h\mathbb{Z}$, then $e_\alpha(t, t_0) := (1 + \alpha h)^{(t-t_0)/h} = (1 - 2h)^{t/h}$ and $\sigma(t) := t + h$.

(ii) If $\mathbb{T} = \mathbb{R}$, then $e_\alpha(t, t_0) := e^{\alpha(t-t_0)} = \mathrm{e}^{-2(t-3)}$ and $\sigma(t) := t$.

The control function $u(t)$ given by

$$u(t) = \begin{cases} \frac{4t+2h+4t^2+2+10\sin(2t^2+1)}{t+1}, & t \in [0, 3]_{h\mathbb{Z}}, \\ \frac{4t+4t^2+2+10\sin(2t^2+1)}{t+1}, & t \in [3, 6]_{\mathbb{R}}. \end{cases}$$

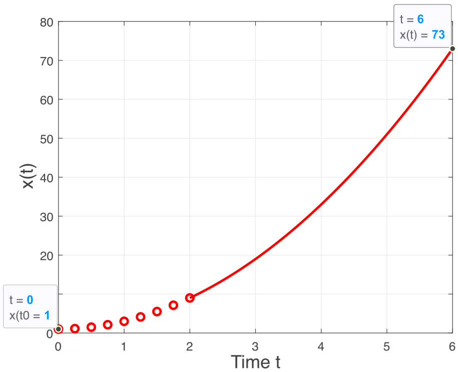


**Figure 15.** Trajectory of the system state $x(t)$ is $[0,3]_{h\mathbb{Z}} \cup [3,6]_{\mathbb{R}}$.

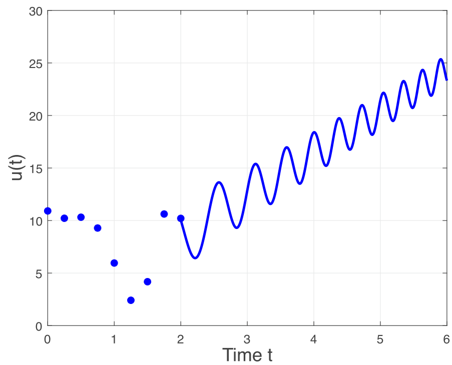


**Figure 16.** Steering control $u(t)$ of the system $[0,3]_{h\mathbb{Z}} \cup [3,6]_{\mathbb{R}}$.

As in Case 3 above, for $t \in [0,3]_{(1/4)\mathbb{Z}}$, the solution follows and it is ignored. Next, for $t \in [3,6]_{\mathbb{R}}$, the solution is given by

$$x(t) = e_{\ominus 2}(t,3)x(3) + \int_3^t e_{\ominus 2}\bigl(t,\sigma(s)\bigr)\bigl(2s + 2\sigma(s) + 4s^2 + 2\bigr)\,\Delta s.$$

Using (ii),

$$\begin{aligned} x(t) &= \mathrm{e}^{-2(t-3)} + \int_3^t \mathrm{e}^{-2(t-s)}\bigl(4s + 4s^2 + 2\bigr)\,\Delta s \\ &= 2t^2 + 1 = z(t) \quad \forall t \in [3,6]_{\mathbb{R}}. \end{aligned}$$

Figures 15 and 16 depict the controlled state trajectory and the input function of system (12).

## 7 Conclusion

In this manuscript, the $T$-controllability of dynamic systems on time scales is formulated. We have used the Gronwall's inequality and Lipschitz-type conditions on the nonlinear term to obtain the $T$-controllability results. Moreover, the theoretical results are illustrated by providing some numerical examples with simulation for distinct time scales, namely, $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = [0,2]_{h\mathbb{Z}} \cup [2,4]_{\mathbb{R}}$, and $\mathbb{T} = [0,3]_{h\mathbb{Z}} \cup [3,6]_{\mathbb{R}}$. Our results contribute to the broader field of dynamic system control by bridging the gap between discrete and continuous dynamics through the lens of time scales, offering new insights and tools for system designers and engineers.

Future research could build on these insights by exploring higher-order dynamic equations on more complex time scales, further expanding the applicability and effectiveness of control methods in real-world scenarios.

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**Conflicts of interest.** The authors declare no conflicts of interest.

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## Appendix A

Here we present some important definitions and results from [19, 24] that were used to prove our main result (Theorem 6).

Let $X$ be a Banach space, and let $X^* = L(X, \mathbb{R})$ be the set of continuous linear functionals $x^* : X \to \mathbb{R}$.

We write $x_n \rightharpoonup x_0$ if a sequence $\{x_n\}$ converges weakly to $x_0$, i.e., if $x^*(x_n - x_0) \to 0$ for all $x^* \in X^*$.

**Definition A1.** A mapping $F : D \subset X \to X$ is said to be weakly continuous at $x_0$ if for every sequence $\{x_n\}$ such that $x_n \rightharpoonup x_0$, we have $Fx_n \rightharpoonup Fx_0$.

**Definition A2.** A mapping $F : D \subset X \to X$ is said to be hemicontinuous if $F(x + t_n, y) \rightharpoonup Fx$ as $t_n \to 0^+$.

**Definition A3.** A mapping $F : X \to X^*$ is said to be of type $(M)$ if the following conditions are met:

- (i) If a sequence $\{x_n\}$ in $X$ converges weakly to $x$ in $X$ and $\{Fx_n\}$ converges weakly to $y$ in $X^*$ and $\lim_n \sup(Fx_n, x_n) \leqslant (y, x)$, then $Fx = y$.
- (ii) $F$ is continuous from finite-dimensional subspaces of $X$ to $X^*$ endowed with a weak* topology.

**Proposition A1.** *Let $A : X \to X^*$ be a linear mapping. Then $A$ is bounded if and only if $A$ satisfies condition* (i) *of the above definition.*

*Proof.* If $A$ is bounded, then (i) is trivially satisfied since all continuous linear operators are weakly continuous. Let $A$ satisfy (i), and let $\{x_n\}$ be any sequence in $X$ such that $x_n \to x$ and $Ax_n \to y$. The result gives $(Ax_n, x_n) \to (y, x)$, and so $Ax = y$ by (i). Thus the graph of $A$ is closed, and hence, $A$ is bounded by the closed graph theorem. □

**Theorem A1.** *Let $F : X \to X^*$ be a mapping of type* M. *If $F$ is coercive, then $F$ is onto, i.e., the range of $F$ is all of $X^*$.*

*Proof.* For any $w \in X^*$, let us define the mapping $F_w : X \to X^*$ by

$$F_w(x) = F(x) - w, \quad x \in X.$$

Since $F_x$ is a bounded coercive mapping of type M, it is sufficient to show that $0 \in R(F)$. Let $\Lambda$ be the set of all finite subsets $D$ of $X$ such that $0 \in D$, and let the convex hull of $D$ be denoted by $\operatorname{conv} D$.

Since the mapping $F$ is coercive and finitely continuous, it follows by Proposition 7.3 of Browder [13] that for each $D \in \Lambda$, there exists a constant $M > 0$ and an element $x_D \in \operatorname{conv} D$ such that $\|x_D\| \leqslant M$ and $(Fx_D,\ x - x_D) \geqslant 0$,

Let $M = \inf\{(Fx_D, x_D)\colon D \in \Lambda\}$. Since the mapping $F$ is bounded and the subset $\{x_D\colon D \in \Lambda\}$ is a bounded subset of $X$, clearly $M > -\infty$. For each $D_0$ in $\Lambda$, set $V_{D_0} = \cup\{x_D\colon D \in \Lambda,\ D \supset D_0\}$. Then $V_{D_0}$ is contained in the ball of radius $R$ in $X$.

Let $\bar{V}_{D_0}$ denote the weak closure of $V_{D_0}$ in $X$. Since the Banach space $X$ is reflexive, the family $\{\bar{V}_{D_0}\colon D_0 \in \Lambda\}$ is a family of weakly closed subsets of $X$, and the family also has the finite intersection property, i.e.,

$$\cap\{\bar{V}_{D_0}\colon D_0 \in \Lambda\} \neq \emptyset.$$

Let $x_0 \in \cap\{\bar{V}_{D_0}\colon D_0 \in \Lambda\}$. We will now show that $Fx_0 = 0$. Suppose that $Fx_0 \neq 0$. Let $x \in X$ be such that $(Fx_0, x) < M$, and let $D_1 \in \Lambda$ be such that $x$ and $x_0$ are in $D_1$. Since $x_0 \in \bar{V}_{D_1}$, it follows by Proposition 7.2 of Browder [13] that we can find a sequence $\{D_n\}_{n=2}^{\infty}$ in $\Lambda$, where $D_n \supset D_1$ for each $n$ such that $x_{D_n} \rightharpoonup x_0$. We may assume that there exists an element $y_0 \in X^*$ such that $Fx_{D_n} \rightharpoonup y_0$. Since the relation $(Fx_D, x_D - u) \leqslant 0$ holds for each $u \in \operatorname{conv} D$ and for every $D \in \Lambda$, it follows that $\lim_n \sup(Fx_{D_n}, x_{D_n}) \leqslant (y_0, u)$ for every $u \in \operatorname{conv} D_1$.

Taking $u = x_0$ and using condition (i) of the Definition A3, we get $Fx_0 = y_0$. Again taking $u = x \in D_1 \subset \operatorname{conv} D_1$, this relation leads to

$$M < \limsup_n (Fx_{D_n}, x_{D_n}) \leqslant (y_0, x) = (Fx_0, x) < M,$$

a contradiction. Thus $Fx_0 = 0$, and the theorem is proved. □

## Appendix B

Here we prove an important result inspired by Theorem 4 of [20] that was used to prove our main result (Theorem 5).

Let us consider the Volterra integral equation defined on time-scale of the form

$$\int_0^t k(t,s)g\big(s, x(s)\big)\,\Delta s = f(t) \quad \text{on } J = [0,a] \tag{B.1}$$

with $f \in C^1(J)$ and $f(0) = 0$, $k$ and $k_t$ continuous on $\Omega = \{(t,s)\colon 0 \leqslant s \leqslant t \leqslant a\}$ and such that $\det k(t,t) \neq 0$ on $J$.

**Theorem B2.** *Assume that $g : J \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous and satisfies*

$$\big(g(t,x) - g(t,y),\ x - y\big) > 0 \quad \forall t \in J,\ x, y \in \mathbb{R}^n,\ x \neq y, \tag{B.2}$$

$$\frac{(g(t,x), x)}{|x|} \to \infty, \quad |x| \to \infty. \tag{B.3}$$

*Then* (B.1) *has a unique solution $x(\cdot) \in \mathrm{C_{rd}}(J)$.*

*Proof.* Differentiating (B.1) leads to the Volterra integral equation of the second kind of the form

$$y(t) + \int_0^t k^{-1}(t,t)k_t(t,s)y(s)\,\Delta s = k^{-1}(t,t)f^{\Delta}(t) \quad \text{on } J$$

for $y(t) = g(t, x(t))$, which has a unique solution $y(\cdot) \in \mathrm{C_{rd}}(J)$. Therefore, it is now sufficient to show that $g(t, x(t)) = y(t)$ has a unique solution $x(\cdot) \in \mathrm{C_{rd}}(J)$. For fixed $t \in J$, the map $g(t, \cdot)$ is rd-continuous and satisfies (B.3), and hence, by an application of the Brouwer degree (see [19, Thm. 3.3]), we have $g(t, \mathbb{R}^n) = \mathbb{R}^n$. Thus, $g(t, z) = y(t)$ has a solution $z = x(t)$, and it is unique by (B.2). Now $t_n \to t_0$ implies that $\{y(t_n)\}$ is bounded, hence, $\{x(t_n)\}$ is bounded by (B.3), and therefore, $x(t_n) \to x(t_0)$ is an obvious consequence of uniqueness. □

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