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Null controllability of Chafee–Infante equation under discrete-time point measurements*

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Abstract. Nonlinear system is one of the main research objects in cybernetics, and it is the main theme of cybernetics in the 21st century. Recently, the control of the reaction–diffusion equation has been widely studied, but the nonlinear reaction–diffusion equation has been rarely studied. This paper will take the Chafee–Infante equation as an example, and the null controllability of this equation will be shown. We consider the null controllability for Chafee–Infante equation with point actuations subject to a known constant delay. The point measurements can be sampled in time and transmitted through a communication network with a time-varying delay. We design an observer for the future value of the state in order to compensate the input delay, then we ensure that the estimation error vanishes exponentially with a desired decay rate by using a time-varying observer gain. By constructing Lyapunov–Krasovskii functional and combining linear matrix inequalities (LIMs), we obtain the convergence conditions. We design the boundary controller and the point controller, and we conclude that both controllers can ensure the exponential stability of the closed-loop system with an arbitrary decay rate, which is smaller than that of the observers estimation error. At last, numerical example is given.

Keywords: Chafee–Infante reaction–diffusion equation, point control, boundary control, input delay.

1 Introduction

In recent years, a lot of research has been done on the soliton and exact solutions of Chafee–Infante equation, which is helpful to apply the reaction–diffusion equation to solve related practical problems. However, there are not many studies on the stability of the Chafee–Infante system, and there are two main ways to stabilize the Chafee–Infante

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system: one is to stabilize the Chafee–Infante system through noise [18], and the other is to use feedback control to stabilize the Chafee–Infante system [24]. Due to the limitations of the current stabilization methods, this paper continues the study of the stability of the Chafee–Infante system.

Recently, one important class of partial differential equations (PDEs) were studied under continuous-time measurements in [1, 19, 29], as well further extended to discrete-time measurements for both static [5,10,26] and dynamic feedback [12]. Through a series of research, an observer, which could predict the future value of the state, was proposed to compensate the input delay [3]. In order to improve the exponential convergence under delayed measurements, a time-varying injection gain was brought in an observer [4].

In [13], introducing a constant input delay can be compensated by representing it as a PDE–PDE cascade in a reaction–diffusion system, which is analysed using the backstepping transformation [14]. While it is difficult to combine with data sampling. Qualitative stability for sampled-data infinite-dimensional systems in general form was studied by many authors; see [15–17]. When the given PDEs are not stable, many researchers discussed how to design a sampled-data control of parabolic PDEs by Lyapunov–Krasovskii functionals to gain LMI-based quantitative stability conditions [2, 25, 27]. It should be point out that above control is proposed by an unbounded operator, whether this direct Lyapunov–Krasovskii functionals can be extended to state-feedback boundary or point control has not been solved. In [28], qualitative stability results are obtained for observer-based design for delayed boundary and in-domain point control. The above methods can solve the problem of delay in linear systems, but cannot solve the influencing factors of nonlinearity.

The main difference from [28] is that the Chafee–Infante reaction–diffusion equation is a nonlinear reaction–diffusion equation, and Selivanov and Fridman considered the linear case. We generalize their results to nonlinear case.

In the present work, point actuation subject to a constant delay is proposed to stabilize Chafee–Infante reaction–diffusion equation under boundary conditions. The boundary control and the point control strategy for Chafee–Infante reaction–diffusion equation are given in Sections 2 and 3, respectively. Numerical example is presented in Section 4.

2 Boundary control

We consider the following system governed by the reaction–diffusion PDE:

$$\begin{aligned} &z_t(x,t) = z_{xx}(x,t) + az(x,t) - z^3(x,t),\\ &d_L z(0,t) + (1-d_L)z_x(0,t) = 0,\\ &d_R z(1,t) + (1-d_R)z_x(1,t) = u(t-r), \end{aligned} \tag{1}$$

where $z : [0,1] \times [0,\infty) \to \mathbb{R}$ is the state, a is the reaction coefficient, u is the boundary control, and $r \geqslant 0$ is a known constant delay. Each of the constants $d_L, d_R \in \{0,1\}$ specifies either the Dirichlet or the Neumann boundary condition. If the reaction coefficient a is large enough and $u(t) = 0$, the system will be unstable.

For boundary control with the input delay, the robustness analysis is more difficult than distributed control; see [8] and [25]. In order to show this, assume that $d_L = d_R = 1$ in (1), then it can be obtained that the backstepping transformation contributes to the target system with the boundary delay [14, Chap. 4]

$$\begin{aligned} &w_t(x,t) = w_{xx}(x,t), \\ &w(0,t) = 0, \qquad w(1,t) = -\int_0^1 l(1,y)\big[w(y,t) - w(y,\, t-r)\big]\,\mathrm{d}y, \end{aligned}$$

where l is the kernel of the inverse transformation. In the case of boundary delay, we find that an appropriate Lyapunov functional for delay-dependent stability is a difficult problem, and Lyapunov-based ISS analysis in the case of boundary disturbances is problematic because the disturbance is multiplied by an unbounded operator [11]. We compensate the input delay by using an appropriate observer in order to avoid these difficulties.

Inspired by [28], we suppose that N in-domain sensors provide point measurements of the state sampled in time and transmitted through a network with a time-varying delay. Thus, the values $z(x_i, s_k)$ are available at time t_k, where

$$\begin{aligned} &0 \leqslant x_1 < x_2 < \cdots < x_N \leqslant 1, \\ &0 = s_0 < s_1 < \cdots, \quad s_{k+1} - s_k \leqslant h, \quad \lim_{k\to\infty} s_k = \infty, \\ &t_k = s_k + \eta_k, \quad \eta_k \in [0, \eta_M] \quad t_k \leqslant t_{k+1}. \end{aligned} \tag{2}$$

2.1 Observer/predictor construction

In order to estimate the future value of the state $\hat{z}(x,t) \approx z(x,\, t+r)$, we construct the following observer:

$$\begin{aligned} &\hat{z}_t(x,t) = \hat{z}_{xx}(x,t) + a\hat{z}(x,t) - \hat{z}^3(x,t) + L\mathrm{e}^{-\alpha_0(t+r-s_k)} \\ &\qquad \times \sum_{i=1}^{N} b_i(x)\big[\hat{z}(x_i,\, s_k - r) - z(x_i,\, s_k)\big], \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}, \\ &d_L\hat{z}(0,t) + (1-d_L)\hat{z}_x(0,t) = 0, \qquad d_R\hat{z}(1,t) + (1-d_R)\hat{z}_x(1,t) = u(t), \\ &\hat{z}(\cdot,t) = 0, \quad t \leqslant t_0. \end{aligned} \tag{3}$$

By shifting system (1) in time by r and introducing a correction term, observer (3) is available. The time-varying injection gain $L\mathrm{e}^{-\alpha_0(t+r-s_k)}$ ensures the observation error decays with the rate α_0 [4]. The following shape functions $b_i \in L^2(0,1)$ are similar to those introduced in [8]:

$$b_i(x) = \begin{cases} 1, & x \in \Omega_i, \\ 0, & x \notin \Omega_i, \end{cases} \tag{4}$$

where $\{\Omega_i\}_{i=1}^{n}$ is a partition of $[0,1]$ such that $x_i \in \Omega_i$.

According to (1), (3), if $u(t) = 0$ for $t < t_0$, the observation/prediction error $\bar{z}(x,t) = \hat{z}(x,\, t-r) - z(x,t)$ will satisfy

$$\begin{aligned}\bar{z}_t(x,t) &= \bar{z}_{xx}(x,t) + a\bar{z}(x,t)\\ &\quad - \bar{z}(x,t)\big[\hat{z}^2(x,\, t-r) + \hat{z}(x,\, t-r)z(x,t) + z^2(x,t)\big], \quad t \in [0,\, t_0+r).\end{aligned}$$

Let

$$\beta(x,t) = \hat{z}^2(x,\, t-r) + \hat{z}(x,\, t-r)z(x,t) + z^2(x,t) \geqslant 0, \tag{5}$$

thus,

$$\begin{aligned}&\bar{z}_t(x,t) = \begin{cases}\bar{z}_{xx}(x,t) + (a-\beta(x,t))\bar{z}(x,t), & t \in [0,\, t_0+r),\\ \bar{z}_{xx}(x,t) + (a-\beta(x,t))\bar{z}(x,t) & \\ +L\mathrm{e}^{-\alpha_0(t-s_k)}\sum_{i=1}^N b_i(x)\bar{z}(x_i, s_k), & t \in [t_k+r,\, t_{k+1}+r),\ k \in \mathbb{Z},\end{cases}\\ &d_L\bar{z}(0,t) + (1-d_L)\bar{z}_x(0,t) = 0, \qquad d_R\bar{z}(1,t) + (1-d_R)\bar{z}_x(1,t) = 0,\\ &\bar{z}(\cdot,0) = -z(\cdot,0).\end{aligned} \tag{6}$$

Studying the well-posedness of (6) for the initial conditions $\bar{z}(\cdot,0) \in X$, where

$$X = \big\{w \in H^1(0,1)\colon\ d_Lw(0) = 0,\ d_Rw(1) = 0\big\}$$

is the state space with the H^1-norm, system (6) can be shown as follows:

$$\dot{\zeta}(t) + A\zeta(t) = \begin{cases}0, & t \in [0,\, t_0+r),\\ f_k(t), & t \in [t_k+r,\, t_{k+1}+r), k \in \mathbb{Z},\end{cases} \tag{7}$$

where $\zeta(t) = \bar{z}(\cdot,t)$, and

$$A : D(A) \to L^2(0,1), \quad Aw = -w'' - (a-\beta)w,$$

is a linear operator on the Hilbert space

$$\begin{aligned}D(A) = \big\{w \in H^2(0,1)\colon\ &d_Lw(0) + (1-d_L)w'(0) = 0,\\ &d_Rw(1) + (1-d_R)w'(1) = 0\big\}\end{aligned}$$

with the inner product $(u,v)_{D(A)} = (Au, Av)_{L^2}$. The functions $f_k \in L^2(t_k+r,\, t_{k+1}+r;\, L^2(0,1))$ are defined by

$$f_k(t) = L\mathrm{e}^{-\alpha_0(t-s_k)}\sum_{i=1}^N b_i(\cdot)\big[\zeta(s_k)\big](x_i).$$

A strong solution of (7) on $[0,T]$ is a function

$$\zeta \in L^2\big(0,T; D(A)\big) \cap C\big([0,T]; X\big) \tag{8}$$

such that $\dot{\zeta} \in L^2(0,T; L^2(0,1))$, and (7) holds almost everywhere on $[0,T]$.

The eigenfunctions of the Sturm–Liouville operator A form a complete orthonormal basis of $L^2(0,1)$ [20, Thm. 7.5.7]. Thus, in a way similar to that in [23, Thm. 7.7], we can show that (7) has a unique strong solution on $[0,\ t_0+r]$ and on each $[t_k+r,\ t_{k+1}+r]$ for the initial conditions $\zeta(0)\in X$, $\zeta(t_k+r)\in X$. We have the strong solution on $[0,\infty)$ for the initial condition $\zeta(0)=\bar z(\cdot,0)\in X$ by taking the endpoint value of the solution on $[t_{k-1}+r,\ t_k+r]$ as the initial condition for the solution on $[t_k+r,\ t_{k+1}+r]$.

Theorem 1. *For positive α_0, α_1, there are a scalar G and positive scalars S_i, R_i, p_i with $i=1,2$ such that*

$$\Phi<0, \qquad \alpha_1 p_2 \leqslant 2p_1, \qquad \begin{pmatrix} R_2 & G\\ G & R_2\end{pmatrix}$$

with $\Phi=\{\phi_{ij}\}$ being a symmetric matrix made up of

$$\begin{gathered}
\phi_{11}=-R_1\mathrm{e}^{-\alpha_1 r}+S_1+2p_1(a+\alpha_0)+\alpha_1-\pi^2(2p_1-\alpha_1p_2)\frac{\max\{d_L,d_R\}}{4-3d_Ld_R},\\
\phi_{12}=1-p_1+p_2(a+\alpha_0), \qquad \phi_{13}=R_1\mathrm{e}^{-\alpha_1 r}, \qquad \phi_{14}=\phi_{16}=p_1L,\\
\phi_{22}=-2p_2+r^2R_1+(h+\eta_M)^2R_2, \qquad \phi_{24}=\phi_{26}=p_2L,\\
\phi_{33}=-(R_1+S_1-S_2)\mathrm{e}^{-\alpha_1 r}-R_2\mathrm{e}^{-\alpha_1\tau_M}, \qquad \phi_{34}=\phi_{45}=(R_2-G)\mathrm{e}^{-\alpha_1\tau_M},\\
\phi_{35}=G\mathrm{e}^{-\alpha_1\tau_M}, \qquad \phi_{44}=2(G-R_2)\mathrm{e}^{-\alpha_1\tau_M}-\alpha_1,\\
\phi_{55}=-(S_2+R_2)\mathrm{e}^{-\alpha_1\tau_M}, \qquad \phi_{66}=-\frac{\alpha_1p_2\pi^2}{4\max_i|\Omega_i|^2},
\end{gathered}$$

where $\tau_M=h+\eta_M+r$. Then system (6) *is exponentially stable with the decay rate α_0, i.e.,*

$$\big\|\bar z(\cdot,t)\big\|_{H^1}\leqslant \bar C\mathrm{e}^{-\alpha_0 t}\big\|\bar z(\cdot,0)\big\|_{H^1}, \quad t\geqslant 0, \tag{9}$$

for some $\bar C>0$. In addition,

$$\big\|\sigma(\cdot,t)\big\|_{L^2}\leqslant C_\sigma\mathrm{e}^{-\alpha_0 t}\big\|z(\cdot,0)\big\|_{H^1}, \quad t\geqslant 0, \tag{10}$$

for some C_σ, where

$$\sigma(x,t)=\sum_{i=1}^N b_i(x)\bar z(x_i,t), \quad x\in[0,1],\ t\geqslant 0. \tag{11}$$

Proof. Let $\zeta(x,t)=\mathrm{e}^{\alpha_0 t}\bar z(x,t)$. For $t\geqslant t_0+r$, due to (6) it implies that

$$\begin{aligned}
&\zeta_t=\zeta_{xx}+\big(a+\alpha_0-\beta(x,t)\big)\zeta+L\sum_{i=1}^N b_i(x)\zeta\big(x_i,\ t-\tau(t)\big),\\
&d_L\zeta(0,t)+(1-d_L)\zeta_x(0,t)=0, \qquad d_R\zeta(1,t)+(1-d_R)\zeta_x(1,t)=0,
\end{aligned} \tag{12}$$

where $\tau(t)=t-s_k$, $t\in[t_k+r,\ t_{k+1}+r)$, $k\in\mathbb{Z}$, $r\leqslant\tau(t)\leqslant\tau_M=r+h+\eta_M$. Construct the Lyapunov–Krasovskii functional

$$V_\zeta=V_1+V_2+V_{S_1}+V_{R_1}+V_{S_2}+V_{R_2},$$

where

$$V_1 = \int_0^1 \zeta^2(x,t)\,\mathrm{d}x, \qquad V_2 = p_2 \int_0^1 \zeta_x^2(x,t)\,\mathrm{d}x,$$

$$V_{S_1} = S_1 \int_0^1 \int_{t-r}^{t} \mathrm{e}^{-\alpha_1(t-s)} \zeta^2(x,s)\,\mathrm{d}s\,\mathrm{d}x,$$

$$V_{R_1} = rR_1 \int_0^1 \int_{-r}^{0} \int_{t+\theta}^{t} \mathrm{e}^{-\alpha_1(t-s)} \zeta_s^2(x,s)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}x,$$

$$V_{S_2} = S_2 \int_0^1 \int_{t-\tau_M}^{t-r} \mathrm{e}^{-\alpha_1(t-s)} \zeta^2(x,s)\,\mathrm{d}s\,\mathrm{d}x,$$

$$V_{R_2} = (h+\eta_M)R_2 \int_0^1 \int_{-\tau_M}^{-r} \int_{t+\theta}^{t} \mathrm{e}^{-\alpha_1(t-s)} \zeta_s^2(x,s)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}x.$$

In order to define V_ζ on $[t_0 + r - \tau_M,\, \infty)$, set $\zeta(\cdot,t) = \zeta(\cdot,0)$ for $t < 0$. Observe that the functional V_ζ is well-defined and continuous for the strong solution (8). For $t \geqslant t_0 + r$,

$$\dot V_1 + \alpha_1 V_1 = 2\int_0^1 \zeta\zeta_t + \alpha_1 \int_0^1 \zeta^2, \qquad \dot V_2 + \alpha_1 V_2 = 2p_2 \int_0^1 \zeta_x \zeta_{xt} + \alpha_1 p_2 \int_0^1 \zeta_x^2,$$

$$\dot V_{S_1} + \alpha_1 V_{S_1} = S_1 \int_0^1 \zeta^2 - S_1 \mathrm{e}^{-\alpha_1 r} \int_0^1 \zeta^2(x,\, t-r)\,\mathrm{d}x,$$

$$\dot V_{S_2} + \alpha_1 V_{S_2} = S_2 \mathrm{e}^{-\alpha_1 r} \int_0^1 \zeta^2(x,\, t-r)\,\mathrm{d}x - S_2 \mathrm{e}^{-\alpha_1 \tau_M} \int_0^1 \zeta^2(x, t-\tau_M)\,\mathrm{d}x.$$

By making use of Jensen's inequality ([9, Prop. B.8]), we have

$$\begin{aligned}
\dot V_{R_1} + \alpha_1 V_{R_1} &= r^2 R_1 \int_0^1 \zeta_t^2(x,t)\,\mathrm{d}x - rR_1 \int_0^1 \int_{t-r}^{t} \mathrm{e}^{-\alpha_1(t-s)} \zeta_s^2(x,s)\,\mathrm{d}s\,\mathrm{d}x \\
&\leqslant r^2 R_1 \int_0^1 \zeta_t^2(x,t)\,\mathrm{d}x - R_1 \mathrm{e}^{-\alpha_1 r} \int_0^1 \bigl(\zeta(x,t) - \zeta(x,\, t-r)\bigr)^2\,\mathrm{d}x.
\end{aligned}$$

Jensen's inequality and reciprocally convex approach [21, Thm. 1] imply that

$$\begin{aligned}
&\dot V_{R_2} + \alpha_1 V_{R_2} \\
&\quad \leqslant (h+\eta_M)^2 R_2 \int_0^1 \zeta_t^2(x,t)\,\mathrm{d}x - \mathrm{e}^{-\alpha_1 \tau_M} \int_0^1 \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} R_2 & G \\ G & R_2 \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} \mathrm{d}x,
\end{aligned}$$

where

$$\varDelta_1 = \zeta(x,\, t-r) - \zeta\bigl(x,\, t-\tau(t)\bigr), \qquad \varDelta_2 = \zeta\bigl(x,\, t-\tau(t)\bigr) - \zeta\bigl(x,\, t-\tau_M\bigr).$$

According to the descriptor method [6], (12) implies that

$$0 = 2\int_0^1 \bigl[p_1\zeta(x,t) + p_2\zeta_t(x,t)\bigr]\Biggl[-\zeta_t(x,t) + \zeta_{xx}(x,t) + \bigl(a+\alpha_0-\beta(x,t)\bigr)\zeta(x,t) + L\sum_{i=1}^N b_i(x)\zeta\bigl(x_i,\, t-\tau(t)\bigr)\Biggr]\,\mathrm{d}x,$$

and its right-hand side will be added to $\dot V_\zeta$. Denoting

$$\kappa(x,t) = \zeta(x_i,t) - \zeta(x,t), \quad x\in\varOmega_i,\ i\in 1,\dots,N, \tag{13}$$

and using (4), we have

$$0 = 2\sum_{i=1}^N \int_{\varOmega_i} [p_1\zeta + p_2\zeta_t]\bigl[-\zeta_t + \zeta_{xx} + \bigl(a+\alpha_0-\beta(x,t)\bigr)\zeta + L\kappa\bigl(x,\, t-\tau(t)\bigr) + L\zeta\bigl(x,\, t-\tau(t)\bigr)\bigr]\,\mathrm{d}x. \tag{14}$$

Integrating by parts and considering the boundary conditions with $d_L, d_R \in \{0,1\}$, we have

$$\begin{aligned} 2p_1\sum_{i=1}^N\int_{\varOmega_i}\zeta\zeta_{xx} &= -2p_1\sum_{i=1}^N\int_{\varOmega_i}\zeta_x^2,\\ 2p_2\sum_{i=1}^N\int_{\varOmega_i}\zeta_t\zeta_{xx} &= -2p_2\int_0^1\zeta_{xt}\zeta_x = -\dot V_2. \end{aligned} \tag{15}$$

Wirtinger's inequality (see [28]) implies from $\alpha_1 p_2 \leqslant 2p_1$ that

$$0 \leqslant (2p_1 - \alpha_1 p_2)\max\{d_L, d_R\}\Biggl[\int_0^1 \zeta_x^2(x,t)\,\mathrm{d}x - \frac{\pi^2}{4-3d_Ld_R}\int_0^1\zeta^2(x,t)\,\mathrm{d}x\Biggr]. \tag{16}$$

Denote $[x_i^L, x_i^R) = \varOmega_i$. Because $\kappa(x_i,t)=0$ and $\kappa_x = -\zeta_x$,

$$\int_{\varOmega_i}\kappa^2 = \int_{x_i^L}^{x_i}\kappa^2 + \int_{x_i^L}^{x_i}\kappa^2 \leqslant \frac{4|\varOmega_i|^2}{\pi^2}\Biggl[\int_{x_i^L}^{x_i}\zeta_x^2 + \int_{x_i}^{x_i^R}\zeta_x^2\Biggr] \leqslant \frac{4\max_i|\varOmega_i|^2}{\pi^2}\int_{\varOmega_i}\zeta_x^2.$$

Thus, for any $\alpha_2 > 0$, we have

$$
\begin{aligned}
&-\alpha_2 \sup_{\theta\in[t-\tau_M,\,t]} V_\zeta(\theta)\\
&\quad\leqslant -\alpha_2 V_\zeta\bigl(t-\tau(t)\bigr)\\
&\quad\leqslant -\alpha_2 \sum_{i=1}^{N}\int\limits_{\Omega_i}\zeta^2\bigl(x,\,t-\tau(t)\bigr)\,\mathrm{d}x-\alpha_2 p_2\sum_{i=1}^{N}\int\limits_{\Omega_i}\zeta_x^2\bigl(x,\,t-\tau(t)\bigr)\,\mathrm{d}x\\
&\quad\leqslant -\alpha_2 \sum_{i=1}^{N}\int\limits_{\Omega_i}\zeta^2\bigl(x,\,t-\tau(t)\bigr)\,\mathrm{d}x-\frac{\alpha_2 p_2\pi^2}{4\max_i|\Omega_i|^2}\sum_{i=1}^{N}\int\limits_{\Omega_i}\kappa^2\bigl(x,\,t-\tau(t)\bigr)\,\mathrm{d}x.
\end{aligned}
$$

Consider the matrix Ψ that coincides with Φ except for

$$
\Psi_{44}=-2(R_2-G)\mathrm{e}^{-\alpha_1\tau_M}-\alpha_2,\qquad \Psi_{66}=-\frac{\alpha_2 p_2\pi^2}{4\max_i|\Omega_i|^2}.
$$

Since $\Phi < 0$ is astrict inequality, $\Psi < 0$ holds for large enough $\alpha_2 < \alpha_1$. Adding the right-hand sides of (14), (16) to $\dot V_\zeta$ and using (5), (15), we get

$$
\begin{aligned}
&\dot V_\zeta+\alpha_1 V_\zeta-\alpha_2\sup_{\theta\in[t-\tau_M,\,t]}V_\zeta(\theta)\\
&\quad\leqslant\biggl[-R_1\mathrm{e}^{-\alpha_1 r}+S_1+2p_1(a+\alpha_0)+\alpha_1-\pi^2(2p_1-\alpha_1 p_2)\frac{\max\{d_L,d_R\}}{4-3d_Ld_R}\biggr]\int\limits_0^1\zeta^2\,\mathrm{d}x\\
&\qquad+2\bigl[1-p_1+p_2(a+\alpha_0)\bigr]\int\limits_0^1\zeta\zeta_t\,\mathrm{d}x+2R_1\mathrm{e}^{-\alpha_1 r}\int\limits_0^1\zeta\zeta(x,\,t-r)\,\mathrm{d}x\\
&\qquad+2p_1L\int\limits_0^1\zeta\zeta\bigl(x,\,t-\tau(t)\bigr)\,\mathrm{d}x+2p_1L\int\limits_0^1\zeta\kappa\bigl(x,\,t-\tau(t)\bigr)\,\mathrm{d}x\\
&\qquad+\bigl[-2p_2+r^2R_1+(h+\eta_M)^2R_2\bigr]\int\limits_0^1\zeta_t^2\,\mathrm{d}x+2p_2L\int\limits_0^1\zeta_t\zeta\bigl(x,\,t-\tau(t)\bigr)\,\mathrm{d}x\\
&\qquad+2p_2L\int\limits_0^1\zeta_t\kappa\bigl(x,\,t-\tau(t)\bigr)\,\mathrm{d}x\\
&\qquad-\bigl[(R_1+S_1-S_2)\mathrm{e}^{-\alpha_1 r}+R_2\mathrm{e}^{-\alpha_1\tau_M}\bigr]\int\limits_0^1\zeta^2(x,\,t-r)\,\mathrm{d}x\\
&\qquad+2(R_2-G)\mathrm{e}^{-\alpha_1\tau_M}\int\limits_0^1\zeta(x,\,t-r)\zeta\bigl(x,\,t-\tau(t)\bigr)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+ 2G\mathrm{e}^{-\alpha_1\tau_M} \int_0^1 \zeta(x,\, t-r)\zeta(x,\, t-\tau_M)\,\mathrm{d}x \\
&+ 2(R_2 - G)\mathrm{e}^{-\alpha_1\tau_M} \int_0^1 \zeta\bigl(x,\, t-\tau(t)\bigr)\zeta(x,\, t-\tau_M)\,\mathrm{d}x \\
&+ \bigl[2(G - R_2)\mathrm{e}^{-\alpha_1\tau_M} - \alpha_2\bigr] \int_0^1 \zeta^2\bigl(x,\, t-\tau(t)\bigr)\,\mathrm{d}x \\
&- (S_2 + R_2)\mathrm{e}^{-\alpha_1\tau_M} \int_0^1 \zeta^2(x,\, t-\tau_M)\,\mathrm{d}x - \frac{\alpha_2 p_2 \pi^2}{4\max_i |\Omega_i|^2} \int_0^1 \kappa^2\bigl(x,\, t-\tau(t)\bigr)\,\mathrm{d}x \\
&- \bigl(1 - \max\{d_L, d_R\}\bigr)(2p_1 - \alpha_1 p_2) \int_0^1 \zeta_x^2\,\mathrm{d}x \\
\leqslant{}& \sum_{i=1}^{N} \int_{\Omega_i} \psi^T(x,t)\Psi\psi(x,t)\,\mathrm{d}x - \bigl(1 - \max\{d_L, d_R\}\bigr)(2p_1 - \alpha_1 p_2)\bigl\|\zeta_x(\cdot,t)\bigr\|_{L^2}^2
\end{aligned}
$$

with $\psi(x,t) = \mathrm{col}\{\zeta, \zeta_t,\, \zeta(x,\, t-r),\, \zeta(x, t-\tau(t)),\, \zeta(x, t-\tau_M),\, \kappa(x, t-\tau(t))\}$. Since $\Psi < 0$ and $2p_1 \geqslant \alpha_1 p_2$,

$$
\dot{V}_\zeta \leqslant -\alpha_1 V_\zeta(t) + \alpha_2 \sup_{\theta\in[t-\tau_M,\, t]} V_\zeta(\theta), \quad t \geqslant t_0 + r.
$$

Using Halanay inequality [7, Lemma 4.2], we obtain

$$
V_\zeta(t) \leqslant \mathrm{e}^{-\bar{\alpha}(t-t_0-r)} \sup_{\theta\in[t_0+r-\tau_M,\, t_0+r]} V_\zeta(\theta), \quad t \geqslant t_0 + r, \tag{17}
$$

where $\bar{\alpha}$ is the unique positive solution of $\bar{\alpha} = \alpha_1 - \alpha_2\mathrm{e}^{\bar{\alpha}\tau_M}$.

For $t \in [0,\, t_0 + r)$, (6) implies (12) with $L = 0$. Then calculations similar to the above imply $\dot{V}_\zeta(t) \leqslant \delta V_\zeta(t)$ for $t \in [0, t_0 + r)$ with large enough δ. Thus, we get

$$
V_\zeta(t) \leqslant \mathrm{e}^{\delta t} V_\zeta(0) \leqslant \mathrm{e}^{\delta(t_0+r)} V_\zeta(0), \quad t \in [0,\, t_0 + r].
$$

Additionally, we set $\zeta(\cdot, t) = \zeta(\cdot, 0)$ for $t < 0$ and then obtain

$$
V_\zeta(t) = V_\zeta(0), \quad t \in [t_0 + r - \tau_M,\, 0].
$$

Thus, it implies that

$$
\sup_{\theta\in[t_0+r-\tau_M,\, t_0+r]} V_\zeta(\theta) \leqslant \mathrm{e}^{\delta(t_0+r)} V_\zeta(0) \leqslant C_V \bigl\|\zeta(\cdot, 0)\bigr\|_{H^1}^2 \tag{18}
$$

for some $C_V > 0$. Considering that $\zeta(x,t) = \mathrm{e}^{\alpha_0 t}\bar{z}(x,t)$, relation (18), together with (17), allows us derive

$$\begin{aligned}\|\bar{z}(\cdot,t)\|_{H^1}^2 &= \mathrm{e}^{-2\alpha_0 t}\|\zeta(\cdot,t)\|_{H^1}^2 \leqslant \frac{\mathrm{e}^{-2\alpha_0 t}}{\min\{1,p_2\}}V_\zeta(t)\\ &\leqslant \bar{C}^2\mathrm{e}^{-2\alpha_0 t}\|\zeta(\cdot,0)\|_{H^1}^2 = \bar{C}^2\mathrm{e}^{-2\alpha_0 t}\|\bar{z}(\cdot,0)\|_{H^1}^2\end{aligned}$$

for $t \geqslant 0$ with some $\bar{C} > 0$. This proves (9).

According to notation (13), $b_i(x)\zeta(x_i,t) = b_i(x)(\zeta(x,t)+\kappa(x,t))$ for any $x \in [0,1]$, and we get

$$\begin{aligned}\mathrm{e}^{2\alpha_0 t}\int_0^1 \sigma^2 &= \int_0^1 \left(\sum_{i=1}^N b_i(x)\zeta(x_i,t)\right)^2 \mathrm{d}x = \int_0^1 \big(\zeta(x,t)+\kappa(x,t)\big)^2\left(\sum_{i=1}^N b_i(x)\right)^2 \mathrm{d}x\\ &\leqslant 2\int_0^1 \kappa^2 + 2\int_0^1 \zeta^2 \leqslant 2\max\left\{1, \frac{4\max_i|\Omega_i|^2}{p_2\pi^2}\right\}V_\zeta(t) \leqslant C_\sigma^2\|\bar{z}(\cdot,0)\|_{H^1}^2\\ &= C_\sigma^2\|z(\cdot,0)\|_{H^1}^2\end{aligned}$$

for $t \geqslant 0$ with some $C_\sigma > 0$. (10) is proofed. □

Using the standard arguments for time-delay systems [7], for any given α_0 and appropriate L, it is easy to get that the LMIs of Theorem 1 are feasible if the delays r, η_M, sampling h, and the maximum subdomain length $\max_i|\Omega_i|$ are small enough.

2.2 Boundary controller synthesis

According to the backstepping transformation in [14, 30], we construct a boundary controller for (1), which is based on the estimation $\hat{z}$,

$$w(x,t) = \hat{z}(x,t) - \int_0^x k(x,y)\hat{z}(y,t)\,\mathrm{d}y, \tag{19}$$

where $k(x,y)$ is the solution of

$$\begin{aligned}&k_{xx}(x,y) - k_{yy}(x,y) = \lambda k(x,y), \quad k(x,x) = -\frac{\lambda}{2}x,\\ &d_L k(x,0) + (1-d_L)k_y(x,0) = 0\end{aligned} \tag{20}$$

with some λ. There exists bounded kernel $k(x,y)$ for any λ ([30, Thm. 2.1]). Let

$$u(t) = \begin{cases}\int_0^1 k(1,y)\hat{z}(y,t)\,\mathrm{d}y & \text{if } d_R = 1,\\ k(1,1)\hat{z}(1,t) + \int_0^1 k_x(1,y)\hat{z}(y,t)\,\mathrm{d}y & \text{if } d_R = 0\end{cases} \tag{21}$$

for $t \geqslant t_0$ and $u(t) = 0$ for $t < t_0$. According to calculations similar to those in [30, Chap. 2.2], it is easy to get

$$\begin{aligned} &w_t(x,t) = w_{xx}(x,t) - \big(\lambda - a + \beta(x,t)\big)w(x,t) + v(x,t),\\ &d_L w(0,t) + (1-d_L)w_x(0,t) = 0,\\ &d_R w(1,t) + (1-d_R)w_x(1,t) = 0,\\ &w(\cdot,t_0) = 0 \end{aligned} \tag{22}$$

for $t \geqslant t_0$, where

$$v(x,t) = L\mathrm{e}^{-\alpha_0(t+r-s_k)}\left[\sigma(x,s_k) - \int_0^x k(x,y)\sigma(y,s_k)\,\mathrm{d}y\right], \quad t \in [t_k, t_{k+1})$$

with $\sigma(x,t)$ defined in (11). The well-posedness of (22) also holds. (19) is invertible, so (3) and (1) have the well-posedness (since $z(x,t) = \hat{z}(x,\,t-r) - \bar{z}(x,t)$).

Theorem 2. *If the assumptions of Theorem* 1 *hold and*

$$\lambda > \alpha_c + a - \frac{\max\{d_L,d_R\}\pi^2}{4 - 3d_L d_R + \pi^2} \tag{23}$$

with $\alpha_c > 0$*, the solutions of system* (22) *will satisfy*

$$\big\|w(\cdot,t)\big\|_{H^1} \leqslant C_w \mathrm{e}^{-\min\{\alpha_0,\alpha_c\}t}\big\|z(\cdot,0)\big\|_{H^1}, \quad t \geqslant t_0 \tag{24}$$

with some $C_w > 0$*.*

Proof. At first, we cconsider $V_w = V_{w_1} + V_{w_2}$ with

$$V_{w_1} = \int_0^1 w^2(x,t)\,\mathrm{d}x, \qquad V_{w_2} = \int_0^1 w_x^2(x,t)\,\mathrm{d}x.$$

By taking the derivative of above, we get

$$\dot{V}_{w_1} = 2\int_0^1 w w_{xx} - 2\big(\lambda - a + \beta(x,t)\big)\int_0^1 w^2 + 2\int_0^1 wv.$$

Integration by parts combined with Young's inequality yields the following result:

$$2\int_0^1 w w_{xx} = -2\int_0^1 w_x^2, \qquad 2\int_0^1 wv \leqslant 2\mu\int_0^1 w^2 + \frac{1}{2\mu}\int_0^1 v^2$$

with an arbitrary $\mu > 0$. By using (5), we get

$$\dot{V}_{w_1} \leqslant -2\int_0^1 w_x^2 - 2(\lambda - a - \mu)\int_0^1 w^2 + \frac{1}{2\mu}\int_0^1 v^2.$$

Employing integration by parts, we obtain

$$\dot V_{w_2} = 2\int_0^1 w_x w_{xt} = -2\int_0^1 w_{xx} w_t$$
$$= -2\int_0^1 w_{xx}^2 + 2\big(\lambda - a + \beta(x,t)\big)\int_0^1 w_{xx} w - 2\int_0^1 w_{xx} v.$$

Due to

$$2\big(\lambda - a + \beta(x,t)\big)\int_0^1 w_{xx} w = -2\big(\lambda - a + \beta(x,t)\big)\int_0^1 w_x^2 \leqslant -2(\lambda - a)\int_0^1 w_x^2,$$
$$-2\int_0^1 w_{xx} v \leqslant 2\int_0^1 w_{xx}^2 + \frac{1}{2}\int_0^1 v^2,$$

we get

$$\dot V_{w_2} \leqslant -2(\lambda - a)\int_0^1 w_x^2 + \frac{1}{2}\int_0^1 v^2.$$

Summing up the above inequalities, for any $\mu > 0$, it follows that

$$\dot V_w + 2\alpha_c V_w \leqslant -2(1 + \lambda - a - \alpha_c)\|w_x\|_{L_2}^2 - 2(\lambda - a - \alpha_c - \mu)\|w\|_{L_2}^2$$
$$+ \left(\frac{1}{2\mu} + \frac{1}{2}\right)\int_0^1 v^2.$$

Condition (23) implies $1 + \lambda - a - \alpha_c > 0$. Applying

$$-\|w_x\|_{L_2}^2 \leqslant -\frac{\max\{d_L, d_R\}\pi^2}{4 - 3d_L d_R}\|w\|_{L_2}^2$$

and (23), for small enough $\mu > 0$, we have

$$\dot V_w \leqslant -2\alpha_c V_w + \left(\frac{1}{2\mu} + \frac{1}{2}\right)\int_0^1 v^2.$$

$k(x,y)$ is bounded, so there is $C_V > 0$ such that

$$\int_0^1 v^2(x,t)\,\mathrm{d}x \leqslant C_V \mathrm{e}^{-2\alpha_0(t - s_k)}\big\|\sigma(\cdot, s_k)\big\|_{L^2}^2 \leqslant C_V C_\sigma^2 \mathrm{e}^{-2\alpha_0 t}\big\|z(\cdot, 0)\big\|_{H^1}^2.$$

Summing up above inequalities, we get that

$$\dot V_w(t) \leqslant -2\alpha_c V_w(t) + \left(\frac{1}{2\mu} + \frac{1}{2}\right)C_V C_\sigma^2 \mathrm{e}^{-2\alpha_0 t}\big\|z(\cdot, 0)\big\|_{H^1}^2.$$

If $\alpha_c \neq \alpha_0$, the comparison principle implies (24) (note that $V_w(t_0) = 0$). If (23) holds for $\alpha_c = \alpha_0$, then for slightly larger $\alpha_c' > \alpha_c$, implying (24) for α_c, it is still true. □

Corollary 1. *Assuming that Theorem* 1 *holds, system* (1) *with the decay rate* $\min\{\alpha_0, \alpha_c\}$ *will be exponentially stabilized by observer-based boundary controller* (3), (20), (21) *with* λ *satisfying* (23), *i.e.,*

$$\left\|z(\cdot,t)\right\|_{H^1} \leqslant C_z \mathrm{e}^{-\min\{\alpha_0,\alpha_c\}t}\left\|z(\cdot,0)\right\|_{H^1}, \quad t \geqslant 0 \tag{25}$$

with some $C_z > 0$.

Proof. Because of transformation (19), there exists an inverse bounded in H^1 norm [30], and there is a constant $\tilde{C}$ such that

$$\left\|\hat{z}(\cdot,t)\right\|_{H^1} \leqslant \tilde{C}\left\|w(\cdot,t)\right\|_{H^1} \leqslant \tilde{C}C_w \mathrm{e}^{-\min\{\alpha_0,\alpha_c\}t}\left\|z(\cdot,0)\right\|_{H^1} \tag{26}$$

for $t \geqslant t_0$. Because of $z(x,t) = \hat{z}(x,\, t-r) - \bar{z}(x,t)$, relation (26), together with (9), implies (25). □

3 Point control

Using the Dirac delta function, we study the point control and consider the system governed by

$$\begin{aligned} &z_t(x,t) = z_{xx}(x,t) + az(x,t) - z^3(x,t) + \sum_{j=1}^{M}\delta(x-\bar{x}_j)u_j(t-r), \\ &d_L z(0,t) + (1-d_L)z_x(0,t) = 0, \\ &d_R z(1,t) + (1-d_R)z_x(1,t) = 0, \end{aligned} \tag{27}$$

where $z : [0,1]\times[0,\infty) \to \mathbb{R}$ is the state, $\delta(x)$ is the Dirac delta function representing the point actuation, $r \geqslant 0$ is a known constant delay, and u_j are the control signals applied at $0 \leqslant \bar{x}_1 < \bar{x}_2 < \dots < \bar{x}_M \leqslant 1$. Note that $\bar{x}_1 = 0$ or $\bar{x}_M = 1$ model boundary actuation. Each of the constants $d_L, d_R \in \{0,1\}$ specifies either the Dirichlet or the Nuemann boundary condition (if $d_L = 1$, then $\bar{x}_1 \neq 0$; if $d_R = 1$ then $\bar{x}_N \neq 1$).

Following Section 3, the values of $z(x_i, s_k)$ are available for observation at time t_k, where x_i, s_k, and t_k satisfy (2). We note that x_i, $\bar{x}_j$ are not related, namely, it is not necessary to collocate the sensors and actuators.

We construct an observer, which is similar to (3), to estimate the future value of the state $\hat{z}(x,t) \approx z(x,\, t+r)$,

$$\begin{aligned} &\hat{z}_t(x,t) = \hat{z}_{xx}(x,t) + a\hat{z}(x,t) - \hat{z}^3(x,t) + \sum_{j=1}^{M}\delta(x-\bar{x}_j)u_j(t) + L\mathrm{e}^{-\alpha_0(t+r-s_k)} \\ &\qquad \times \sum_{i=1}^{N} b_i(x)\big[\hat{z}(x_i,\, s_k - r) - z(x_i, s_k)\big], \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}, \\ &d_L\hat{z}(0,t) + (1-d_L)\hat{z}_x(0,t) = 0, \qquad d_R\hat{z}(1,t) + (1-d_R)\hat{z}_x(1,t) = 0, \\ &\hat{z}(\cdot,t) = 0, \quad t \leqslant t_0 \end{aligned} \tag{28}$$

with the shape functions $b_i \in L^2(0,1)$ in (4). The control signals are as follows:

$$u_j(t) = -K_j \hat{z}(\bar{x}_j, t), \quad j = 1, \dots, M. \tag{29}$$

According to (27)-(29), the observation/prediction error $\bar{z}(x,t) = \hat{z}(x, t-r) - z(x,t)$ satisfies (6). As a result, (9), (10) are established under the assumptions of Theorem 1.

Because the solutions of (28), (29) should be understood in the weak sense, we define the state space as follows:

$$X = \{ w \in H^1(0,1) \colon d_L w(0) = 0,\ d_R w(1) = 0 \}$$

with the H^1-norm. Let X^* be its dual space. A weak solution of (28), (29) on $[t_0, T]$ is a function

$$\hat{z} \in L^2(t_0, T; X) \cap C([t_0, T]; L^2(0,1)) \tag{30}$$

such that $\hat{z}_t \in L^2(t_0, T; X^*)$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \hat{z}(\xi,t)\varphi(\xi)\,\mathrm{d}\xi = {}& -\int_0^1 \hat{z}_\xi(\xi,t)\varphi_\xi(\xi)\,\mathrm{d}\xi + (a - \beta(x,t)) \int_0^1 \hat{z}(\xi,t)\varphi(\xi)\,\mathrm{d}\xi \\ & - \sum_{j=1}^M K_j \hat{z}(\bar{x}_j, t)\varphi(\bar{x}_j) + \int_0^1 g(\xi,t)\varphi(\xi)\,\mathrm{d}\xi \end{aligned} \tag{31}$$

for any $\varphi \in X$ and almost all $t \in [t_0, T]$, where $g \in L^\infty_{\mathrm{loc}}(t_0, T; L^2(0,1))$ is as follows:

$$g(\xi,t) = L\mathrm{e}^{-\alpha_0(t+r-s_k)} \sum_{i=1}^N b_i(\xi)\bar{z}(x_i, s_k), \quad t \in [t_k, t_{k+1}).$$

Since $\bar{z}$ is the strong solution of (6), g is a well-defined inhomogeneity.

By the integration by parts, condition (31) is motivated. According to the standard Galerkin approximation procedure [22], we can obtain that (28), (29) has a unique weak solution on $[t_0, \infty)$ for any initial conditions $\hat{z}(\cdot, t_0) \in L^2(0,1)$, including $\hat{z}(\cdot, t_0) = 0$ required in (28).

Theorem 3. *Under the assumptions of Theorem* 1, *if*

$$2(a + \alpha_c) \max_{j=1,\dots,M} |\Delta_j^L|^2 < \frac{\pi^2}{4}, \qquad 2(a + \alpha_c) \max_{j=1,\dots,M} |\Delta_j^R|^2 < \frac{\pi^2}{4}, \tag{32}$$

where $\{\Delta_j^L, \Delta_j^R\}$ *is a partition of* $[0,1]$, *then the solutions of system* (28) *under the controllers* (29) *with*

$$K_j \in \begin{cases} [2(a+\alpha_c)|\Delta_j^L|,\ \frac{\pi^2}{4|\Delta_j^L|}] & \text{if } |\Delta_j^R| = 0, \\ [2(a+\alpha_c)|\Delta_j^R|,\ \frac{\pi^2}{4|\Delta_j^R|}] & \text{if } |\Delta_j^L| = 0, \\ [2(a+\alpha_c)(|\Delta_j^L| + |\Delta_j^R|),\ [\frac{\pi^2}{4}(\frac{1}{|\Delta_j^L|} + \frac{1}{|\Delta_j^R|}] & \text{otherwise} \end{cases} \tag{33}$$

satisfy

$$\left\|\hat{z}(\cdot,t)\right\|_{L^2} \leqslant \hat{C}\mathrm{e}^{-\min\{\alpha_0,\alpha_c\}t}\left\|z(\cdot,0)\right\|_{H^1}, \quad t \geqslant t_0, \tag{34}$$

with some $\hat{C} > 0$*.*

Proof. We consider the Lyapunov functional $\hat{V} = \int_0^1 \hat{z}^2(x,t)\,\mathrm{d}x$, which is well-defined and continuous for the weak solution (30). Applying (31) with $\varphi(\xi) = \hat{z}(\xi,t)$ and taking the derivative of $\dot{\hat{V}}$ along (28), (29), we get

$$\dot{\hat{V}} = -2\int_0^1 \hat{z}_x^2 + 2\big(a - \beta(x,t)\big)\int_0^1 \hat{z}^2 - 2\sum_{j=1}^M K_j\hat{z}^2(\bar{x}_j,t) + 2\int_0^1 \hat{z}\hat{v},$$

where

$$\hat{v}(x,t) = L\mathrm{e}^{-\alpha_0(t+r-s_k)}\sigma(x,s_k), \quad t \in [t_k, t_{k+1}),$$

σ is defined in (11). By using (5) and (33), for $\epsilon > 0$, we have

$$\begin{aligned} &2(a - \beta(x,t) + \alpha_c + \epsilon)\big(\big|\Delta_j^L\big| + \big|\Delta_j^R\big|\big) \\ &\quad < 2(a + \alpha_c + \epsilon)\big(\big|\Delta_j^L\big| + \big|\Delta_j^R\big|\big) < K_j, \quad j = 1,\dots,M. \end{aligned}$$

Young's inequality implies that

$$2\int_0^1 \hat{z}\hat{v} \leqslant 2\epsilon\int_0^1 \hat{z}^2 + \frac{1}{2\epsilon}\int_0^1 \hat{v}^2.$$

Therefore, it implies that

$$\dot{\hat{V}} + 2\alpha_c\hat{V} \leqslant -2\int_0^1 \hat{z}_x^2 + 2(a + \alpha_c + \epsilon)\int_0^1 \hat{z}^2 - 2\sum_{j=1}^M K_j\hat{z}^2(\bar{x}_j,t) + \frac{1}{2\epsilon}\int_0^1 \hat{v}^2.$$

Let $|\Delta_j^L||\Delta_j^R| \neq 0$. According to Proposition 2 (with $\nu = 2$ for simplicity), for any $\mu_j \in (0,1)$, we have

$$\begin{aligned} -2K_j\hat{z}^2(\bar{x}_j,t) &= -2\mu_j K_j\hat{z}^2(\bar{x}_j,t) - 2(1-\mu)K_j\hat{z}^2(\bar{x}_j,t) \\ &\leqslant -\frac{\mu_j K_j}{|\Delta_j^L|}\int_{\Delta_j^L}\hat{z}^2 + \mu_j K_j\frac{8|\Delta_j^L|}{\pi^2}\int_{\Delta_j^L}\hat{z}_x^2 \\ &\quad - \frac{(1-\mu_j)K_j}{|\Delta_j^R|}\int_{\Delta_j^R}\hat{z}^2 + (1-\mu_j)K_j\frac{8|\Delta_j^R|}{\pi^2}\int_{\Delta_j^R}\hat{z}_x^2, \end{aligned}$$

which contributes to

$$\dot{\hat{V}} + 2\alpha_c \hat{V} \leqslant \sum_{j=1}^{M}\left(-2 + \mu_j K_j \frac{8|\Delta_j^L|}{\pi^2}\right)\int\limits_{\Delta_j^L} \hat{z}_x^2 + \sum_{j=1}^{M}\left(-2 + (1-\mu_j) K_j \frac{8|\Delta_j^R|}{\pi^2}\right)\int\limits_{\Delta_j^R} \hat{z}_x^2$$
$$+ \sum_{j=1}^{M}\left(2(a+\alpha_c+\epsilon) - \frac{\mu_j K_j}{|\Delta_j^L|}\right)\int\limits_{\Delta_j^L} \hat{z}^2$$
$$+ \sum_{j=1}^{M}\left(2(a+\alpha_c+\epsilon) - \frac{(1-\mu_j) K_j}{|\Delta_j^R|}\right)\int\limits_{\Delta_j^R} \hat{z}^2 + \frac{1}{2\epsilon}\int\limits_0^1 \hat{v}^2.$$

According to (32) and (33), Proposition 3 with

$$\underline{a} = 2(a+\alpha_c+\epsilon)\Delta_j^L, \qquad \bar{a} = \frac{\pi^2}{4\Delta_j^L},$$
$$\underline{b} = 2(a+\alpha_c+\epsilon)\Delta_j^R, \qquad \bar{b} = \frac{\pi^2}{4\Delta_j^R}$$

ensures the existence of $\mu_j \in (0,1)$ such that

$$\dot{\hat{V}} + 2\alpha_c \hat{V} \leqslant \frac{1}{2\epsilon}\int\limits_0^1 \hat{v}^2.$$

If $|\Delta_j^L| = 0$ (with $\mu_j = 0$) or $|\Delta_j^R| = 0$ (with $\mu_j = 1$), then the calculations will be similar. Applying the definition of $\hat{v}$,

$$\int\limits_0^1 \hat{v}^2 \leqslant L^2 \mathrm{e}^{-2\alpha_0(t+r-s_k)} \left\|\sigma(\cdot, s_k)\right\|_{L^2}^2 \leqslant \mathrm{e}^{-2\alpha_0 t} C \left\|z(\cdot, 0)\right\|_{H^1}^2$$

with $C = L^2 \mathrm{e}^{-2\alpha_0 r} C_\sigma^2$. Thus, we have

$$\dot{\hat{V}} \leqslant -2\alpha_c \hat{V} + \mathrm{e}^{-2\alpha_0 t}\frac{C}{2\epsilon}\left\|z(\cdot, 0)\right\|_{H^1}^2.$$

If $\alpha_c \neq \alpha_0$, the comparison principle implies (34) (note that $\hat{V}(t_0) = 0$). If the conditions of Theorem 3 are satisfied for $\alpha_c = \alpha_0$, they will be true for slightly larger $\alpha_c' > \alpha_c$, implying (34) for α_c. □

Corollary 2. *Assuming that Theorem* 1 *and* (32) *are satisfied, system* (27) *with the decay rate* $\min\{\alpha_0, \alpha_c\}$ *will be exponentially stabilized by observer-based point controller* (28)*,* (29)*,* (33)*, i.e.,*

$$\left\|z(\cdot, t)\right\|_{L^2} \leqslant C_z \mathrm{e}^{-\min\{\alpha_0, \alpha_c\} t}\left\|z(\cdot, 0)\right\|_{H^1}, \quad t \geqslant 0,$$

with some $C_z > 0$.

4 Numerical example

Consider system (1) and choose $a = 10$, $r = 0.05$, $d_L = 1$, $d_R = 0$. We find that it is unstable when $u(t - r) = 0$. Let the following assumption holds:

- There exist $N = 10$ in-domain sensors transmitting point measurements at $x_i = (2i-1)/(2N)$, $i = 1, \dots, N$, with the sampling period $h = 0.01$ and time-varying network delay $\eta_k \leqslant \eta_M = 0.01$, and the conditions of Theorem 1 are satisfied with $L = -10$, $\alpha_0 = 0.5$, $\alpha_1 = 1$.

Then observer (3) will be able to predict the future state, which converges with the rate α_0.

By taking $\alpha_c = 0.5$, we obtain the boundary controller (21) with

$$k(1,1) = -\frac{\lambda}{2}, \qquad k_x(1,y) = -\lambda y \frac{I_2(\sqrt{\lambda(1-y^2)})}{1-y^2},$$

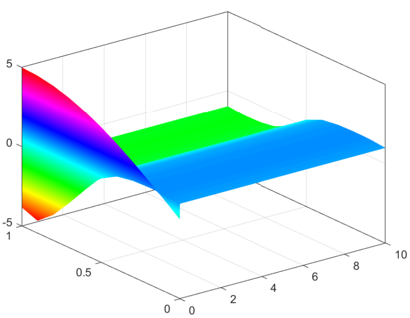
where $\lambda = a + \alpha_0 - \pi^2/(4+\pi^2) + 10^{-5}$, and I_2 is the modified Bessel function. Corollary 1 makes exponential stability of the system with the decay rate $\min \alpha_0, \alpha_c = 0.5$ come true.

The numerical simulations are chosen as $z(x,0) = 5\sin(\pi x/2)$, and $\eta_k \in [0.01]$ are randomly selected such that $t_k \leqslant t_{k+1}$. The results are presented in Fig. 1.

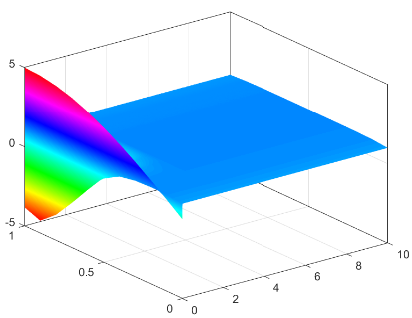
Consider system (27) and observe (28) similar to above. Setting (29) at $\bar{x}_1 = 0.25$, $\bar{x}_2 = 0.75$ and choosing the partition of $[0,1]$ to be uniform, i.e., $\Delta_j^L = \Delta_j^R = 0.25$ for $j = 1, 2$, conditions (32) will be satisfied with $\alpha_c = 0.5$. Then Corollary 2 makes the exponential stability of the closed-loop system (27)–(29) come true with

$$K_j = \frac{\pi^2}{4}\left(\frac{1}{|\Delta_j^L|} + \frac{1}{|\Delta_j^R|}\right) = 2\pi^2, \quad j = 1, 2.$$

The numerical simulations are chosen as $z(x,0) = 5\sin(\pi x/2)$, and $\eta_k \in [0.01]$ are randomly selected such that $t_k \leqslant t_{k+1}$. The results are presented in Fig. 2.

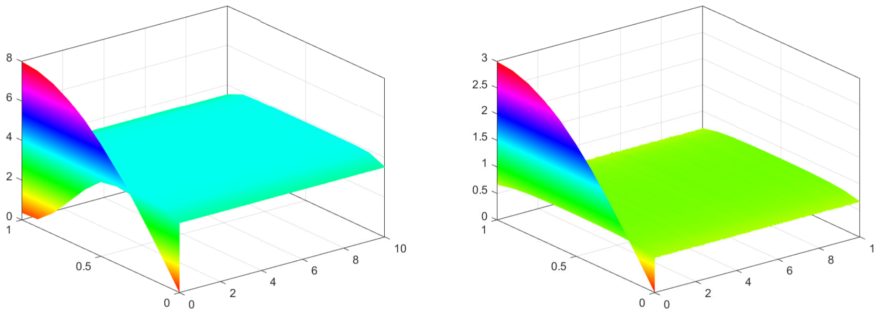


(a) State $z(x,t)$



(b) Estimation/prediction $\hat{z}(x,t)$

Figure 1. Boundary control.



(a) State $z(x,t)$ (b) Estimation/prediction $\hat{z}(x,t)$

Figure 2. Point control.

5 Conclusion

Delayed boundary and in-domain point controllers for Chafee–Infante equation under the discrete-time point measurements are designed by using observers that estimate the future value of the state. LMI-based conditions are provided for the number of point measurements and the maximum delays and time-sampling intervals that preserve the stability of the closed-loop system. For the future research, the result may be extended to time-varying delays.

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