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Relative controllability of damped fractional differential system with distributed delays and impulses*

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Abstract. In this paper, we deal with the relative controllability of linear and nonlinear damped fractional differential system with distributed delays and impulses. The sufficient and necessary conditions for the relative controllability of the linear system under consideration are given by using the controllability Gramian matrix. We also prove a sufficient condition on the nonlinear term to ensure that the above system is relative controllable. Two instances are provided to verify that our theoretical results are accurate.

Keywords: relative controllability, fractional calculus, distributed delays, impulses, Mittag-Leffler function.

1 Introduction

Fractional differential equations (FDEs) have emerged as a fundamental mathematical tool for modeling hereditary phenomena across physical and engineering systems with theoretical foundations rigorously established in the seminal works of Oldham and Spanier [22] on fractional calculus, followed by comprehensive developments in solution existence theory by Kilbas et al. [11], numerical analysis frameworks by Diethelm [8], and systematic methodologies by Zhou et al. [28]. The controllability problem originating from Kalman's classical theory has been substantially extended to FDE contexts through three dominant methodological branches: (i) Gramian matrix techniques pioneered by Balachandran and Kokila [6] for single-delay systems and later generalized to multiple delays [5], (ii) fixed-point theorem approaches initiated by Li et al. [15] for pure-delay systems and subsequently refined for Riemann–Liouville [17] and Mittag-Leffler (M-L) function-based [27] formulations, and (iii) adjoint operator methods developed by

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Wang et al. [26] for stochastic systems and conformable derivatives [16]. In [4], authors study the controllability of fractional damped systems (FDS) in finite-dimensional space using the M-L matrix function in conjunction with an iteration technique. Liu et al. [18], consider the fractional Caputo derivative of any order with the difference that Schaefer's fixed point theorem is used to introduce controllability results. Sathiyaraj et al. [23] used M-L matrix function, controllability Gramian matrix, and fixed point technique to establish complete controllability.

Time delay is an inescapable concept in real life research, which often appears in various systems such as chemical processes, biomedical systems, etc. In some other systems, distributed delays exist in the control, therefore it is necessary to study the dynamic systems involving distributed delays [12, 25], and distributed delays systems have a wide range of applications [1, 24]. In [3, 7], Balachandran consider the relative controllability of FDE with distributed delays in control. Arthi investiges the controllability results of FDDS with distributed delays in [2]. On the other hand, amid the research on natural disasters, biotechnology, and economics, FDE with impulses are becoming more and more important. In many systems, due to rapid changes at one point in time, the duration is so short compared to the total exposure time that it can be considered as impulses. Kumar et al. [13] considered the investigation of fractional damped system with impulses. Recent advances in fractional controllability have further refined Gramian matrix techniques, while novel treatments of impulsive effects and distributed delays provide enhanced frameworks for coupled dynamics like those considered here.

Previous studies on fractional-order controllability have mainly focused on systems with delays [4] or impulses [13]. However, in practical systems such as chemical reactors (due to distributed delays caused by material transfer) and neural regulatory systems (affected by sudden energy injections as impulses), both of these effects exist simultaneously. This coupling presents unique challenges. Recent advances in fractional-order control systems have yielded significant progress in several directions: ψ-Caputo derivatives offer new control paradigms for time-delay systems [19], Gramian matrix methods address complex Langevin system dynamics [14], Hilfer derivatives extend control theory under integral boundary conditions [9], and nonlocal techniques advance impulsive control [20]. However, these approaches remain limited in handling coupled distributed delay-damping-impulse systems, which constitutes the key innovation of our study. Consequently, this paper aims to explore these outcomes.

The structure of this paper is as follows: Section 2 introduces several key concepts and notations. In Section 3, we derive the solution expressions for both linear and semilinear systems. Section 4 focuses on establishing the relative controllability of linear FDS. In Section 5, we develop criteria for the relative controllability of semilinear FDS. Finally, Section 6 provides two illustrative examples.

2 Preliminaries

In this section, serval basic concepts and lemmas are presented that are used throughout this study.

Definition 1. (See [6].) The fractional derivative of order α ($0 \leqslant n \leqslant \alpha \leqslant n+1$) in the Caputo sense for a function $\chi : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$ {}^{C}\mathcal{D}^{\alpha}\chi(\rho) = \frac{1}{\Gamma(n-\alpha+1)} \int_0^{\rho} \frac{\chi^{(n+1)}(s)}{(\rho - s)^{\alpha-n}} \,\mathrm{d}s, $$

where Γ is the Gamma function.

The definition of its Laplace transform is given as

$$ \mathcal{L}\{\chi(\rho);(s)\} = G(s), $$
$$ \mathcal{L}\{{}^{C}\mathcal{D}^{\alpha}\chi\}(s) = s^{\alpha}G(s) - \sum_{i=0}^{k-1} \chi^{i}(0^+)s^{\alpha-i-1}. $$

In particular, if $0 < \alpha_2 \leqslant 1$, then

$$ \mathcal{L}\{{}^{C}\mathcal{D}^{\alpha_2}\chi\}(s) = s^{\alpha_2}G(s) - s^{\alpha_2-1}\chi(0^+), $$

and if $1 < \alpha_1 \leqslant 2$, then

$$ \mathcal{L}\{{}^{C}\mathcal{D}^{\alpha_1}\chi\}(s) = s^{\alpha}G(s) - s^{\alpha_1-1}\chi(0^+) - s^{\alpha_1-2}\chi'(0^+). $$

Definition 2. (See [6].) Defines the M-L function with two parameters as

$$ E_{\alpha_1,\alpha_2}(r) = \sum_{k=0}^{\infty} \frac{r^k}{\Gamma(k\alpha_1+\alpha_2)}, \quad \alpha_1, \alpha_2 > 0,\ r \in \mathbb{C}. $$

In the special case $\alpha_2 = 1$, we have

$$ E_{\alpha_1,1}(r) = \sum_{k=0}^{\infty} \frac{r^k}{\Gamma(k\alpha_1+1)}, \quad \alpha_1 > 0, $$
$$ E_{\alpha_1}\left(\lambda r^{\alpha_1}\right) = \sum_{k=0}^{\infty} \frac{\lambda^k r^{\alpha_1 k}}{\Gamma(k\alpha_1+1)}, \quad \alpha_1 > 0,\ r \in \mathbb{C}. $$

The Laplace transform of the M-L function is

$$ \mathcal{L}\{\rho^{\alpha_2-1}E_{\alpha_1,\alpha_2}(\pm m\rho^{\alpha_1})\}(s) = \frac{s^{\alpha_1-\alpha_2}}{s^{\alpha_1} \mp m}. $$

For an $n \times n$ matrix $\mathbf{A}$, we define

$$ E_{\alpha_1,\alpha_2}(\mathbf{A}) = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{\Gamma(k\alpha_1+\alpha_2)} $$

and

$$ \mathcal{L}\{\rho^{\alpha_2-1}E_{\alpha_1,\alpha_2}(\pm\mathbf{A}\rho^{\alpha_1})\}(s) = \frac{s^{\alpha_1-\alpha_2}}{s^{\alpha_1}\mathbf{I} \mp \mathbf{A}}. $$

3 The solution of systems

First, we consider the solutions of damped FDS with distributed delays

$$
\begin{aligned}
&{}^{C}\mathcal{D}^{\alpha_1}\chi(\rho) - \mathbf{A}\,{}^{C}\mathcal{D}^{\alpha_2}\chi(\rho) = \int_{-\theta}^{0} \mathrm{d}_\varsigma \mathbf{B}(\rho,\varsigma)u(\rho+\varsigma), \quad \rho \geqslant 0,\\
&\chi(0) = \chi_0, \quad \chi'(0) = \chi_1, \qquad u(\rho) = \varphi(\rho), \quad -\theta \leqslant \rho \leqslant 0,
\end{aligned} \tag{1}
$$

where $0 < \alpha_2 \leqslant 1 < \alpha_1 \leqslant 2$, $\chi \in \mathbb{R}^n$ is a state vector. Assume that $\varsigma > 0$ is the time delay and $u(\rho) \in \mathbb{R}^m$ is a control vector, $\mathbf{A}$ is $n \times n$ constant matrices, $\mathbf{B}(\rho,\varsigma)$ is an $n \times m$ dimensional and continuous from in ρ for fixed ς and is of bounded variation in ς on $(-\theta, 0)$, θ is a negative constant. ${}^{C}\mathcal{D}^{\alpha_1}$ and ${}^{C}\mathcal{D}^{\alpha_2}$ denote the Caputo fractional derivatives of order $1 < \alpha_1 \leqslant 2$ and $0 < \alpha_2 \leqslant 1$, respectively. We make the following assumptions in order to establish the main conclusion of this article: the maps $I_i : J_i \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, \dots, m$, are continuous, and there exist positive constants such that $\|I_i(\chi(\rho_i)) - I_i(y(\rho_i))\| \leqslant L_i\|\chi - y\|$. Also, there exist positive constants K_i such that $\|I_i(\chi(\rho))\| \leqslant K_i$ for all $\rho \in J$.

Lemma 1. *Assume that* $0 < \alpha_2 \leqslant 1$, $1 < \alpha_1 \leqslant 2$. *Then the solution of system* (1) *has the form*

$$
\begin{aligned}
\chi(\rho) &= \chi_0 + \rho E_{\alpha_1-\alpha_2,\,2}\bigl(\mathbf{A}\rho^{\alpha_1-\alpha_2}\bigr)\chi_1\\
&\quad + \int_0^{\rho} (\rho-s)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}\bigl(\mathbf{A}(\rho-s)^{\alpha_1-\alpha_2}\bigr)\left[\int_{-\theta}^{0} \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)\right]\mathrm{d}s.
\end{aligned}
$$

Proof. We consider the equation

$$
{}^{C}\mathcal{D}^{\alpha_1}\chi(\rho) - \mathbf{A}\,{}^{C}\mathcal{D}^{\alpha_2}\chi(\rho) = \int_{-\theta}^{0} \mathrm{d}_\sigma \mathbf{B}(\rho,\varsigma)u(\rho+\varsigma).
$$

By applying the Laplace transformation to both sides of the above equation, we obtain

$$
\begin{aligned}
&s^{\alpha_1}\mathcal{L}\bigl[\chi(s)\bigr] - s^{\alpha_1-1}\chi_0 - s^{\alpha_1-2}\chi_1 - s^{\alpha_2}\mathbf{A}\mathcal{L}\bigl[\chi(s)\bigr] + \mathbf{A}s^{\alpha_2-1}\chi_0\\
&\quad = \mathcal{L}\left[\int_{-\theta}^{0} \mathrm{d}_\tau \mathbf{B}(s,\varsigma)u(s+\varsigma)\right],
\end{aligned}
$$

then

$$
\begin{aligned}
\mathcal{L}\bigl[\chi(s)\bigr] &= \frac{s^{\alpha_1-\alpha_2-1}}{s^{\alpha_1-\alpha_2}\mathbf{I} - \mathbf{A}}\chi_0 + \frac{s^{\alpha_1-\alpha_2-2}}{s^{\alpha_1-\alpha_2}\mathbf{I} - \mathbf{A}}\chi_1 - \frac{s^{-1}}{s^{\alpha_1-\alpha_2}\mathbf{I} - \mathbf{A}}\chi_0\\
&\quad + \frac{s^{-\alpha_2}}{s^{\alpha_1-\alpha_2}\mathbf{I} - \mathbf{A}}\mathcal{L}\left[\int_{-\theta}^{0} \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)\right].
\end{aligned}
$$

Taking inverse Laplace transform on both sides and employing M-L function, we have

$$\chi(\rho) = \chi_0 + \rho E_{\alpha_1-\alpha_2,\,2}\big(\mathbf{A}\rho^{\alpha_1-\alpha_2}\big)\chi_1 + \int_0^\rho (\rho-s)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}(\rho-s)^{\alpha_1-\alpha_2}\big)\left[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)\right]\mathrm{d}s. \quad \square$$

Then we consider the solutions of damped FDS with distributed delays and impulses as given by

$$\begin{aligned} &{}^{C}\mathcal{D}^{\alpha_1}\chi(\rho) - \mathbf{A}\,{}^{C}\mathcal{D}^{\alpha_2}\chi(\rho) \\ &\quad= \int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(\rho,\varsigma)u(\rho+\varsigma), \quad \rho \in J := [0,T] \setminus \{\rho_1,\rho_2,\dots,\rho_k\}, \\ &\Delta\chi(\rho_i) = \chi\big(\rho_i^+\big) - \chi\big(\rho_i^-\big) = I_i\big(\chi(\rho_i)\big), \quad i = 1,2,\dots,k, \\ &\chi(0) = \chi_0, \quad \chi'(0) = \chi_1, \qquad u(\rho) = \varphi(\rho), \quad -\theta \leqslant \rho \leqslant 0, \end{aligned} \tag{2}$$

where $f : J \times \mathbb{R}^n \to \mathbb{R}^m$,

$$\chi\big(\rho_i^+\big) = \lim_{\varepsilon\to 0^+}\chi(\rho_i+\varepsilon), \qquad \chi\big(\rho_i^-\big) = \lim_{\varepsilon\to 0^-}\chi(\rho_i+\varepsilon)$$

represent the right and left limits of $\chi(\rho)$ at the discontinuity points $\rho = \rho_i$,

$$\rho_{i-1} < i\varsigma < \rho_i, \quad i = 1,2,\dots,k.$$

Lemma 2. *The solution of system* (2) *is given by*

$$\chi(\rho) = \begin{cases} \chi_0 + \rho E_{\alpha_1-\alpha_2,\,2}(\mathbf{A}\rho^{\alpha_1-\alpha_2})\chi_1 \\ \quad + \int_0^\rho (\rho-s)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}(\mathbf{A}(\rho-s)^{\alpha_1-\alpha_2})[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\tau)u(s+\tau)]\,\mathrm{d}s, \\ \quad \rho \in (0,\rho_1], \\ \chi_0 + \rho E_{\alpha_1-\alpha_2,\,2}(\mathbf{A}\rho^{\alpha_1-\alpha_2})\chi_1 + I_1(\chi(\rho_1^-)) \\ \quad + \int_0^\rho (\rho-s)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}(\mathbf{A}(\rho-s)^{\alpha_1-\alpha_2})[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)]\,\mathrm{d}s, \\ \quad \rho \in (\rho_1,\rho_2], \\ \dots, \\ \chi_0 + \rho E_{\alpha_1-\alpha_2,\,2}(\mathbf{A}\rho^{\alpha_1-\alpha_2})\chi_1 + \sum_{j=1}^k I_j(\chi(\rho_j^-)) \\ \quad + \int_0^\rho (\rho-s)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}(\mathbf{A}(\rho-s)^{\alpha_1-\alpha_2})[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)]\,\mathrm{d}s, \\ \quad \rho \in (\rho_i,\rho_{i+1}]. \end{cases} \tag{3}$$

Proof. Let $0 < \alpha_2 \leqslant 1 < \alpha_1 \leqslant 2$, and let $u \in C_p([0,T],\mathbb{R}^m)$ be the control function. The state response representation of system (2) can be defined as follows: for $\rho \in (0,\rho_1]$,

$$\chi(\rho) = \chi_0 + \rho E_{\alpha_1-\alpha_2,2}(\mathbf{A}\rho^{\alpha_1-\alpha_2})\chi_1 + \int_0^\rho (\rho-s)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\bigl(\mathbf{A}(\rho-s)^{\alpha_1-\alpha_2}\bigr)\Biggl[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)\Biggr]\mathrm{d}s.$$

Take $\rho = \rho_1$,

$$\chi(\rho_1) = \chi_0 + \rho_1 E_{\alpha_1-\alpha_2,2}(\mathbf{A}\rho_1^{\alpha_1-\alpha_2})\chi_1 + \int_0^{\rho_1} (\rho_{1-t})^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\bigl(\mathbf{A}(\rho_1-s)^{\alpha_1-\alpha_2}\bigr)\Biggl[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)\Biggr]\mathrm{d}s.$$

For $t \in (\rho_1,\rho_2]$,

$$\begin{aligned}\chi(\rho) &= \chi(\rho_1^+)\chi_0 - \rho_1 E_{\alpha_1-\alpha_2,2}(\mathbf{A}\rho_1^{\alpha_1-\alpha_2})\chi_1 \\ &\quad - \int_0^{\rho_1} (\rho_1-s)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\bigl(\mathbf{A}(\rho_1-s)^{\alpha_1-\alpha_2}\bigr)\Biggl[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)\Biggr]\mathrm{d}s \\ &\quad + \rho E_{\alpha_1-\alpha_2,2}(\mathbf{A}\rho^{\alpha_1-\alpha_2})\chi_1 \\ &\quad + \int_0^{\rho} (\rho-s)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\bigl(\mathbf{A}(\rho-s)^{\alpha_1-\alpha_2}\bigr)\Biggl[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\tau)u(s+\varsigma)\Biggr]\mathrm{d}s \\ &= \chi_0 + I_1(\chi(\rho_1^-)) + \rho E_{\alpha_1-\alpha_2,2}\bigl(\mathbf{A}\rho^{\alpha_1-\alpha_2}\bigr)\chi_1 \\ &\quad + \int_0^{\rho} (\rho-t)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\bigl(\mathbf{A}(\rho-s)^{\alpha_1-\alpha_2}\bigr)\Biggl[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)\Biggr]\mathrm{d}s.\end{aligned}$$

Then, with $\rho = \rho_2$,

$$\chi(\rho_2) = \chi_0 + \rho_2 E_{\alpha_1-\alpha_2,2}\bigl(\mathbf{A}\rho_2^{\alpha_1-\alpha_2}\bigr)\chi_1 + \int_0^{\rho_2} (\rho_2-s)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\bigl(\mathbf{A}(\rho_2-s)^{\alpha_1-\alpha_2}\bigr)\Biggl[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)\Biggr]\mathrm{d}s.$$

For $t \in (\rho_1,\rho_2]$,

$$\begin{aligned}\chi(\rho) &= \chi(\rho_2^+)\chi_0 - \rho_2 E_{\alpha_1-\alpha_2,2}\bigl(\mathbf{A}\rho_2^{\alpha_1-\alpha_2}\bigr)\chi_1 \\ &\quad - \int_0^{\rho_2} (\rho_2-s)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\bigl(\mathbf{A}(\rho_2-s)^{\alpha_1-\alpha_2}\bigr)\Biggl[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)\Biggr]\mathrm{d}s\end{aligned}$$

$$+ \rho E_{\alpha_1-\alpha_2,\,2}\big(\mathbf{A}\rho^{\alpha_1-\alpha_2}\big)\chi_1 + \int_0^\rho (\rho - s)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}(\rho - s)^{\alpha_1-\alpha_2}\big)\Bigg[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)\Bigg]\mathrm{d}s$$

$$= \chi_0 + I_1\big(\chi(\rho_1^-)\big) + I_2\big(\chi(\rho_2^-)\big) + \rho E_{\alpha_1-\alpha_2,\,2}(\mathbf{A}\rho^{\alpha_1-\alpha_2})\chi_1 + \int_0^\rho (\rho - t)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}(\rho - s)^{\alpha_1-\alpha_2}\big)\Bigg[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)\Bigg]\mathrm{d}s.$$

Similarly, for $\rho \in (\rho_i,\ \rho_i + 1]$, $i = 1, 2, \dots, k$, $\chi(\rho)$ can be written as

$$\chi(\rho) = \chi_0 + \rho E_{\alpha_1-\alpha_2,\,2}\big(\mathbf{A}\rho^{\alpha_1-\alpha_2}\big)\chi_1 + \sum_{j=1}^k I_j\big(\chi(\rho_j^-)\big) + \int_0^\rho (\rho - s)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}(\rho - s)^{\alpha_1-\alpha_2}\big)\Bigg[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)\Bigg]\mathrm{d}s. \quad \square$$

4 Linear system

This section focuses on the relative controllability of system (2).

Definition 3. System (2) is said to be relatively controllable on J if for every set of vectors $\chi_0, \chi_1, \chi_2 \in \mathbb{R}^n$, there exists a control function $u(\rho) \in L^2(J, \mathbb{R}^m)$ such that the solution of system (2) satisfies $\chi(T) = \chi_2$.

Now, using the asymmetric Fubini theorem to change the order of the integral to the last term of (3), we get $\rho \in (0, \rho_1]$,

$$\chi(\rho) = \chi_0 + \rho E_{\alpha_1-\alpha_2,\,2}\big(\mathbf{A}\rho^{\alpha_1-\alpha_2}\big)\chi_1 + \int_0^\rho (\rho - s)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}(\rho - s)^{\alpha_1-\alpha_2}\big)\Bigg[\int_{-\theta}^0 \mathrm{d}_\varsigma \mathbf{B}(s,\varsigma)u(s+\varsigma)\Bigg]\mathrm{d}s$$

$$= \chi_0 + \rho E_{\alpha_1-\alpha_2,\,2}\big(\mathbf{A}\rho^{\alpha_1-\alpha_2}\big)\chi_1 + \int_{-\theta}^0 \mathrm{d}\mathbf{B}_\varsigma \Bigg[\int_\tau^0 \big(\rho - (s-\varsigma)\big)^{\alpha_1-1} \times E_{\alpha_1-\alpha_2,\,\alpha_1}\Big(\mathbf{A}\big(\rho - (s-\varsigma)\big)^{\alpha_1-\alpha_2}\Big)\mathbf{B}(s-\varsigma,\,\varsigma)\varphi(t)\,\mathrm{d}t\Bigg] + \int_0^\rho \Bigg[\int_{-\theta}^0 \big(\rho - (s-\varsigma)\big)^{\alpha_1-1} \times E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}(\rho - (s-\varsigma))^{\alpha_1-\alpha_2}\big)\,\mathrm{d}_\varsigma \mathbf{B}_\rho(s-\varsigma,\,\varsigma)\Bigg] u(t)\,\mathrm{d}t.$$

The state vector of system (2) can be expressed as follows:

$$\chi(\rho) = \begin{cases} \chi_0 + \rho E_{\alpha_1-\alpha_2,\,2}(\mathbf{A}\rho^{\alpha_1-\alpha_2})\chi_1 \\ \quad + \int_{-\theta}^{0} \mathrm{d}\mathbf{B}_\varsigma [\int_\varsigma^0 (\rho-(s-\varsigma))^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}(\mathbf{A}(\rho-(s-\tau))^{\alpha_1-\alpha_2}) \\ \qquad\qquad \times \mathbf{B}(s-\varsigma,\,\varsigma)\varphi(s)\,\mathrm{d}s] \\ \quad + \int_0^\rho [\int_{-\varsigma}^0 (\rho-(s-\varsigma))^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}(\mathbf{A}(\rho-(s-\varsigma))^{\alpha_1-\alpha_2}) \\ \qquad\qquad \times \mathrm{d}_\varsigma \mathbf{B}_\rho(s-\varsigma,\,\varsigma)]u(s)\,\mathrm{d}s, \quad \rho \in (0,\rho_1], \\ \chi_0 + \rho E_{\alpha_1-\alpha_2,\,2}(\mathbf{A}\rho^{\alpha_1-\alpha_2})\chi_1 \\ \quad + \int_{-\theta}^{0} \mathrm{d}\mathbf{B}_\varsigma [\int_\varsigma^0 (\rho-(s-\varsigma))^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}(\mathbf{A}(\rho-(s-\varsigma))^{\alpha_1-\alpha_2}) \\ \qquad\qquad \times \mathbf{B}(s-\varsigma,\varsigma)\varphi(s)\,\mathrm{d}s] + I_1(\chi(\rho_1^-)) \\ \quad + \int_0^\rho [\int_{-\theta}^0 (\rho-(s-\varsigma))^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}(\mathbf{A}(\rho-(s-\tau))^{\alpha_1-\alpha_2}) \\ \qquad\qquad \times \mathrm{d}_\varsigma \mathbf{B}_\rho(s-\varsigma,\,\varsigma)]u(s)\,\mathrm{d}s, \quad \rho \in (\rho_1,\rho_2], \\ \dots, \\ \chi_0 + \rho E_{\alpha_1-\alpha_2,\,2}(\mathbf{A}\rho^{\alpha_1-\alpha_2})\chi_1 \\ \quad + \int_{-\theta}^{0} \mathrm{d}\mathbf{B}_\varsigma [\int_\varsigma^0 (\rho-(s-\varsigma))^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}(\mathbf{A}(\rho-(s-\varsigma))^{\alpha_1-\alpha_2}) \\ \qquad\qquad \times \mathbf{B}(s-\varsigma,\varsigma)\varphi(s)\,\mathrm{d}s] + \sum_{j=1}^k I_j(\chi(\rho_j^-)) \\ \quad + \int_0^\rho [\int_{-\theta}^0 (\rho-(s-\varsigma))^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}(\mathbf{A}(\rho-(s-\varsigma))^{\alpha_1-\alpha_2}) \\ \qquad\qquad \times \mathrm{d}_\varsigma \mathbf{B}_\rho(s-\varsigma,\,\varsigma)]u(s)\,\mathrm{d}s, \quad \rho \in (\rho_i,\rho_{i+1}], \end{cases}$$

where

$$\mathbf{B}_\rho(s,\varsigma) = \begin{cases} \mathbf{B}(s,\varsigma), & s \leqslant \rho, \\ 0, & s > \rho. \end{cases}$$

We define the controllability Gramian matrix

$$\mathcal{W} = \int_0^T \left[\int_{-\theta}^0 \big(T-(s-\varsigma)\big)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}(T-(s-\varsigma)\big)^{\alpha_1-\alpha_2}\big)\mathbf{B}(s-\varsigma,\varsigma)\,\mathrm{d}\mathbf{B}_\sigma \right] \\ \times \left[\int_{-\theta}^0 \big(T-(s-\varsigma)\big)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}\big(T-(s-\varsigma)\big)^{\alpha_1-\alpha_2}\big)\mathbf{B}(s-\varsigma,\varsigma)\,\mathrm{d}\mathbf{B}_\varsigma \right]^* \mathrm{d}s.$$

Theorem 1. *The linear system is controllable on $(0,T]$ if and only if the controllability Gramian $\mathcal{W} = \int_0^T \mathcal{G}(T,s)\mathcal{G}^*(T,s)\,\mathrm{d}s$ is positive definite for some $T > 0$.*

Proof. We prove the sufficiency. Assume that $\mathcal{W}$ is positive definite, it is nonsingular, and its inverse is well defined. We can define the control function as

$$u(\rho) = \begin{cases} \mathcal{G}^*(T,\rho)\mathcal{W}^{-1}\xi, & \rho \in (0,\rho_1], \\ \mathcal{G}^*(T,\rho)\mathcal{W}^{-1}\{\xi - I_1(\chi(\rho_1^-))\}, & \rho \in (\rho_1,\rho_2], \\ \dots, \\ \mathcal{G}^*(T,\rho)\mathcal{W}^{-1}\{\xi - \sum_{j=1}^k I_j(\chi(\rho_j^-))\}, & \rho \in (\rho_i,\rho_{i+1}], \end{cases} \tag{4}$$

where

$$\mathcal{G}(T,\rho) = \int_{-\theta}^{0} \big(T-(s-\varsigma)\big)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}\big(T-(s-\varsigma)\big)^{\alpha_1-\alpha_2}\big) \times \mathbf{B}(s-\varsigma,\,\varsigma)\,\mathrm{d}\mathbf{B}_\varsigma,$$

$$\xi = \chi_2 - \chi_0 - TE_{\alpha_1-\alpha_2,\,2}\big(\mathbf{A}T^{\alpha_1-\alpha_2}\big)\chi_1 - \int_{-\theta}^{0} \mathrm{d}\mathbf{B}_\varsigma \int_{\varsigma}^{0} \big(T-(s-\varsigma)\big)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}\big(T-(s-\varsigma)\big)^{\alpha_1-\alpha_2}\big) \times \mathbf{B}(s-\varsigma,\,\varsigma)\varphi(s)\,\mathrm{d}s.$$

If $\rho = T \in (0,\rho_1]$, we get

$$\begin{aligned}\chi(T) &= \chi_0 + TE_{\alpha_1-\alpha_2,\,2}\big(\mathbf{A}T^{\alpha_1-\alpha_2}\big)\chi_1 + \int_0^T \mathcal{G}(T,s)\mathcal{G}^*(T,s)W^{-1}\xi\,\mathrm{d}s \\ &\quad + \int_{-\theta}^{0} \mathrm{d}\mathbf{B}_\varsigma \int_{\varsigma}^{0} \big(T-(s-\varsigma)\big)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}\big(T-(s-\varsigma)\big)^{\alpha_1-\alpha_2}\big) \times \mathbf{B}(s-\varsigma,\,\varsigma)\varphi(s)\,\mathrm{d}s \\ &= \chi_2.\end{aligned}$$

So, by Definition 3 of relative controllability, system is relative controllable on $(0,T]$, $T \in (0,\rho_1]$. We assume that $\rho \in (\rho_i,\rho_{i+1}]$, $i = 1,2,\dots,\rho_k$. Then, with $\rho = T$, and using Eq. (4), we obtain

$$\begin{aligned}\chi(\rho) &= \chi_0 + \rho E_{\alpha_1-\alpha_2,\,2}\big(\mathbf{A}\rho^{\alpha_1-\alpha_2}\big)\chi_1 \\ &\quad + \sum_{j=1}^{k} I_j\big(\chi(\rho_j^-)\big) + \int_0^\rho \mathcal{G}(T,s)\mathcal{G}^*(T,s)\mathcal{W}^{-1}\Bigg(\xi - \sum_{j=1}^{k} I_j\big(\chi(\rho_j^-)\big)\Bigg)\,\mathrm{d}s \\ &\quad + \int_{-\theta}^{0} \mathrm{d}\mathbf{B}_\varsigma \int_{\varsigma}^{0} \big(\rho-(s-\varsigma)\big)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}\big(\rho-(s-\varsigma)\big)^{\alpha_1-\alpha_2}\big) \times \mathbf{B}(s-\varsigma,\,\varsigma)\varphi(s)\,\mathrm{d}s \\ &= \chi_2.\end{aligned}$$

Thus, system is relative controllable on $(0,T]$, $T \in (\rho_i,\rho_{i+1}]$, $i = 1,2,\dots,\rho_k$.

Furthermore, we prove the necessity. Suppose the controllability Gramian matrix is singular. For $T \in (\rho_i,\rho_{i+1}]$, $i = 1,2,\dots,\rho_k$, there exists a vector $z \neq 0$ such that

$$z^*\mathcal{W}z = z^* \int_0^T \mathcal{G}(T,s)\mathcal{G}^*(T,s)z\,\mathrm{d}t = 0.$$

Thus

$$z^*\mathcal{G}(T,s) = 0.$$

Then we get for $\rho \in T$,

$$z^*\big(T-(s-\varsigma)\big)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}\big(T-(s-\varsigma)\big)^{\alpha_1-\alpha_2}\big)\mathbf{B}(s-\varsigma,\,\varsigma)=0.$$

Consider $\chi_0=\chi_1=0$ and $\chi_2=z$. The system is relative controllable, there exists a control function $u(\rho)$ on T that steers the response from 0 to $\chi_2=z$ at $\rho=T$. Thus,

$$\begin{aligned}\chi_2&=z\\&=\int_{-\theta}^{0}\mathrm{d}\mathbf{B}_\varsigma\Bigg[\int_{\varsigma}^{0}\big(\rho-(s-\varsigma)\big)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}\big(\rho-(s-\varsigma)\big)^{\alpha_1-\alpha_2}\big)\\&\qquad\times\mathbf{B}(s-\varsigma,\,\varsigma)\varphi(s)\,\mathrm{d}s\Bigg]\\&\quad+\int_0^{\rho}\Bigg[\int_{-\theta}^{0}\big(\rho-(s-\varsigma)\big)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}\big(\rho-(s-\varsigma)\big)^{\alpha_1-\alpha_2}\big)\\&\qquad\times\mathbf{B}(s-\varsigma,\,\varsigma)\,\mathrm{d}\mathbf{B}_\varsigma\Bigg]u(s)\,\mathrm{d}s.\end{aligned}$$

Multiplying both side by z^*, we obtain $z^*z=0$, which implies $z=0$, a contradiction. Therefore, without loss of generality, we conclude that $\mathcal{W}$ is nonsingular. $\square$

5 Semilinear system

Consider the semilinear fractional damped system of the form

$$\begin{aligned}&{}^C\mathcal{D}^{\alpha_1}\chi(\rho)-\mathbf{A}\,{}^C\mathcal{D}^{\alpha_2}\chi(\rho)\\&\quad=\int_{-\theta}^{0}\mathrm{d}_\varsigma\mathbf{B}(\rho,\varsigma)u(\rho+\varsigma)+g\big(\rho,\chi(\rho),u(\rho)\big),\quad \rho\in[0,T]\setminus\{\rho_1,\rho_2,\dots,\rho_k\},\\&\Delta\chi(\rho_i)=\chi\big(\rho_i^+\big)-\chi\big(\rho_i^-\big)=I_i\big(\chi(\rho_i)\big),\quad i=1,2,\dots,\rho_k,\\&\chi(0)=\chi_0,\quad \chi'(0)=\chi_1,\qquad u(\rho)=\varphi(\rho),\quad \theta\leqslant\rho\leqslant 0.\end{aligned}\tag{5}$$

The solution of the corresponding linear system is obtained by using the formula, which is given by

$$\begin{aligned}\chi(\rho)&=\chi_0+\rho E_{\alpha_1-\alpha_2,\,2}\big(\mathbf{A}\rho^{\alpha_1-\alpha_2}\big)\chi_1\\&\quad+\int_0^{\rho}(\rho-s)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}(\rho-s)^{\alpha_1-\alpha_2}\big)\Bigg[\int_{-\theta}^{0}\mathrm{d}_\varsigma\mathbf{B}(s,\varsigma)u(s+\varsigma)\Bigg]\,\mathrm{d}s\\&\quad+\int_0^{\rho}(\rho-s)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\,\alpha_1}\big(\mathbf{A}(\rho-s)^{\alpha_1-\alpha_2}\big)g\big(s,y(s),v(s)\big)\,\mathrm{d}s.\end{aligned}$$

Theorem 2. *Suppose there exists a continuous function f that satisfies the condition*

$$\lim_{|\chi,u|\to\infty}\frac{|g(\rho,\chi,u)|}{|\chi,u|}=0$$

uniformly in $\rho\in[0,T]$, and let system (2) *be relatively controllable. Then system* (5) *is relatively controllable on* $[0,T]$.

Proof. Assume that system (2) is relatively controllable and $\chi_0,\chi_1,\chi_2\in\mathbb{R}^n$. Let $\mathfrak{P}$ denote the Banach space of continuous functions $(\chi,u):[0,T]\times[0,T]\to\mathbb{R}^n\times\mathbb{R}^n$ with the norm defined by

$$\big\|(\chi,u)\big\|=\|\chi\|+\|u\|,$$

where

$$\|\chi\|=\sup\big|\chi(\rho)\big|,\quad \|u\|=\sup\big|u(\rho)\big|\quad \text{for } \rho\in[0,T].$$

Define a continuous operator $\Psi:\mathfrak{P}\to\mathfrak{P}$ by

$$\Psi(y,v)=(\chi,u),$$

where

$$\begin{aligned}
u(\rho)=\mathcal{G}(T,\rho)W^{-1}\bigg\{&\chi_2-\chi_0-\varsigma_{j=1}^{k}I_j\big(\chi(\rho_j^-)\big)-TE_{\alpha_1-\alpha_2,2}\big(\mathbf{A}T^{\alpha_1-\alpha_2}\big)\chi_1\\
&-\int_{-\theta}^{0}\mathrm{d}\mathbf{B}_\varsigma\bigg[\int_{\varsigma}^{0}\big(T-(s-\varsigma)\big)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\big(\mathbf{A}\big(T-(t-\tau)\big)^{\alpha_1-\alpha_2}\big)\\
&\qquad\times\mathbf{B}(s-\varsigma,\varsigma)\varphi(s)\bigg]\,\mathrm{d}s\\
&-\int_{0}^{\rho}(T-s)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\big(\mathbf{A}(T-s)^{\alpha_1-\alpha_2}\big)g\big(s,y(s),v(s)\big)\,\mathrm{d}t\bigg\}
\end{aligned}$$

and

$$\begin{aligned}
\chi(\rho)=\chi_0&+\rho E_{\alpha_1-\alpha_2,2}\big(\mathbf{A}\rho^{\alpha_1-\alpha_2}\big)\chi_1\\
&+\int_{0}^{\rho}(\rho-s)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\big(\mathbf{A}(\rho-s)^{\alpha_1-\alpha_2}\big)\bigg[\int_{-\theta}^{0}\mathrm{d}_\varsigma\mathbf{B}(s,\varsigma)u(s+\varsigma)\bigg]\mathrm{d}s\\
&+\int_{0}^{\rho}(\rho-s)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\big(\mathbf{A}(\rho-s)^{\alpha_1-\alpha_2}\big)g\big(s,y(s),v(s)\big)\,\mathrm{d}s.
\end{aligned}$$

For simplification, take

$$a_1 = \left\| \sum_{j=1}^{k} I_j\bigl(\chi(\rho_j^-)\bigr) + TE_{\alpha_1-\alpha_2,\,2}\bigl(\mathbf{A}T^{\alpha_1-\alpha_2}\bigr)\chi_1 \right\|,$$

$$a_2 = \left\| \int_{-\theta}^{0} \mathrm{d}\mathbf{B}_\varsigma \left[\int_{\tau}^{0} \bigl(T-(s-\tau)\bigr)^{\alpha_1-1} E_{\alpha_1-\alpha_2,\,\alpha_1}\bigl(\mathbf{A}\bigl(T-(s-\varsigma)\bigr)^{\alpha_1-\alpha_2}\bigr) \times \mathbf{B}(s-\varsigma,\,\varsigma)\varphi(s) \right] \mathrm{d}s \right\|,$$

$$a_3 = \sup_{s\in[0,T]} \bigl\| E_{\alpha_1-\alpha_2,\,\alpha_1}\bigl(\mathbf{A}(T-s)^{\alpha_1-\alpha_2}\bigr)\bigr\|, \qquad a_4 = \sup_{s\in[0,T]} \bigl\|\mathcal{G}(T,\rho)\bigr\|,$$

$$a = \max_{s\in[0,T]} \bigl\{T\bigl\|\mathcal{G}(T,\rho)\bigr\|, 1\bigr\},$$

$$b_1 = 4a_1\bigl|W^{-1}\bigr|\bigl(|\chi_2| + |\chi_0| + a_1 + a_2\bigr), \qquad b_2 = 4(a_1+a_2),$$

$$c_1 = 4a_3a_4T^{\alpha_1}\bigl|W^{-1}\bigr|\alpha_1^{-1}, \qquad c_2 = 4a_3T^{\alpha_1}\alpha_1^{-1},$$

$$b = \max\{ac_1, c_2\}, \qquad c = \max\{ab_1, b_2\}, \qquad \sup|g| = \sup\bigl\{\bigl|g(s,y,v)\bigr|\bigr\}.$$

We have

$$\begin{aligned}
\bigl|u(\rho)\bigr| &\leqslant \bigl\|\mathcal{G}(T,\rho)\bigr\|\bigl|W^{-1}\bigr|\left\{|\chi_2| + |\chi_0| + a_1 + a_2 + \int_0^\rho (\rho-t)^{\alpha_1-1}a_3\sup|g|\,\mathrm{d}t\right\} \\
&\leqslant a_4\bigl|W^{-1}\bigr|\bigl(|\chi_2|+|\chi_0|+a_1+a_2\bigr) + \bigl\|\mathcal{G}(T,\rho)\bigr\|\bigl|W^{-1}\bigr|\bigl(a_3T^{\alpha_1}\alpha_1^{-1}\sup|g|\bigr) \\
&\leqslant \frac{b_1}{4} + \frac{c_1}{4}\sup|g|
\end{aligned}$$

and

$$\begin{aligned}
\bigl|\chi(\rho)\bigr| &\leqslant |\chi_0| + a_1 + a_3T^{\alpha_1}\alpha_1^{-1}\sup|g| + \int_0^\rho \bigl\|\mathcal{G}(T,s)\bigr\|\bigl|u(s)\bigr|\,\mathrm{d}s \\
&\leqslant \frac{b_2}{4} + a\left(\frac{b_1}{4} + \frac{c_1}{4}\sup|g|\right) + a_3T^{\alpha_1}\alpha_1^{-1}\sup|g| \\
&\leqslant \frac{b}{2} + \frac{c}{2}\sup|g|.
\end{aligned}$$

Therefore, Ψ maps $\mathfrak{P}(r)$ into itself. To demonstrate that the operator Ψ possesses a fixed point within $\mathfrak{P}(r)$, we will leverage the Arzelà–Ascoli theorem, which establishes the complete continuity of Ψ. Additionally, we will prove the fact that the continuity of f directly implies the continuity of Ψ. Because $\mathfrak{P}(r)$ is closed, bounded, and convex, the Schauder fixed-point theorem guarantees that $\mathfrak{P}$ has a fixed point $(y,v)\in\mathfrak{P}(r)$ such that $\mathfrak{P}(y,v) = (y,v) = (\chi,u)$. Hence, the control function $u(\rho)$ steers the system from the initial complete state $\chi(T)$ to χ_2 on T. The system is relatively controllable on $[0,T]$. □

6 Example

Two examples are presented to demonstrate the relative controllability based on the proposed criteria.

Example 1. Consider the following linear damped fractional system with distributed delay and impulse for $0 < \alpha_2 \leqslant 1 < \alpha_1 \leqslant 2$:

$$\begin{aligned}
&{}^{C}\mathcal{D}^{\alpha_1}\chi(\rho) - \mathbf{A}{}^{C}\mathcal{D}^{\alpha_2}\chi(\rho) = \int_{-\theta}^{0} \mathrm{d}_\tau \mathbf{B}(\rho,\varsigma)u(\rho+\varsigma) \quad \rho \in [0,3]-1,\\
&\Delta\chi(\rho_i) = \chi(\rho_i^+) - \chi(\rho_i^-) = I_i(\chi(\rho_i)), \quad i = 1,\\
&\chi(0) = \chi_0, \quad \chi'(0) = \chi_1, \qquad u(\rho) = \varphi(\rho), \quad -\theta \leqslant \rho \leqslant 0.
\end{aligned} \tag{6}$$

Let

$$\alpha_1 = 1.5, \qquad \alpha_2 = 0.5,$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{B}(s,\varsigma) = \begin{pmatrix} -\cos(s+\varsigma) & \sin(s+\varsigma) \\ -\sin(s+\varsigma) & -\cos(s+\varsigma) \end{pmatrix}.$$

The M-L matrix function is given by

$$E_{\alpha,\beta}(\mathbf{A}\rho^\alpha) = \begin{pmatrix} \sum_{\iota=0}^\infty \frac{(-1)^\iota \rho^{2\iota\alpha}}{\Gamma(2\iota\alpha+\beta)} & \sum_{\iota=0}^\infty \frac{(-1)^\iota \rho^{2\iota+1)\alpha}}{\Gamma(2\iota+1)\alpha+\beta)} \\ -\sum_{\iota=0}^\infty \frac{(-1)^\iota \rho^{(2\iota+1)\alpha}}{\Gamma(2\iota+1)\alpha+\beta)} & \sum_{\iota=0}^\infty \frac{(-1)^\iota \rho^{2\iota\alpha}}{\Gamma(2\iota\alpha+\beta)} \end{pmatrix}.$$

Then

$$E_{1,\,1.5}(\mathbf{A}(T-\rho)) = \begin{pmatrix} \sum_{\iota=0}^\infty \frac{(-1)^\iota (T-\rho)^{2\iota}}{\Gamma(2\iota+1.5)} & \sum_{\iota=0}^\infty \frac{(-1)^\iota (T-\rho)^{2\iota+1)}}{\Gamma(2\iota+2.5)} \\ -\sum_{\iota=0}^\infty \frac{(-1)^\iota (T-\rho)^{(2\iota+1)}}{\Gamma(2\iota+2.5)} & \sum_{\iota=0}^\infty \frac{(-1)^\iota (T-\rho)^{2\iota}}{\Gamma(2\iota+2.5)} \end{pmatrix}$$

and

$$E_{1,\,1.5}(\mathbf{A}(T-(\rho-\sigma))) = \begin{pmatrix} \sum_{\iota=0}^\infty \frac{(-1)^\iota (T-(\rho-\varsigma)^{2\iota}}{\Gamma(2\iota+1.5)} & \sum_{\iota=0}^\infty \frac{(-1)^\iota (T-(\rho-\tau)^{2\iota+1)}}{\Gamma(2\iota+2.5)} \\ -\sum_{\iota=0}^\infty \frac{(-1)^\iota (T-(\rho-\varsigma)^{(2\iota+1)}}{\Gamma(2\iota+2.5)} & \sum_{\iota=0}^\infty \frac{(-1)^\iota (T-(\rho-\varsigma)^{2\iota}}{\Gamma(2\iota+2.5)} \end{pmatrix}.$$

Further,

$$(T-(\rho-\varsigma))^{0.5} E_{1,\,1.5}(\mathbf{A}(T-(\rho-\varsigma))) = \begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ -\mathbf{C}_3 & \mathbf{C}_4 \end{pmatrix},$$

where

$$\mathbf{C}_1 = \mathbf{C}_4 = \sum_{\iota=0}^\infty \frac{(-1)^\iota (T-(\rho-\varsigma)^{2\iota+0.5}}{\Gamma(2\iota+1.5)},$$

$$\mathbf{C}_2 = \mathbf{C}_3 = \sum_{\iota=0}^\infty \frac{(-1)^\iota (T-(\rho-\varsigma)^{2\iota+1.5}}{\Gamma(2\iota+2.5)}.$$

Then

$$\begin{aligned}\mathcal{G}(T,\rho) &= \int_{-1}^{0} \big(T-(s-\varsigma)\big)^{0.5} E_{1,1.5}\big(\mathbf{A}\big(T-(s-\varsigma)\big)\big)\mathbf{B}(s-\varsigma,\,\varsigma)\,\mathrm{d}\mathbf{B}_\varsigma \\ &= \begin{pmatrix} \mathbf{P}(s) & \mathbf{Q}(s) \\ -\mathbf{Q}(s) & \mathbf{P}(s) \end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}\mathbf{P}(s) &= \int_{-1}^{0} \big[\mathbf{C}_1\cos(T+\varsigma) - \mathbf{C}_3\sin(T+\varsigma)\big]\,\mathrm{d}\varsigma, \\ \mathbf{Q}(s) &= \int_{-1}^{0} \big[\mathbf{C}_2\cos(T+\varsigma) + \mathbf{C}_4\sin(T+\varsigma)\big]\,\mathrm{d}\varsigma.\end{aligned}$$

We get the controllability matrix

$$\mathcal{W} = \int_{0}^{T+\varsigma} \mathcal{G}(T,s)\mathcal{G}^*(T,s)\,\mathrm{d}s = \int_{0}^{T+\varsigma} \begin{pmatrix} \mathbf{P}(s) & \mathbf{Q}(s) \\ -\mathbf{Q}(s) & \mathbf{P}(s) \end{pmatrix} \begin{pmatrix} \mathbf{P}(s) & \mathbf{Q}(s) \\ -\mathbf{Q}(s) & \mathbf{P}(s) \end{pmatrix}^* > 0$$

for any $T > 0$, the matrix is positive define. Then the linear system is relative controllable.

Example 2. The problem of nonlinear fractional damped differential system is as follows:

$$\begin{aligned}&{}^C\mathcal{D}^{\alpha_1}\chi(\rho) - \mathbf{A}\,{}^C\mathcal{D}^{\alpha_2}\chi(\rho) \\ &\quad= \int_{-\theta}^{0} \mathrm{d}_\varsigma\mathbf{B}(\rho,\varsigma)u(\rho+\varsigma) + g\big(\rho,\chi(\rho),u(\rho)\big), \quad \rho\in[0,T]\setminus\{\rho_1,\rho_2,\dots,\rho_k\}, \\ &\varDelta\chi(\rho_i) = \chi(\rho_i^+) - \chi(\rho_i^-) = I_i\big(\chi(\rho_i)\big), \quad i=1,2,\dots,\rho_k, \\ &\chi(0)=\chi_0, \quad \chi'(0)=\chi_1, \qquad u(\rho)=\varphi(\rho), \quad -\theta\leqslant\rho\leqslant 0,\end{aligned} \tag{7}$$

where $\alpha_1 = 1.5$, $\alpha_2 = 0.5$,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{B}(s,\varsigma) = \begin{pmatrix} -\mathrm{e}^\varsigma\cos(\rho) & \mathrm{e}^\varsigma\sin(\rho) \\ -\mathrm{e}^\varsigma\sin(\rho) & -\mathrm{e}^\varsigma\cos(\rho) \end{pmatrix},$$

and

$$g\big(\rho,\chi(\rho)\big) = \begin{pmatrix} \frac{1+\chi_1(\rho)}{1+\chi_0^2(\rho)} \\ \frac{1+\chi_0(\rho)}{1+\chi_1^2(\rho)} \end{pmatrix}.$$

It is easy to prove that $g(\chi)$ is continuous on $\mathbb{R}^2$, satisfying the conditions in Theorem 2, so the nonlinear system is relatively controllable.

7 Conclusions

This study focuses on the controllability criterion of fractional damped differential systems incorporating distributed delays and impulses, encompassing both linear and nonlinear systems. Initially, the solutions for both linear and nonlinear systems are derived through the application of the Laplace transform and its inverse. Using the controllability matrix and fixed-point techniques, the controllability of the considered system is established under certain assumptions. Additionally, two numerical examples are presented at the conclusion of this study to illustrate and validate the derived theoretical framework. Future work could extend this study in two directions: (i) incorporating ψ-Hilfer derivatives [21] into stochastic control to generalize the current deterministic framework; (ii) applying delayed Mittag-Leffler function perturbations [10] to enhance the analysis of impulse response.

Conflicts of interest.

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