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Exact controllability of conformable linear systems with semilinear boundary control

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Abstract. In this manuscript, we investigate the exact controllability of a class of linear systems governed by conformable fractional derivatives of order α in $(0, 1]$ subject to semilinear boundary control in Banach spaces. We first establish the existence of mild solutions to the associated fractional Cauchy problems. We then derive sufficient conditions ensuring the exact controllability of these conformable linear systems under semilinear boundary control actions. An abstract model of an age-structured population dynamics system is provided to illustrate the applicability of the theoretical results.

Keywords: conformable derivative, controllability, fixed point theorem, semilinear boundary control, fractional semigroups.

1 Introduction

In the past few years, there has been a significant surge in research within the field of control theory dedicated to the exploration of fractional differential equations. This burgeoning interest can be attributed to their growing relevance in a multitude of scientific domains, ranging from physics to biology, as well as their superior ability to accurately and generically model real-world phenomena when compared to traditional derivatives. Numerous references in the literature, such as [20], underscore the expanding scope of applications for fractional differential equations. Despite the recognized importance of fractional derivatives, it is crucial to acknowledge that there is no single, universally accepted definition of the fractional derivative, and prominent variants include the Riemann–Liouville and Caputo definitions; see [20].

One notable characteristic of fractional derivatives is that they often deviate from the familiar rules and properties associated with their integer-order counterparts. Recognizing this discrepancy, several researchers have embarked on the endeavor of introducing novel definitions of fractional derivatives that align more closely with the conventions and behaviors of integer derivatives. In this regard, the works of Abdeljawad and Khalil [1,19], among others, have made significant contributions. They have introduced a distinctive type of fractional derivative that are based on limit principles, known as the conformable derivative. This particular fractional derivative strives to bridge the gap between the fractional and integer-order derivatives, rendering it a subject of considerable interest and ongoing research within the realm of fractional calculus.

The primary benefits of fractional derivatives, as compared to integer derivatives, lie in their flexibility and nonlocal nature. Fractional derivatives exhibit greater flexibility in approximating real data than their classical counterparts, and they account for nonlocal effects, which classical derivatives fail to capture. Fractional derivatives also have properties similar to integer derivatives, such as linearity and the Leibniz rule. In some cases, fractional derivatives and integer derivatives can be exchanged. One of the advantages of the conformable derivative is that it is flexible due to the presence of the fractional number α, like other fractional derivatives. The conformable derivative generalizes the usual derivative, which corresponds to the case $\alpha = 1$. This new definition of the fractional derivative has the advantage of preserving some classical properties of the integer derivative.

In control theory, a highly significant concept is controllability. Controllability is defined as the ability to manipulate a dynamic system to reach a specific state or trajectory using appropriate control inputs or actions. In other words, a system is considered controllable if it is possible to influence its evolution in such a way that it can be steered from an initial state to a desired final state by following a set of defined rules or commands. Controllability may vary depending on the system, its characteristics, and associated constraints, and it serves as a fundamental concept for the design and analysis of control systems.

At present, extensive research is dedicated to the exploration of conformable derivatives in finite- and infinite-dimensional spaces. These investigations encompass a wide range of topics, including solutions to the Cauchy problem, stability, observability, and controllability; see [5, 6, 17, 26, 31]. In the context of infinite-dimensional systems, the authors in [11] introduced some characterisations for the approximate and exact controllability of conformable linear systems. Recent works have applied fractional and nonlocal operators to complex systems, including fractal modeling of radioactive decay [28], numerical analysis of fractional Burgers equations [14], and existence results for impulsive delay systems with Mittag-Leffler kernels [3]. Furthermore, in [12], they established a classical duality relation between controllability and observability for linear systems, thus providing criteria for exact and approximate observability for some of these systems. For finite-dimensional systems, in [27], the authors studied the controllability and observability.

Simultaneously, various notions of controllability, such as approximate controllability, approximate positive controllability, null controllability, and stabilization, have been introduced for abstract classical boundary control problems in the works of [8, 9, 22, 23]. Moreover, the controllability of specific classes of partial differential equations has been extensively explored by numerous researchers with notable contributions from Lions and Zuazua [15, 34].

In [17], the authors studied the controllability of mild solutions of a class of nonlocal conformable Cauchy problem, which are represented by the following equation:

$$\begin{aligned} D_t^\alpha x(t) &= Ax(t) + f\big(t, x(t)\big) + Bu(t), \quad t \in [0, T], \\ x(0) &= x_0 + g(x), \end{aligned}$$

where B is the control operator, D_t^α is the conformal vector fractional derivative of order $\alpha \in (0, 1]$ in a Banach space X, A is the α-infinitesimal generator of a α-semigroup, $x_0 \in X$, and f, g are given functions; for more details, see [17]. In a parallel context, in the case of state-dependent delay, the authors in [29] have investigated the controllability of Caputo fractional impulsive quasilinear differential systems.

Concerning the boundary fractional differential equations, the authors in [13, 32], have studied the approximate boundary controllability of fractional diffusion equations. Additionally, in the context of interior approximate controllability, the authors in [18] explored fractional wave equations, focusing on cases involving a nonnegative self-adjoint operator that satisfies the unique continuation property.

Additionally, stabilization of fractional boundary control systems has garnered significant attention from various authors, particularly for specific classes of boundary control systems. For instance, in [33], the authors utilized the backstepping method to address the boundary feedback stabilization of fractional reaction-diffusion equations. Additionally, the work presented in [24] also touches upon the topic of boundary stabilization in the context of fractional reaction-diffusion equations.

The goal of this manuscript is to investigate the exact controllability of mild solution of an abstract fractional linear Cauchy problem by semilinear boundary control in Banach spaces with conformable derivative of order α in $(0, 1]$. To the best of our knowledge, this concept has not yet been explored in the broader context of fractional systems with boundary control. By using the fixed point theorem, we develop a fractional semigroup approach for the existence and uniqueness of a mild solution of this system. The mild solution in this paper is given by a variation of parameters formula. Furthermore, we provide sufficient conditions that ensure the exact controllability for this class of system with semilinear boundary control. In comparison with the work of Hannabou et al. [17], it is crucial to highlight that boundary control problems are related to unbounded control systems, emphasizing the complexity associated with studying such systems compared to distributed parameter systems with internal control.

The paper's structure is outlined as follows. In the following section, we introduce some preliminaries on conformable derivative and α-semigroup. In Section 3, we give sufficient conditions to guarantee the existence of mild solutions of conformable Cauchy problem of order α in $(0, 1]$ with semilinear boundary control in Banach spaces and the expression of these solutions. In Section 4, we give the main result on the exact controllability. Finally, in the last section, the abstract result is applied to a model for an age-dependent population equation.

2 A background on conformable derivative

This section introduces some key definitions and findings from existing literature that hold significant importance in the context of this study.

Definition 1. (See [1, 19].) Let X be a Banach space and $\alpha > 0$. We define the conformable derivative of a function $\psi : [0, \infty) \to X$ of order α as follows:

$$D_t^\alpha \psi(t) = \lim_{\varepsilon \to 0} \frac{\psi(t + \varepsilon t^{1-\alpha}) - \psi(t)}{\varepsilon}$$

for all $t > 0$.

If $D_t^\alpha \psi(t)$ exists on $(0, T)$, then, for some $T > 0$, $D_t^\alpha \psi(0) = \lim_{t \to 0} D_t^\alpha \psi(t)$.

The definition of the conformable fractional integral of a function $\psi : [0, \infty) \to X$ of order $\alpha \in (0, 1]$ is as follows:

$$I_\alpha \psi(t) = \int_0^t \psi(s) \,\mathrm{d}_\alpha s, \quad \text{where } \mathrm{d}_\alpha s := s^{\alpha-1} \,\mathrm{d}s.$$

The fractional C_0-α-semigroup corresponding to the conformable derivative is presented in [2].

Definition 2. Let $\alpha \in (0, 1]$. A family $(S_\alpha(t))_{t \geqslant 0} \subset \mathcal{L}(X, X)$ is called a fractional C_0-α-semigroup of operators if:

- $S_\alpha(0) = I$;
- $S_\alpha((t + s)^{1/\alpha}) = S_\alpha(t^{1/\alpha}) S_\alpha(s^{1/\alpha})$ for every t and $s \geqslant 0$;
- $\lim_{t \to 0^+} S_\alpha(t)x = x$, for all $x \in X$.

The infinitesimal generator of the above C_0-α-semigroup is given by the following definition.

Definition 3. Let $(S_\alpha(t))_{t \geqslant 0}$ be a C_0-α-semigroup on the Banach space X. We define the operator $(A, D(A))$ as the α-infinitesimal generator of $(S_\alpha(t))_{t \geqslant 0}$, and it is given by

$$D(A) = \Big\{ x \in X \colon \lim_{t \to 0^+} D_t^\alpha S_\alpha(t)x \text{ exists} \Big\},$$
$$Ax = \lim_{t \to 0^+} D_t^\alpha S_\alpha(t)x \quad \text{for any } x \in D(A).$$

Some basic properties of the fractional C_0-α-semigroup, which we will use, either in a direct or indirect way, are provided by the following theorems for $\alpha \in (0, 1]$.

Theorem 1. (*See* [25].) *Considering $(S_\alpha(t))_{t \geqslant 0}$ as a C_0-α-semigroup on X with A being its α-infinitesimal generator, we can state the following properties*:

(i) *For any given $x \in X$,*

$$\int_0^t S_\alpha(s)x \,\mathrm{d}_\alpha s \in D(A) \quad \textit{and} \quad A\Bigg(\int_0^t S_\alpha(s)x \,\mathrm{d}_\alpha s \Bigg) = S_\alpha(t)x - x.$$

(ii) *If $x \in D(A)$, we get $S_\alpha(t)x \in D(A)$ and*

$$D_t^\alpha S_\alpha(t)x = AS_\alpha(t)x = S_\alpha(t)Ax.$$

(iii) *In the case where $x \in D(A)$,*

$$S_\alpha(t)x - S_\alpha(s)x = \int_s^t S_\alpha(r)Ax \,\mathrm{d}_\alpha r = \int_s^t AS_\alpha(r)x \,\mathrm{d}_\alpha r.$$

Theorem 2. (*See* [25].) *Let $((S_\alpha(t))_{t\geqslant 0}$ be a C_0-α-semigroup on X. There exist $M > 0$ and $\omega > 0$ such that*

$$\big\|S_\alpha(t)\big\| \leqslant M\mathrm{e}^{\omega t^\alpha}.$$

Furthermore, for any $x \in X$, the function $t \longmapsto S_\alpha(t)x$ is in $\mathcal{C}([0,\infty), X)$.

To verify that an operator generates a C_0-α-semigroup, one can simply apply Hille–Yosida's theorem tailored for conformable fractional semigroups of operators; see [4].

Let A be an α-infinitesimal generator of a C_0-α-semigroup $(S_\alpha(t))_{t\geqslant 0}$. For any $\lambda \in \rho(A)$ and $x \in X$, we set $\|x\|_{-1,\lambda} := \|(\lambda - A)^{-1}x\|$. The space obtained by extending the norm $\|\cdot\|_{-1,\lambda}$ to the completion of X is known as the extrapolation space linked to both X and A. This space is denoted by X_{-1} and is not conditioned by the choice of λ. However, we can show that the unique extension of S_α on X_{-1} forms a C_0-α-semigroup denoted by $S_{-1}^\alpha := (S_{-1}^\alpha(t))_{t\geqslant 0}$, and its corresponding generator is denoted by A_{-1}. For additional insights, please refer to [30].

3 Well-posedness of the problem

In this section, we consider the following fractional semilinear boundary control problem:

$$\begin{aligned} D_t^\alpha x(t) &= A_m x(t), \quad \alpha \in (0,1],\ t \geqslant 0, \\ Qx(t) &= f\big(t, x(t)\big) + Bu(t), \quad t \geqslant 0, \\ x(0) &= x_0. \end{aligned} \tag{1}$$

Here $x(\cdot) \in L^q_{\mathrm{loc}}([0,\infty), X)$ is the state, f is a function in $L^1([0,T] \times X, \partial X)$ for all $T > 0$, $B \in \mathcal{L}(U, \partial X)$ represents the control operator, $u(\cdot) \in L^p_{\mathrm{loc}}([0,\infty), U)$ is the boundary control with Banach spaces X, ∂X, and U are called the state space, the boundary, and the control space, respectively, with $1 \leqslant p < \infty$ and $1 \leqslant q < \infty$. $A_m : D(A_m) \subset X \to X$ is a closed, densely linear operator defined on its domain $D(A_m)$, $Q \in \mathcal{L}([D(A_m)], \partial X)$ is a boundary operator, where $[D(A_m)] := (D(A_m), \|\cdot\|_{A_m})$ and $\|x\|_{A_m} = \|x\| + \|A_m x\|$ for any $x \in D(A_m)$.

In the sequel, we introduce the following assumptions:

(H1) $A := A_m$ with $D(A) := \ker Q$ generates a C_0-α-semigroup $(S_\alpha(t))_{t\geqslant 0}$ on X;
(H2) $Q : D(A_m) \to \partial X$ is surjective, and there exist $\gamma > 0$ and $\lambda_0 \in \mathbb{R}$ satisfying $\|Qx\| \geqslant \gamma^{-1}\|\lambda x\|$ for any $\lambda > \lambda_0$ and $x \in \ker(\lambda - A)$.

Remark 1. Assumptions (H1) and (H2) together form the structural basis for modeling boundary control problems within a semigroup framework. In particular, (H1) ensures that the uncontrolled system admits a well-posed evolution governed by a strongly continuous semigroup, while (H2) guarantees that any boundary input can be mapped consistently into the system through the surjectivity of the control operator Q. These assumptions reflect typical physical configurations, where boundary actuation directly influences the state evolution, such as in thermal systems, chemical reactors, or population dynamics.

With the above hypotheses, [16] has established the following properties, which will be crucial for examining the well-posedness of the system (1).

Lemma 1. *Under conditions* (H1)–(H2)*, the following statements hold true for every* $\lambda \in \rho(A)$:

(i) $D(A_m) = D(A) \oplus \ker(\lambda - A_m)$;
(ii) $Q|_{\ker(\lambda - A_m)}$ *is bijective, and* $Q_\lambda := (Q|_{\ker(\lambda - A_m)})^{-1} : \partial X \to \ker(\lambda - A_m) \subset X$ *is bounded*;
(iii) $P_\lambda := Q_\lambda Q \in \mathcal{L}([D(A_m)])$ *is a projection on* $\ker(\lambda - A_m)$ *along* $D(A) = \ker Q$;
(iv) $R(\mu, A)Q_\lambda = (1/(\lambda - \mu))(Q_\mu - Q_\lambda) = R(\lambda, A)Q_\mu$ *for all* $\lambda, \mu \in \rho(A)$*,* $\lambda \neq \mu$;
(v) *There exist* $\gamma > 0$ *and* $\omega > 0$ *such that* $\|\lambda Q_\lambda\| \leqslant \gamma$ *for all* $\lambda > \omega$*.*

Definition 4. Let $u \in U$ and $f \in L^1([0,T] \times X, \partial X)$. A function $x(\cdot) := x(\cdot, x_0, u) \in C^1([0,T], X)$ is called a classical solution of (1) if $x(t) \in D(A_m)$ for all $t \geqslant 0$ and satisfies (1).

In this section, for the sake of simplicity, we denote $f(t, x(t)) + Bu(t)$ by $h(t, x(t), u(t))$ and define $B_\lambda := Q_\lambda B \in \mathcal{L}(U, \ker(\lambda - A_m))$ for all $\lambda \in \rho(A)$.

Remark 2. Since $B \in \mathcal{L}(U, \partial X)$ and $L^p_{\mathrm{loc}}(\mathbb{R}_+, U) \subset L^1_{\mathrm{loc}}(\mathbb{R}_+, U)$, we can show that $f(\cdot, x(\cdot)) \in L^1((0,T), \partial X)$ for all $x \in L^q_{\mathrm{loc}}(\mathbb{R}_+, X)$ if and only if $h(\cdot, x(\cdot), u(\cdot)) \in L^1((0,T), \partial X)$ for all $u \in L^p_{\mathrm{loc}}(\mathbb{R}_+, U)$ and $x \in L^q_{\mathrm{loc}}(\mathbb{R}_+, X)$.

Now, consider $x(\cdot)$ as a classical solution to (1). Then, by applying conditions (i)–(iii) in Lemma 1, we obtain

$$x(t) - Q_\lambda h\big(t, x(t), u(t)\big) \in D(A)$$

for all $t \in (0,T]$ and

$$\begin{aligned} D_t^\alpha x(t) &= A_m x(t), \\ &= A_m\big(x(t) - Q_\lambda h\big(t, x(t), u(t)\big)\big) + A_m Q_\lambda h\big(t, x(t), u(t)\big), \\ &= A\big(x(t) - Q_\lambda h\big(t, x(t), u(t)\big)\big) + \lambda Q_\lambda h\big(t, x(t), u(t)\big), \\ &= A_{-1} x(t) + (\lambda - A_{-1}) Q_\lambda h\big(t, x(t), u(t)\big). \end{aligned} \tag{2}$$

Therefore, if $x(\cdot) = x(\cdot, x_0, u)$ represents a classical solution for (1), it also serves as a solution for (2). We make the assumption that $h(\cdot, x(\cdot), u(\cdot)) \in L^1([0,T] \times X \times U, \partial X)$.

By applying the variation of constants formula, we obtain

$$
\begin{aligned}
x(t) &= S_{-1}^{\alpha}(t)x_0 + \int_0^t S_{-1}^{\alpha}\big(\big(t^{\alpha}-s^{\alpha}\big)^{1/\alpha}\big)(\lambda - A_{-1})Q_{\lambda}h\big(s,x(s),u(s)\big)\,\mathrm{d}_{\alpha}s,\\
&= S_{\alpha}(t)x_0 + (\lambda - A_{-1})\int_0^t S_{-1}^{\alpha}\big(\big(t^{\alpha}-s^{\alpha}\big)^{1/\alpha}\big)Q_{\lambda}h\big(s,x(s),u(s)\big)\,\mathrm{d}_{\alpha}s,\\
&= S_{\alpha}(t)x_0 + (\lambda - A_{-1})\int_0^t S_{\alpha}\big(\big(t^{\alpha}-s^{\alpha}\big)^{1/\alpha}\big)Q_{\lambda}h\big(s,x(s),u(s)\big)\,\mathrm{d}_{\alpha}s. \qquad (3)
\end{aligned}
$$

Generally, this solution may not be within the space of X. The boundary control system defined in (1) is considered well-posed when $B_{-1} := (\lambda - A_{-1})Q_{\lambda} \in \mathcal{L}(\partial X, X_{-1})$ qualifies as an (p,q)-admissible control operator for S_{α}, i.e., if

$$
(\lambda - A_{-1})\int_0^{\tau} S_{\alpha}\big(\big(t^{\alpha}-s^{\alpha}\big)^{1/\alpha}\big)Q_{\lambda}h\big(s,x(s),u(s)\big)\,\mathrm{d}_{\alpha}s \in X
$$

for all $\tau \in (0,T]$, $u \in L^p_{\mathrm{loc}}(\mathbb{R}_+,U)$, and $x \in L^q_{\mathrm{loc}}(\mathbb{R}_+,X)$. It is well known that $A_{-1}x \in X$ if and only if $x \in D(A)$. This property is essential to ensure that all mild solutions of (2) or (1) take values in X.

Following [30], we therefore introduce the following definition of admissible boundary control operator.

Definition 5. We assume that assumptions (H1)–(H2) are satisfied, $T > 0$, $h \in L^1([0,T] \times X \times U, \partial X)$, and $\lambda \in \rho(A)$. The operator $Q_{\lambda} \in \mathcal{L}(\partial X, \ker(\lambda - A_m))$ or B_{-1} is designated as an (p,q)-admissible control operator for S_{α} when, for $\tau \in (0,T]$,

$$
\int_0^{\tau} S_{\alpha}\big(\big(t^{\alpha}-s^{\alpha}\big)^{1/\alpha}\big)Q_{\lambda}h\big(s,x(s),u(s)\big)\,\mathrm{d}_{\alpha}s \in D(A)
$$

for every $u \in L^p_{\mathrm{loc}}(\mathbb{R}_+,U)$ and $x \in L^q_{\mathrm{loc}}(\mathbb{R}_+,X)$.

Note that from (iv) in Lemma 1 the admissibility of Q_{λ} remains consistent regardless of the selection of λ from $\rho(A)$. On the other hand, if $f(\cdot, x(\cdot)) \in L^1((0,T),\partial X)$ for all $x \in L^q_{\mathrm{loc}}(\mathbb{R}_+,X)$, then from Remark 2 and the assertion (v) in lemma 1, according to [9, Lemmas A.2 and A.3], we can show that Q_{λ} is (p,q)-admissible control operator for S_{α}. Hence the solution given by (3) have values in X and becomes

$$
x(t) = S_{\alpha}(t)x_0 + (\lambda - A)\int_0^t S_{\alpha}\big(\big(t^{\alpha}-s^{\alpha}\big)^{1/\alpha}\big)Q_{\lambda}h\big(s,x(s),u(s)\big)\,\mathrm{d}_{\alpha}s \qquad (4)
$$

for all $t \geqslant 0$, and if it is continuous in t, it will be called a mild solution of (1). Furthermore, by (iv) in Lemma 1, we can show that the mild solution (4) of (1) is not influenced by the particular selection of λ within $\rho(A)$. By (iv) in Lemma 1, Eq. (4) implies

$$\begin{aligned}&\mu R(\mu, A)x(t)\\ &\quad= \mu R(\mu, A)S_\alpha(t)x_0 + \int_0^t S_\alpha\big(\big(t^\alpha - s^\alpha\big)^{1/\alpha}\big)\mu Q_\mu h\big(s, x(s), u(s)\big)\,\mathrm{d}_\alpha s\end{aligned}$$

for all $\mu, \lambda \in \rho(A)$. Given that $\lim_{\mu\to+\infty} \mu R(\mu, A)x = x$ for every $x \in X$, we can obtain

$$x(t) = S_\alpha(t)x_0 + \lim_{\mu\to+\infty}\int_0^t S_\alpha\big(\big(t^\alpha - s^\alpha\big)^{1/\alpha}\big)\mu Q_\mu h\big(s, x(s), u(s)\big)\,\mathrm{d}_\alpha s.$$

Definition 6. Let $f \in L^1([0,T] \times X, \partial X)$. A function $x \in C([0,T], X)$ is called a mild solution of system (1) if $f(\cdot, x(\cdot)) \in L^1((0,T), \partial X)$ and it satisfies the integral equation

$$x(t) = S_\alpha(t)x_0 + \lim_{\mu\to+\infty}\int_0^t S_\alpha\big(\big(t^\alpha - s^\alpha\big)^{1/\alpha}\big)\mu Q_\mu h\big(s, x(s), u(s)\big)\,\mathrm{d}_\alpha s \tag{5}$$

for any $t \in [0,T]$.

To ensure the existence of solutions for (1), we need the following hypothesis:

(H3) There exists $k > 0$ such that for a.e. $t \in [0,T]$ and $x, y \in X$,

$$\big\|f(t,x) - f(t,y)\big\|_{\partial X} \leqslant k\|x - y\|_X.$$

Remark 3.

- When f is identically zero, i.e., $h(s, x(s), u(s)) = Bu(s)$, result (v) in Lemma 1 implies the well-posedness of system (1).
- Assumption (H3) implies that, for a.e. $t \in [0,T]$, $u \in U$, and $x, y \in X$,

$$\big\|h(t,x,u) - h(t,y,u)\big\|_{\partial X} \leqslant k\|x - y\|_X.$$

Theorem 3. *Under assumptions* (H1)–(H3)*, the semilinear boundary system* (1) *admits a unique mild solution in the space* $C([0,T], X)$*. In other words, the integral equation* (5) *admits a unique solution in* $C([0,T], X)$*. This mild solution is given by*

$$\begin{aligned}x(t) = S_\alpha(t)x_0 &+ \lim_{\mu\to\infty}\int_0^t S_\alpha\big(\big(t^\alpha - s^\alpha\big)^{1/\alpha}\big)\mu Q_\mu f\big(s, x(s)\big)\,\mathrm{d}_\alpha s\\ &+ \lim_{\mu\to\infty}\int_0^t S_\alpha\big(\big(t^\alpha - s^\alpha\big)^{1/\alpha}\big)\mu B_\mu u(s)\,\mathrm{d}_\alpha s.\end{aligned}$$

Proof. For every $x_0 \in X$, we consider the application Φ defined as follows:

$$\big(\Phi(x)\big)(t) = S_\alpha(t)x_0 + \lim_{\mu\to+\infty} \int_0^t S_\alpha\big(\big(t^\alpha - s^\alpha\big)^{1/\alpha}\big)\mu Q_\mu h\big(s, x(s), u(s)\big)\,\mathrm{d}_\alpha s$$

for all $t \in [0,T]$ and $x \in C([0,T],X)$, where the space $C([0,T],X)$ is equipped with the norm $\|x\|_\infty = \sup_{t\in[0,T]}\|x(t)\|$. We observe that a function x in $C([0,T],X)$ is a solution of (5) if and only if it is a fixed point of Φ. For this end, it is enough to show that Φ has a unique fixed point. First of all, we need to verify that Φ is defined from $C([0,T],X)$ to $C([0,T],X)$, i.e., $(\Phi(x)) \in C([0,T],X)$ for all $x \in C([0,T],X)$. Indeed, for any $\mu \in \rho(A)$ and $x \in C([0,T],X)$, we define

$$F_\mu(t) = \int_0^t S_\alpha\big(\big(t^\alpha - s^\alpha\big)^{1/\alpha}\big)\mu Q_\mu h\big(s, x(s), u(s)\big)\,\mathrm{d}_\alpha s$$

for all $t \in [0,T]$. We have $(\Phi(x))(t) = S_\alpha(t)x_0 + \lim_{\mu\to+\infty} F_\mu(t)$. The function $t \to S_\alpha(t)x_0$ is continuous. Therefore, it suffices to verify that $\lim_{\mu\to+\infty} F_\mu(\cdot)$ is continuous on $[0,T]$.

Let $t, t' \in [0,T]$ and $t < t'$. Using Definition 2, we get

$$\begin{aligned}
&F_\mu(t') - F_\mu(t)\\
&\quad= \int_0^t \big[S\big(\big(t'^\alpha - s^\alpha\big)^{1/\alpha}\big) - S\big(\big(t^\alpha - s^\alpha\big)^{1/\alpha}\big)\big]\mu Q_\mu h\big(s, x(s), u(s)\big)\,\mathrm{d}_\alpha s\\
&\qquad+ \int_t^{t'} S\big(\big(t'^\alpha - s^\alpha\big)^{1/\alpha}\big)\mu Q_\mu h\big(s, x(s), u(s)\big)\,\mathrm{d}_\alpha s,\\
&\quad= \big[S\big(\big(t'^\alpha - t^\alpha\big)^{1/\alpha}\big) - I\big] \int_0^t S\big(\big(t^\alpha - s^\alpha\big)^{1/\alpha}\big)\mu Q_\mu h\big(s, x(s), u(s)\big)\,\mathrm{d}_\alpha s\\
&\qquad+ \int_t^{t'} S\big(\big(t'^\alpha - s^\alpha\big)^{1/\alpha}\big)\mu Q_\mu h\big(s, x(s), u(s)\big)\,\mathrm{d}_\alpha s.
\end{aligned}$$

Then, according to (v) in Lemma 1 and Theorem 2, we obtain

$$\begin{aligned}
&\big\|F_\mu(t') - F_\mu(t)\big\|\\
&\quad\leqslant \big\|S\big(\big(t'^\alpha - t^\alpha\big)^{1/\alpha}\big) - I\big\|M\gamma \int_0^t \mathrm{e}^{\omega(t^\alpha - s^\alpha)}\big\|f\big(s, x(s)\big) + Bu(s)\big\|\,\mathrm{d}_\alpha s\\
&\qquad+ M\gamma \int_t^{t'} \mathrm{e}^{\omega(t'^\alpha - s^\alpha)}\big\|f\big(s, x(s)\big) + Bu(s)\big\|\,\mathrm{d}_\alpha s.
\end{aligned}$$

Also,

$$\begin{aligned}
&\Big\| \lim_{\mu\to+\infty} F_\mu(t') - \lim_{\mu\to+\infty} F_\mu(t) \Big\| \\
&\quad \leqslant \big\| S\big((t'^\alpha - t^\alpha)^{1/\alpha} \big) - I \big\| M\gamma \int_0^t \mathrm{e}^{\omega(t^\alpha - s^\alpha)} \big\| f\big(s, x(s)\big) + Bu(s) \big\| \, \mathrm{d}_\alpha s \\
&\qquad + M\gamma \int_t^{t'} \mathrm{e}^{\omega(t'^\alpha - s^\alpha)} \big\| f\big(s, x(s)\big) + Bu(s) \big\| \, \mathrm{d}_\alpha s.
\end{aligned}$$

Using the third criterion in Definition 2, we obtain

$$\lim_{t\to t'} \Big\| \lim_{\mu\to+\infty} F_\mu(t') - \lim_{\mu\to+\infty} F_\mu(t) \Big\| = 0.$$

Therefore, we deduce that $(\Phi(x)) \in C([0,T], X)$.

Now, let x_1 and x_2 be in $C([0,T], X)$ and t in $[0,T]$. We have

$$\begin{aligned}
&\big\| \big(\Phi(x_1)\big)(t) - \big(\Phi(x_2)\big)(t) \big\| \\
&\quad \leqslant \lim_{\mu\to\infty} \int_0^t \big\| S_\alpha\big((t^\alpha - s^\alpha)^{1/\alpha} \big) \mu Q_\mu \big\| \\
&\qquad \times \big\| h\big(s, x_1(s), u(s)\big) - h\big(s, x_2(s), u(s)\big) \big\|_{\partial X} \, \mathrm{d}_\alpha s.
\end{aligned}$$

Applying assumption (H3), statement (v) in Lemma 1, and Theorem 2, we obtain

$$\begin{aligned}
\big\| \big(\Phi(x_1)\big)(t) - \big(\Phi(x_2)\big)(t) \big\| &\leqslant M\mathrm{e}^{\omega T^\alpha} \gamma k \int_0^t \big\| x_1(s) - x_2(s) \big\|_X \, \mathrm{d}_\alpha s, \\
&\leqslant \frac{M\mathrm{e}^{\omega T^\alpha} \gamma k t^\alpha}{\alpha} \|x_1 - x_2\|_\infty.
\end{aligned}$$

Thus

$$\big\| \Phi(x_1) - \Phi(x_2) \big\|_\infty \leqslant \frac{\gamma k T^\alpha M \mathrm{e}^{\omega T^\alpha}}{\alpha} \|x_1 - x_2\|_\infty.$$

It is well known that by induction we get

$$\big\| \Phi^n(x_1) - \Phi^n(x_2) \big\|_\infty \leqslant \frac{\gamma^n k^n T^{n\alpha} M^n \mathrm{e}^{n\omega T^\alpha}}{\alpha^n n!} \|x_1 - x_2\|_\infty$$

for every $n \in \mathbb{N}$. Given that $\lim_{n\to\infty} a^n/n! = 0$ for all $a \in \mathbb{R}$, we can select n such that $\gamma^n k^n T^{n\alpha} M^n \mathrm{e}^{n\omega T^\alpha}/(\alpha^n n!) < 1$. As a result, Φ becomes a contraction, ensuring the existence of a unique fixed point for Φ. This completes the proof. $\square$

Remark 4. Based on the proof of Theorem 3, one can extend the analysis to the following fractional semilinear Cauchy problem with nonlinear boundary control:

$$\begin{aligned} D_t^\alpha x(t) &= A_m x(t) + g\big(t, x(t)\big), \quad \alpha \in (0,1],\ t \geqslant 0,\\ Qx(t) &= h\big(t, x(t), u(t)\big), \quad t \geqslant 0,\\ x(0) &= x_0, \end{aligned} \tag{6}$$

which has a unique solution, provided that assumptions (H1)–(H3) are fulfill and $g \in L^1([0,T] \times X, X)$ satisfies

$$\big\|g(t,x) - g(t,y)\big\|_X \leqslant k_g \|x - y\|_X$$

for a.e. $t \in [0,T]$ and $x, y \in X$, for some constant $k_g > 0$. The corresponding mild solution is given by

$$\begin{aligned} x(t) = S_\alpha(t)x_0 &+ \int_0^t S_\alpha\big(\big(t^\alpha - s^\alpha\big)^{1/\alpha}\big) g\big(s, x(s)\big)\,\mathrm{d}_\alpha s\\ &+ \lim_{\mu\to+\infty} \int_0^t S_\alpha\big(\big(t^\alpha - s^\alpha\big)^{1/\alpha}\big)\mu Q_\mu h\big(s, x(s), u(s)\big)\,\mathrm{d}_\alpha s. \end{aligned}$$

Moreover, our analysis can be combined with the results established in [17] to address the well-posedness of nonlocal Cauchy problems involving conformable derivatives.

4 Controllability of fractional semilinear boundary control

In this section, we investigate the exact controllability of the conformable semilinear boundary control system given by (1).

We state the definition of the exact controllability as follows:

Definition 7. Let $T > 0$. System (1) is called p-exactly controllable on $[0,T]$ if, for every state x^* in X, there is a control $u \in L^p([0,T], U)$ such that the corresponding mild solution $x(t) := x(t, x_0, u)$ of (1) satisfies $x(T) = x^*$.

This means that, starting from any initial state, it is possible to steer the system to any desired final state in the space X at time T by choosing a suitable boundary control input. A schematic illustration of this controllability property is provided in Fig. 1.

Another way to express exact p-controllability is through the controllability map denoted as W, which belongs to $\mathcal{L}(L^p([0,T], U), X)$. This map is defined as follows:

$$W(u) = \lim_{\mu\to\infty} \int_0^T S_\alpha\big(\big(T^\alpha - s^\alpha\big)^{1/\alpha}\big)\mu B_\mu u(s)\,\mathrm{d}_\alpha s \tag{7}$$

for all $u \in L^p([0,T], U)$.

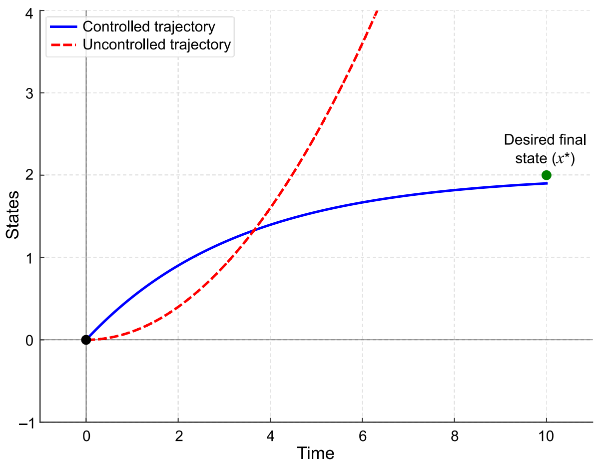

Figure 1. Illustration of exact controllability: controlled vs uncontrolled trajectories.

Hence, system (1) is p-exactly controllable on $[0,T]$ if and only if, for any $x^* \in X$, there exists a control u in $L^p([0,T],U)$ such that $x(\cdot) := x(\cdot, x_0, u)$, the corresponding mild solution of (1), satisfies

$$W(u) = x^* - S_\alpha(T)x_0 - \lim_{\mu\to\infty} \int_0^T S_\alpha\big((T^\alpha - s^\alpha)^{1/\alpha}\big)\mu Q_\mu f\big(s, x(s)\big)\,\mathrm{d}_\alpha s.$$

In the remainder of this article, for the study of p-exact controllability, we will need the following assumption:

(H4) The operator W has an induced inverse operator W^{-1}, whose values can be found in $L^p([0,T],U)/\ker(\mathrm{W})$, and $\|W^{-1}\| < \infty$.

Remark 5. If $(S_\alpha(t))_{t\geqslant 0}$ is a C_0-α-semigroup on X and $u(\cdot) \in W^{1,p}_{\mathrm{loc}}([0,\infty),U)$, then we get

$$\begin{aligned} W(u) &= B_\lambda u(T) - S_\alpha(T)B_\lambda u(0) \\ &\quad + \int_0^T S_\alpha\big((T^\alpha - s^\alpha)^{1/\alpha}\big)B_\lambda\big[\lambda s^{\alpha-1}u(s) - \dot{u}(s)\big]\,\mathrm{d}s \end{aligned}$$

for $T > 0$ and $\lambda \in \rho(A)$. Indeed, we have

$$\begin{aligned} W(u) &= \lim_{\mu\to\infty} \int_0^T S_\alpha\big((T^\alpha - s^\alpha)^{1/\alpha}\big)\mu B_\mu u(s)\,\mathrm{d}_\alpha s, \\ &= \lim_{\mu\to\infty} \int_0^T S_\alpha\big((T^\alpha - s^\alpha)^{1/\alpha}\big)(\lambda - A)R(\lambda, A)\mu B_\mu u(s)\,\mathrm{d}_\alpha s, \end{aligned}$$

$$= \lim_{\mu\to\infty} \int_0^T S_\alpha\big(\big(T^\alpha - s^\alpha\big)^{1/\alpha}\big)\lambda R(\lambda, A)\mu B_\mu u(s)\,\mathrm{d}_\alpha s$$
$$- \lim_{\mu\to\infty} \int_0^T S_\alpha\big(\big(T^\alpha - s^\alpha\big)^{1/\alpha}\big)AR(\lambda, A)\mu B_\mu u(s)\,\mathrm{d}_\alpha s.$$

From (iv) in Lemma 1 we get

$$W(u) = \lim_{\mu\to\infty} \int_0^T S_\alpha\big(\big(T^\alpha - s^\alpha\big)^{1/\alpha}\big)\mu R(\mu, A)\lambda B_\lambda u(s)\,\mathrm{d}_\alpha s$$
$$+ \lim_{\mu\to\infty} \int_0^T D_s^\alpha\big(S_\alpha\big(\big(T^\alpha - s^\alpha\big)^{1/\alpha}\big)\big)\mu R(\mu, A)B_\lambda u(s)\,\mathrm{d}_\alpha s.$$

By using integration by parts, we obtain

$$W(u) = \lim_{\mu\to\infty} \mu R(\mu, A)\Bigg(B_\lambda u(T) - S_\alpha(T)B_\lambda u(0)$$
$$+ \int_0^T S_\alpha\big(\big(T^\alpha - s^\alpha\big)^{1/\alpha}\big)B_\lambda\big[\lambda s^{\alpha-1}u(s) - \dot{u}(s)\big]\,\mathrm{d}s\Bigg).$$

Then the result is obtained by applying the property $\lim_{\mu\to\infty} \mu R(\mu, A)x = x$, where $x \in X$.

Now, we introduce our result on the exact controllability of system (1).

Theorem 4. *If assumptions* (H1)–(H4) *are satisfied, then system* (1) *is p-exactly controllable on* $[0, T]$ *if*

$$\frac{M\gamma kT^\alpha}{\alpha}\left(1 + \frac{T^\alpha M\gamma\|B\|\|W^{-1}\|}{\alpha}\right) < 1,$$

where $M = \sup_{s\in[0,T]} \|S_\alpha(s)\|$.

Proof. Consider an arbitrary point $x^* \in X$. The following control law (feedback) can be proposed:

$$u(t) = W^{-1}\Bigg[x^* - S_\alpha(T)x_0$$
$$- \lim_{\mu\to\infty} \int_0^T S_\alpha\big(\big(T^\alpha - s^\alpha\big)^{1/\alpha}\big)\mu Q_\mu f\big(s, x(s)\big)\,\mathrm{d}_\alpha s\Bigg](t) \tag{8}$$

for all t in $[0,T]$, where with this feedback, $x(t)$ is the solution of (1). Also, for this control, we consider the operator $\Psi : C([0,T],X) \to C([0,T],X)$ given by

$$(\Psi x)(t) = S_\alpha(t)x_0 + \lim_{\mu\to\infty} \int_0^t S_\alpha\bigl((t^\alpha - s^\alpha)^{1/\alpha}\bigr)\bigl(\mu Q_\mu f\bigl(s,x(s)\bigr) + \mu B_\mu u(s)\bigr)\,\mathrm{d}_\alpha s.$$

Now, our objective is to demonstrate the existence of a fixed point x for the operator Ψ (that is the solution of system (1) corresponding to the control law u given in (8)), which satisfies $x(T) = x^*$. For this end, it is only needed to show that Ψ is a contraction. Indeed, let $x_1(t) := x(t,x_0,u_1)$ and $x_2(t) := x(t,x_0,u_2)$ be two solutions of (1) corresponding to feedbacks u_1 and u_2 (that they are defined in (8)), respectively. So, for all $t \in [0,T]$, we have

$$\begin{aligned}(\Psi x_1)(t) - (\Psi x_2)(t) &= \lim_{\mu\to\infty} \int_0^t S_\alpha\bigl((t^\alpha - s^\alpha)^{1/\alpha}\bigr)\mu Q_\mu\bigl(f\bigl(s,x_1(s)\bigr) - f\bigl(s,x_2(s)\bigr)\bigr)\,\mathrm{d}_\alpha s \\ &\quad + \lim_{\mu\to\infty} \int_0^t S_\alpha\bigl((t^\alpha - s^\alpha)^{1/\alpha}\bigr)\mu B_\mu\bigl(u_1(s) - u_2(s)\bigr)\,\mathrm{d}_\alpha s,\end{aligned}$$

and for any $s \in [0,t]$,

$$u_1(s) - u_2(s) = -W^{-1}\Biggl[\lim_{\mu\to\infty} \int_0^T S_\alpha\bigl((T^\alpha - \theta^\alpha)^{1/\alpha}\bigr)\mu Q_\mu\bigl(f\bigl(\theta,x_1(\theta)\bigr) - f\bigl(\theta,x_2(\theta)\bigr)\bigr)\,\mathrm{d}_\alpha \theta\Biggr](s).$$

Using assumptions (H2) and (H3), we obtain

$$\bigl\|u_1(s) - u_2(s)\bigr\| \leqslant \bigl\|W^{-1}\bigr\|M\gamma k \int_0^T \bigl\|x_1(\theta) - x_2(\theta)\bigr\|\,\mathrm{d}_\alpha\theta \leqslant \frac{\|W^{-1}\|M\gamma k T^\alpha}{\alpha}\|x_1 - x_2\|_\infty$$

and

$$\begin{aligned}\bigl\|(\Psi x_1)(t) - (\Psi x_2)(t)\bigr\| &\leqslant M\gamma k \int_0^t \bigl\|x_1(s) - x_2(s)\bigr\|\,\mathrm{d}_\alpha s + M\gamma\|B\| \int_0^t \bigl\|u_1(s) - u_2(s)\bigr\|\,\mathrm{d}_\alpha, \\ &\leqslant \frac{M\gamma k T^\alpha}{\alpha}\|x_1 - x_2\|_\infty + \frac{M\gamma\|B\|T^\alpha}{\alpha}\frac{\|W^{-1}\|M\gamma k T^\alpha}{\alpha}\|x_1 - x_2\|_\infty.\end{aligned}$$

Then it follows that

$$\left\|\Psi x_1 - \Psi x_2\right\|_\infty \leqslant \frac{M\gamma k T^\alpha}{\alpha}\left(1 + \frac{T^\alpha M\gamma\|B\|\|W^{-1}\|}{\alpha}\right)\|x_1 - x_2\|_\infty.$$

Hence, if $(M\gamma k T^\alpha/\alpha)(1 + T^\alpha M\gamma\|B\|\|W^{-1}\|/\alpha) < 1$, then Ψ is a contraction. This completes the proof. □

Remark 6. For the problem mentioned in Remark 4, we can determine a sufficient condition similar to the one in the previous theorem, which ensures the exact controllability of (6). On the other hand, the nonlocal condition $x(0) = g(x) + x_0$ may find more effective application in physics compared to the classical Cauchy problem with the initial condition $x(0) = x_0$, as discussed in [7]. For instance, based on [17] and Theorem 4, we can establish a sufficient condition, which guarantees the exact controllability of the following nonlocal conformable Cauchy problem with semilinear boundary control

$$\begin{aligned} D_t^\alpha x(t) &= A_m x(t), \quad t \geqslant 0, \\ Qx(t) &= f\big(t, x(t)\big) + Bu(t), \quad t \geqslant 0, \\ x(0) &= x_0 + g(x), \end{aligned}$$

where g is uniformly lipschitzien function.

5 Application

In order to illustrate how the abstract results in this manuscript can be applied to real-world scenarios, we investigate the exact controllability of an age-dependent population described by a conformable fractional equation of order $\alpha \in (0, 1]$ defined by the following model (a similar problem is considered in [10]):

$$\begin{aligned} D_t^\alpha p(t,a) &= -\frac{\partial}{\partial a} p(t,a) - \mu(a)p(t,a), \quad a \geqslant 0,\ t \in [0,T], \\ p(t,0) &= \int_0^\infty \beta(a)p(t,a)\,\mathrm{d}a + g\big(t, p(t,\cdot)\big) + u(t), \quad t \in [0,T], \\ p(0,a) &= p_0(a), \quad a \geqslant 0. \end{aligned} \tag{9}$$

Here t and a represent time and age, respectively. The function $p(t,\cdot)$ characterizes the age distribution of a population at time t with the initial age structure at $t = 0$ denoted by $p_0 \in L^1(\mathbb{R}^+, \mathbb{R})$. The fertility rate $\beta \in L^\infty(\mathbb{R}^+, \mathbb{R})$, the mortality rate $\mu \in L^\infty(\mathbb{R}^+, \mathbb{R})$ such that $\mu(a) \geqslant 0$ for all $a \geqslant 0$. $g : [0,T] \times L^1(\mathbb{R}^+, \mathbb{R}) \to \mathbb{R}$ is a boundary nonlinear perturbation and satisfies (H3) for all $T > 0$ with Lipschitz constant $k_g > 0$.

To transform (9) into (1), we take into account the Banach space $X := L^1(\mathbb{R}^+, \mathbb{R})$. Here, on this space, we define

$$\begin{aligned} A_m f &:= -f' - \mu(\cdot)f, \quad f \in D(A_m) := W^{1,1}\big(\mathbb{R}^+, \mathbb{R}\big), \\ Qf &:= f(0), \quad f \in W^{1,1}\big(\mathbb{R}^+, \mathbb{R}\big). \end{aligned}$$

$B = I_{\mathbb{R}}$ and the function f is given by

$$f(t,x) = \int_0^\infty \beta(a)x(a)\,\mathrm{d}a + g(t,x)$$

for all $t \geqslant 0$, $x \in L^1(\mathbb{R}^+, \mathbb{R})$.

In the context of the population model (9), the boundary operator Q represents the evaluation at age zero, that is, $Qf = f(0)$ for any $f \in D(A_m) \subset X$. This reflects the biological reality that new individuals enter the population at age zero, and their number is influenced by the birth process. The boundary condition $p(t,0) = \int_0^\infty \beta(a)p(t,a)\,\mathrm{d}a + g(t, p(t,\cdot)) + u(t)$ models the recruitment rate of newborns, combining the total fertility contribution (via the integral), a nonlinear perturbation g, and an external control action $u(t)$.

Thus, the control acts explicitly and exclusively at the boundary $a = 0$, influencing the population inflow of newborns. The operator Q captures this mechanism abstractly by linking boundary values to the admissible control inputs through the equation $Qx(t) = h(t, x(t), u(t))$. This setup is particularly appropriate for systems such as age-structured population dynamics, where control interventions (e.g., birth policies or external stocking) naturally occur at the origin of the domain.

It is easy to verify assumption (H1). Moreover, $A := A_m|_{\ker Q}$ generates a fractional α-semigroup $(S_\alpha(t))_{t\geqslant 0}$ given by

$$S_\alpha(t)f(a) = \begin{cases} 0, \quad 0 \leqslant a < \frac{t^\alpha}{\alpha}, \\ \mathrm{e}^{-\int_{a-t^\alpha/\alpha}^{a} \mu(s)\,\mathrm{d}s} f(a - \frac{t^\alpha}{\alpha}), \quad a \geqslant \frac{t^\alpha}{\alpha},\ f \in X. \end{cases}$$

It is well know that $\|S_\alpha(t)\| \leqslant 1$ and $M := \sup_{s\in[0,T]} \|S_\alpha(s)\| \leqslant 1$.

On the other hand, assumption (H2) is satisfied. To clarify, let $\lambda \in \rho(A)$ and $f \in \ker(\lambda - A_m)$. We have

$$\begin{aligned} \|Qf\|_{\partial X} &= \bigl|f(0)\bigr| = \left| \int_0^\infty f'(y)\,\mathrm{d}y \right| \\ &\leqslant \int_0^\infty \bigl|-f'(y) - \mu(y)f(y)\bigr|\,\mathrm{d}y + \int_0^\infty \bigl|\mu(y)f(y)\bigr|\,\mathrm{d}y \\ &\leqslant c\|f\|_{D(A_m)}, \end{aligned}$$

where $c = \max\{1, \|\mu\|_\infty\}$ Then Q is surjective linear bounded operator. Moreover, we have

$$\begin{aligned} \|Qf\|_{\partial X} &= \bigl|f(0)\bigr| = \int_0^\infty \bigl|f'(y)\bigr|\,\mathrm{d}y = \int_0^\infty \bigl|\bigl(\lambda + \mu(y)\bigr)f(y)\bigr|\,\mathrm{d}y, \\ &\geqslant \lambda\|f\|_X. \end{aligned}$$

Hence, property (v) in Lemma 1 is true for $\gamma = 1$. Moreover, for all $\lambda \in \rho(A)$, the operator Q_λ is given by

$$(Q_\lambda y)(a) := y q_\lambda(a) := y \mathrm{e}^{-\int_0^a (\lambda+\mu(s))\,\mathrm{d}s}$$

for all $a \geqslant 0$ and $y \in \mathbb{R}$. Assumption (H3) holds for all $T > 0$ with $k = \|\beta\|_\infty + k_g$.

Now, for the linear operator W defined in (7), from the Remark 5, for all $u \in W^{1,2}([0,T],\mathbb{R})$ and $a \geqslant 0$, we have

$$\begin{aligned}
&(Wu)(a) \\
&\quad = B_\lambda u(T) - S_\alpha(T) B_\lambda u(0) + \int_0^T S_\alpha\big(\big(T^\alpha - s^\alpha\big)^{1/\alpha}\big) B_\lambda\big[\lambda s^{\alpha-1} u(s) - \dot{u}(s)\big]\,\mathrm{d}s, \\
&\quad = u(T)\mathrm{e}^{-\int_0^a (\lambda+\mu(r))\,\mathrm{d}r} - u(0)\mathrm{e}^{-\int_0^{a-T^\alpha/\alpha}(\lambda+\mu(r))\,\mathrm{d}r}\mathrm{e}^{-\int_{a-T^\alpha/\alpha}^a \mu(r)\,\mathrm{d}r}\mathbf{1}_{\{a \geqslant T^\alpha/\alpha\}} \\
&\qquad + \int_0^T \big[\lambda s^{\alpha-1}u(s) - \dot{u}(s)\big]\mathrm{e}^{-\int_0^{a-T^\alpha-s^\alpha/\alpha}(\lambda+\mu(r))\,\mathrm{d}r}\mathrm{e}^{-\int_{a-T^\alpha-s^\alpha/\alpha}^a \mu(r)\,\mathrm{d}r} \\
&\qquad\qquad \times \mathbf{1}_{\{a \geqslant T^\alpha - s^\alpha/\alpha\}}\,\mathrm{d}s, \\
&\quad = u(T)\mathrm{e}^{-\int_0^a (\lambda+\mu(r))\,\mathrm{d}r} - u(0)\mathrm{e}^{-\int_0^a \mu(r)\,\mathrm{d}r}\mathrm{e}^{-\lambda(a-T^\alpha/\alpha)}\mathbf{1}_{\{a \geqslant T^\alpha/\alpha\}} \\
&\qquad + \mathrm{e}^{-\int_0^a \mu(r)\,\mathrm{d}r}\int_0^T \big[\lambda s^{\alpha-1}u(s) - \dot{u}(s)\big]\mathrm{e}^{-\lambda(a-T^\alpha-s^\alpha/\alpha)}\mathbf{1}_{\{a \geqslant T^\alpha - s^\alpha/\alpha\}}\,\mathrm{d}s.
\end{aligned}$$

If $a \geqslant T^\alpha/\alpha$, by using integration by parts, we get

$$\begin{aligned}
(Wu)(a) &= u(T)\mathrm{e}^{-\int_0^a (\lambda+\mu(r))\,\mathrm{d}r} - u(0)\mathrm{e}^{-\int_0^a \mu(r)\,\mathrm{d}r}\mathrm{e}^{-\lambda(a-T^\alpha/\alpha)} \\
&\quad + \mathrm{e}^{-\int_0^a \mu(r)\,\mathrm{d}r}\int_0^T \big[\lambda s^{\alpha-1}u(s) - \dot{u}(s)\big]\mathrm{e}^{-\lambda(a-T^\alpha-s^\alpha/\alpha)}\,\mathrm{d}s \\
&= 0,
\end{aligned}$$

and if $a < T^\alpha/\alpha$, we have

$$\begin{aligned}
(Wu)(a) &= u(T)\mathrm{e}^{-\int_0^a (\lambda+\mu(r))\,\mathrm{d}r} - 0 \\
&\quad + \mathrm{e}^{-\int_0^a \mu(r)\,\mathrm{d}r}\int_{(T^\alpha-\alpha a)^{1/\alpha}}^T \big[\lambda s^{\alpha-1}u(s) - \dot{u}(s)\big]\mathrm{e}^{-\lambda(a-T^\alpha-s^\alpha/\alpha)}\,\mathrm{d}s \\
&= \mathrm{e}^{-\int_0^a \mu(r)\,\mathrm{d}r} u\big(\big(T^\alpha - \alpha a\big)^{1/\alpha}\big).
\end{aligned}$$

Then, because W is linear and $W^{1,2}([0,T],\mathbb{R})$ is dense in $L^2([0,T],\mathbb{R})$, we get

$$(Wu)(a) = \begin{cases} \mathrm{e}^{-\int_0^a \mu(r)\,\mathrm{d}r} u((T^\alpha - \alpha a)^{1/\alpha}), & 0 \leqslant a < \frac{T^\alpha}{\alpha}, \\ 0, & a \geqslant \frac{T^\alpha}{\alpha}, \end{cases}$$

for $a \geqslant 0$ and $u \in L^2([0,T],\mathbb{R})$. Therefore, the inverse operator $W^{-1} : L^1(\mathbb{R}^+,\mathbb{R}) \to L^2([0,T],\mathbb{R})/\ker(\mathrm{W})$ is given by

$$(W^{-1}f)(s) = \mathrm{e}^{\int_0^{(T^\alpha - s^\alpha)/\alpha} \mu(r)\,\mathrm{d}r} f\left(\frac{T^\alpha - s^\alpha}{\alpha}\right)$$

for every $f \in L^1(\mathbb{R}^+,\mathbb{R})$ and $s \in [0,T]$. Moreover, $\|W^{-1}\| \leqslant \mathrm{e}^{\int_0^{T^\alpha/\alpha} \mu(r)\,\mathrm{d}r}$. Then it follows that assumption (H4) is true.

Therefore, by Theorem 4, for $\alpha \in]0,1]$, $T > 0$, $\mu(\cdot)$, $\beta(\cdot)$, and $g(\cdot,\cdot)$ such that

$$T^\alpha\left(\frac{\|\beta\|_\infty + k_g}{\alpha}\right)\left(1 + \frac{T^\alpha \mathrm{e}^{\int_0^{T^\alpha/\alpha} \mu(r)\,\mathrm{d}r}}{\alpha}\right) < 1, \tag{10}$$

the abstract age-dependent population equation (9) is 2-exactly controllable on $[0,T]$. It is clear that condition (10) is true if T is a small positive constant. For a numerical example, we assume that $\beta(a) = \mathrm{e}^{-\beta_0 a}\mathbf{1}_{\{a>a_0\}}(a)$, $\mu(a) = \mu_0$ for all $a \geqslant 0$ and $g \equiv 0$, where $a_0, \beta_0, \mu_0 > 0$. Then condition (10) becomes

$$T^\alpha\left(\alpha + T^\alpha \mathrm{e}^{\mu_0 T^\alpha/\alpha}\right) < \alpha^2 \mathrm{e}^{\beta_0 a_0}.$$

6 Conclusion

In this contribution, we study the controllability of infinite-dimensional linear systems with semilinear boundary control, employing conformable derivatives for α in $(0,1]$. Applying the fixed-point theorem within a Banach space, we proved the existence of mild solutions for a class of conformable differential equations of order α in $(0,1]$ with semilinear boundary control. Furthermore, the exact controllability for these mild solutions has been established. The results obtained in this work are supported by a concrete application. In the future work, we intend to study the controllability of the boundary bilinear control systems discussed in [21] with the fractional derivative.

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Conflicts of interest. The authors declare no conflicts of interest.

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