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Monotone iterative sequences for positive solutions of a p-Laplacian Hadamard fractional boundary value problem on an unbounded domain*

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Abstract. In this paper, we ensure the existence and uniqueness of positive solutions for a Hadamard fractional boundary value problem with the p-Laplacian operator on an unbounded domain. The problem is formulated as a nonlinear differential equation involving a fractional derivative of order $\ell \in (n-1, n]$, $n \in \mathbb{N}$, along with boundary conditions of the Hadamard fractional integral and derivative. Using the monotone iterative technique, we establish the existence of positive solutions by constructing monotone sequences that approach the solution. An error estimation formula is provided. An example is also discussed to illustrate the main result.

Keywords: existence results, infinite interval, fractional derivative.

1 Introduction

The study of fractional calculus has proven to be a versatile mathematical approach for analyzing systems characterized by memory and hereditary effects. Its growing application spans areas such as physics, engineering, economics, biomedical sciences, and signal processing. Unlike traditional integer-order calculus, fractional derivatives and integrals provide a nonlocal perspective, making them ideal for modeling phenomena like disease models [3,5,7,23], bioengineering applications, financial mathematics, chaos [9, 20, 30], and ecological models [14]. For the fundamental theories of fractional calculus, its wide-ranging applications in various scientific and technological fields and

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significant developments in fractional differential equations in recent years, we refer to the works of Kilbas et al. [17], Miller and Ross [21], Podlubny [22], Samko et al. [25]. The properties of Hadamard fractional integral and derivative can be found in [16]. The monograph by Lakshmikantham et al. [18] establishes a rigorous theoretical framework for fractional dynamic systems, including existence theorems and stability analysis for fractional-order differential equations.

The Hadamard fractional derivative (HFD for short) is particularly noteworthy for its distinctive logarithmic kernel among the diverse definitions of fractional derivatives. First introduced in 1892 [13], the HFD sets itself apart from the commonly used Riemann–Liouville and Caputo fractional derivatives. The HFD involves a logarithmic function with an arbitrary exponent in the integral kernel, making it particularly relevant in fields like fracture analysis and other mechanical problems. While most of the recent research has focused on Riemann–Liouville and Caputo fractional differential equations, the study of Hadamard fractional equations remains in its infancy and presents a promising area for further investigation. In [11], a modified Lamb–Bateman equation with the Hadamard derivative was analyzed. The authors introduced some operators that include Hadamard derivatives and demonstrated the benefits of their approach through some examples. Garra et al. [10] proposed a new method using linear integro-differential operators with logarithmic kernel based on Hadamard fractional calculus for a generalization of the Lomnitz logarithmic creep law. Hadamard fractional calculus shows the mathematical essential of this creep law more exactly [10]. At the same time, Hadamard derivatives provide stable formulations for expansion and work well on half-open intervals; see [25]. For our half-open interval problem, we employ Hadamard fractional derivatives.

Recent advancements in Hadamard fractional boundary value problems rely on mathematical tools, including fixed point theorems, the monotone iterative method, and topological degree techniques. These methods play a crucial role in establishing the existence, uniqueness, and extremal properties of solutions. Lately, Ahmad, Ntouyas, and Alsaedi [1] considered boundary value problem of Hadamard-type fractional differential inclusions. They developed various findings for the inclusion problem by treating the multivalued map as either convex-valued or nonconvex-valued. In [4], Benhamida, Graef, and Hamani applied Banach's fixed point theorem, Schaefer's fixed point theorem, and the Leray–Schauder nonlinear alternative to obtain the existence and uniqueness of solutions for the studied problem. Senlik Cerdik [26] ensured some sufficient conditions to obtain the existence of nontrivial solutions to a new kind of Hadamard fractional boundary value problem on an unbounded domain by using the nonlinear alternative theorem of Leray–Schauder and the Avery–Peterson fixed point theorem. In [2, 27, 29], by applying fixed point theory and monotone iterative method, the existence of solutions to nonlinear Hadamard fractional differential equations was verified for bounded-unbounded domains.

Also, the study of solutions to the Hadamard fractional boundary value problem with the p-Laplacian operator remains an ongoing research area with increasing interest from researchers [12, 15, 24, 28]. Additionally, the nonlinear part f, which incorporates the HFD, has been investigated by only a limited number of researchers, as noted in [6, 12].

Despite notable advancements, the exploration of Hadamard fractional differential equations is still in its infancy. According to our knowledge, there are no existing studies specifically dedicated to the investigation of positive solutions for Hadamard fractional differential equations on infinite intervals with the p-Laplacian. This paper aims to contribute to the advancement of this field by addressing key aspects of Hadamard fractional differential equations such as existence of solutions, iterative techniques, and practical example.

In [8], Du et al. considered the following problem:

$$\begin{aligned} &D_z^{a_1}x(z) + f_1\big(z, x(z), v(z), D_z^{a_1-1}x(z), D_z^{a_2-1}v(z)\big) = 0, \quad z \in (1,\infty), \\ &D_z^{a_2}v(z) + f_2\big(z, x(z), v(z), D_z^{a_1-1}x(z), D_z^{a_2-1}v(z)\big) = 0, \quad z \in (1,\infty), \\ &x(1) = 0, \qquad D_z^{a_1-1}x(\infty) = \sum_{i=1}^{m} \lambda_{1i} I_z^{\beta_{1i}} v(\xi_i) + \sum_{j=1}^{n} b_j v(\eta_j), \\ &v(1) = 0, \qquad D_z^{a_2-1}v(\infty) = \sum_{i=1}^{m} \lambda_{2i} I_z^{\beta_{2i}} x(\xi_i) + \sum_{j=1}^{n} b_j x(\eta_j) \end{aligned}$$

in which D_z^{ak} are Hadamard fractional derivatives of order ak, and $I_z^{\beta_{ki}}$ is the Hadamard fractional integral of order β_{ki}, $k = 1, 2$, $i = 1, 2, \dots, m$. This paper focused on the existence of positive solutions for a nonlinear system of coupled Hadamard fractional differential equations.

In [19], the researchers investigated the following problem:

$$\begin{aligned} &{}^H D_{1+}^{\wp} x(z) + r(z) f\big(z, x(z), {}^H D_{1+}^{\wp-2} x(z), {}^H D_{1+}^{\wp-1} x(z)\big) = 0, \quad z \in (1,\infty), \\ &x(1) = x'(1) = 0, \qquad {}^H D_{1+}^{\wp-1} x(\infty) = \int_1^{\infty} \omega(z) x(z) \frac{\mathrm{d}z}{z} + \sum_{i=1}^{m} \beta_i x(\eta_i) \end{aligned}$$

in which ${}^H D_{1+}^{\wp}$ is the Hadamard fractional derivatives of order $\wp$, $2 < \wp \leqslant 3$, $f \in \mathcal{C}([1,\infty) \times [0,\infty) \times [0,\infty), [0,\infty))$. Solutions of Hadamard fractional differential equations were analyzed on an infinite interval by applying the monotone iterative technique.

In [32], Zhang et al. studied the following problem:

$$\begin{aligned} &{}^H D_{1+}^{\wp} x(z) + a(z) f\big(z, x(z)\big) = 0, \quad 2 < \wp < 3, \ z \in (1,\infty), \\ &x(1) = x'(1) = 0, \qquad {}^H D_{1+}^{\wp-1} x(\infty) = \sum_{i=1}^{m} \alpha_i {}^H I_{1+}^{\beta_i} x(\eta) + b \sum_{j=1}^{n} \sigma_j x(\xi_j), \end{aligned}$$

where ${}^H D_{1+}^{\wp}$ is the Hadamard fractional derivative of order $\wp$, ${}^H I_{1+}^{\beta_i}$ is the Hadamard fractional integral of order $\beta_i > 0$, $i = 1, 2, \dots, m$. The authors explored a class of Hadamard fractional differential equations subject to nonlocal boundary conditions defined on an infinite interval. Via the use of fixed point theorems, they established novel results regarding the existence, uniqueness, and multiplicity of positive solutions.

In [31], by applying two fixed point theorems of a sum operator, the authors examined the local existence and uniqueness of positive solution for the Hadamard fractional boundary value problem in the following form:

$$
\begin{aligned}
&{}^{H}D_{1^+}^{\wp}x(z)+a(z)f\big(z,x(z)\big)+b(z)g\big(z,x(z)\big)=0, \quad z\in(1,\infty),\\
&x(1)=x'(1)=0, \quad {}^{H}D_{1^+}^{\wp-1}x(\infty)=\sum_{i=1}^{m}\alpha_i\,{}^{H}I_{1^+}^{\beta_i}x(\eta)+c\sum_{j=1}^{n}\sigma_j x(\xi_j),
\end{aligned}
$$

where ${}^{H}D_{1^+}^{\wp}$ is the Hadamard fractional derivative of order $\wp$, $2<\wp<3$, ${}^{H}I_{1^+}^{\beta_i}$ is the Hadamard fractional integral of order $\beta_i>0$, $i=1,2,\dots,m$.

Motivated by the aforementioned papers, we derive new results for the following p-Laplacian fractional boundary value problem involving the HFD:

$$
\begin{aligned}
&\varphi_p\big({}^{H}\mathfrak{D}_{1^+}^{\ell}\varpi(z)\big)\\
&\quad+\tau\big(z,\varpi(z),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(z),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(z)\big)=0, \quad z\in[1,\infty),\\
&\varpi^{(k)}(1)=0, \quad 0\leqslant k\leqslant n-2, \qquad (1)\\
&{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(\infty)=\sum_{i=1}^{m}a_i^{H}\mathfrak{D}_{1^+}^{r}\varpi(\xi_i)+\sum_{j=1}^{k}b_j^{H}I_{1^+}^{c}\varpi(\eta_j),
\end{aligned}
$$

where $n\in\mathbb{N}$, $n\geqslant 3$, $\ell\in(n-1,\,n]$, ${}^{H}\mathfrak{D}_{1^+}^{\ell}$ is the Hadamard fractional derivative of order ℓ, ${}^{H}I_{1^+}^{c}$ is the Hadamard fractional integral of order c, $r\in[0,\,\ell-1]$, $c>0$, $a_i,b_i\geqslant 0$ with $1\leqslant i\leqslant m$, $1\leqslant j\leqslant k$, and $1<\xi_1<\xi_2<\dots<\xi_m<\infty$, $1<\eta_1<\eta_2<\dots<\eta_k<\infty$ are given constants. $\varphi_p(s)$ is the p-Laplacian operator, i.e., $\varphi_p(s)=|s|^{p-2}s$, $p>1$, $\varphi_p^{-1}=\varphi_q$, $1/p+1/q=1$.

This paper aims to provide a comprehensive review of the Hadamard fractional boundary value problem with the p-Laplacian operator on an infinite interval, focusing on the Green's function method, the monotone iterative method, and the solution of problem (1) converted into an integral equation. The proof of our main result is established using the monotone iterative method, which provides two key advantages. First, by selecting a suitably simple initial function, we ensure the existence of solutions through an effective and concise process. Second, this method enables us to approximate a positive solution with adjustable precision levels, adding significant practical value. It is worth mentioning that this technique has not been previously applied to Hadamard fractional boundary value problems with the nonlinear term τ containing lower-order HFD operators ${}^{H}\mathfrak{D}_{1^+}^{\ell-1},\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}$, $n\geqslant 3$, on an unbounded domain. Finally, we present an error estimate formula. In comparison to previous studies [1,12,27,29,32], our problem is more general than those studied problems. To the best of the authors' knowledge, the boundary value problem for Hadamard fractional differential equations with the p-Laplacian operator on infinite intervals has not been studied before. This work aims to fill that gap in the existing literature.

2 Preliminaries

In this section, we revisit essential definitions and fundamental results concerning fractional calculus theory. For details on fractional calculus, please refer to [17].

Definition 1. (See [17].) For a function $a:[1,\infty)\to\mathbb{R}$, the HFD of fractional order y is expressed as

$$ {}^{H}\mathfrak{D}_{1+}^{y}a(z)=\frac{1}{\Gamma(k-y)}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{k}\int\limits_{1}^{z}\left(\log\frac{z}{s}\right)^{k-y-1}a(s)\frac{\mathrm{d}s}{s},\quad k-1<y<k, $$

where $k=[y]+1$, $[y]$ shows the integer part of the real number y, and $\log(\cdot)=\log_{\mathrm{e}}(\cdot)$.

Definition 2. (See [17].) The Hadamard fractional integral of order y for a function $a:[1,\infty)\to\mathbb{R}$ is expressed as

$$ {}^{H}I_{1+}^{y}a(z)=\frac{1}{\Gamma(y)}\int\limits_{1}^{z}\left(\log\frac{z}{s}\right)^{y-1}a(s)\frac{\mathrm{d}s}{s},\quad y>0, $$

in the event that the associated integral exists.

Lemma 1. (*See* [17].) *Let $a\in\mathcal{C}[1,\infty)\cap L^{1}[1,\infty)$ and $y>0$, then the solution of Hadamard fractional differential equation ${}^{H}\mathfrak{D}_{1+}^{y}a(z)=0$ is expressed as*

$$ a(z)=\sum_{j=1}^{k}t_{j}(\log z)^{y-j}, $$

and

$$ {}^{H}I_{1+}^{y}\,{}^{H}\mathfrak{D}_{1+}^{y}a(z)=a(z)+\sum_{j=1}^{k}t_{j}(\log z)^{y-j}, $$

where $t_j\in\mathbb{R}$, $j=1,2,\dots,k$, $k-1<y<k$, $k=[y]+1$.

Lemma 2. (*See* [17].) *If $\mu>y>0$, $a>0$, and $f\in L^{1}(a,\infty)$, then the compositions between the Hadamard fractional integration and differentiation*

$$ {}^{H}\mathfrak{D}_{a}^{y}\,{}^{H}I_{a}^{\mu}f={}^{H}I_{a}^{\mu-y}f $$

holds. We note that, if $a,y,\mu>0$, then

$$ {}^{H}\mathfrak{D}_{a}^{y}\left(\log\frac{z}{a}\right)^{\mu-1}(x)=\frac{\Gamma(\mu)}{\Gamma(\mu-y)}\left(\log\frac{x}{a}\right)^{\mu-y-1} $$

holds.

Now, let us denote

$$ \begin{aligned} X=\biggl\{\varpi\colon\ &\varpi,\ {}^{H}\mathfrak{D}_{1^{+}}^{\ell-1}\varpi,\ {}^{H}\mathfrak{D}_{1^{+}}^{\ell-i}\varpi\in\mathcal{C}\bigl([1,\infty)\bigr),\ \left\|\frac{\varpi}{1+(\log z)^{\ell-1}}\right\|_{\infty}<\infty,\\ &\bigl\|{}^{H}\mathfrak{D}_{1^{+}}^{\ell-1}\varpi\bigr\|_{\infty}<\infty,\ \left\|\frac{{}^{H}\mathfrak{D}_{1^{+}}^{\ell-i}\varpi}{1+(\log z)^{i-1}}\right\|_{\infty}<\infty,\ i=2,\dots,n-1\biggr\}, \end{aligned} $$

equipped with the norm

$$\|\varpi\|_X = \max\left\{\left\|\frac{\varpi}{1+(\log z)^{\ell-1}}\right\|_\infty, \left\|{}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi\right\|_\infty, \left\|\frac{{}^H\mathfrak{D}_{1^+}^{\ell-i}\varpi}{1+(\log z)^{i-1}}\right\|_\infty, \ i=2,\dots,n-1\right\},$$

where $\|\varpi\|_\infty = \sup_{z\in[1,\infty)} |\varpi(z)|$. It is obvious that

$$\varphi_p(z+s) = \begin{cases} 2^{p-1}(\varphi_p(z)+\varphi_p(s)), & p \geqslant 2,\ z,s>0,\\ \varphi_p(z)+\varphi_p(s), & 1<p<2,\ z,s>0,\end{cases}$$

$$\varphi_p^{-1}(z+s) = \begin{cases} 2^{1/(p-1)}(\varphi_p^{-1}(z)+\varphi_p^{-1}(s)), & p \geqslant 2,\ z,s>0,\\ \varphi_p^{-1}(z)+\varphi_p^{-1}(s), & 1<p<2,\ z,s>0.\end{cases}$$

We adopt the following assumptions that we will use in the forthcoming section:

(K_0) $\tau \in \mathcal{C}([1,\infty)\times[0,\infty)^n, [0,\infty))$, $\tau(z,0,\dots,0)$ is not equal to 0 for $z\in[1,\infty)$;

(K_1) There exist nonnegative functions $r_0(z), r_i(z) \in L^1[1,\infty)$ and constants $\lambda_i \geqslant 0$ such that

$$\tau(z,\varpi_1,\dots,\varpi_n) \leqslant r_0(z) + \sum_{i=1}^n r_i(z)\,\varphi_p\bigl(\varpi_i^{\lambda_i}\bigr)$$

for any $z\in[1,\infty)$ and $\varpi_i\in[0,\infty)$, $i=1,\dots,n$, and note

$$r_0 = \int_1^\infty \varphi_q\bigl(r_0(z)\bigr)\frac{\mathrm{d}z}{z} < \infty, \quad r_1 = \int_1^\infty \varphi_q\bigl(r_1(z)\bigr)\bigl(1+(\log z)^{\ell-1}\bigr)^{\lambda_1}\frac{\mathrm{d}z}{z} < \infty,$$

$$r_i = \int_1^\infty \varphi_q\bigl(r_i(z)\bigr)\bigl(1+(\log z)^{i-2}\bigr)^{\lambda_i}\frac{\mathrm{d}z}{z} < \infty, \quad i=2,3,\dots,n;$$

(K_2) $\tau(z,u_0,u_1,\dots,u_{n-1}) \leqslant \tau(z,\varpi_0,\varpi_1,\dots,\varpi_{n-1})$ if and only if $u_i\leqslant\varpi_i$ for any $z\in[1,\infty)$, $u_i,\varpi_i\in[0,\infty)$, $i=0,1,\dots,n-1$.

Lemma 3. *Let $h\in\mathcal{C}[1,\infty)$ with $\int_1^\infty \varphi_q(h(s))\,\mathrm{d}s/s<\infty$ and*

$$\Delta = \Gamma(\ell) - \sum_{i=1}^m \frac{a_i\Gamma(\ell)}{\Gamma(\ell-r)}(\log\xi_i)^{\ell-r-1} - \sum_{j=1}^k \frac{b_j\Gamma(\ell)}{\Gamma(\ell+c)}(\log\eta_j)^{\ell+c-1}$$

with $\Delta>0$. Then $\varpi\in X$ is a solution of

$$\begin{aligned} &\varphi_p\bigl({}^H\mathfrak{D}_{1^+}^{\ell}\varpi(z)\bigr) + h(z) = 0, \quad z\in[1,\infty),\\ &\varpi^{(k)}(1)=0, \quad 0\leqslant k\leqslant n-2,\\ &{}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(\infty) = \sum_{i=1}^m a_i\,{}^H\mathfrak{D}_{1^+}^{r}\varpi(\xi_i) + \sum_{j=1}^k b_j\,{}^HI_{1^+}^{c}\varpi(\eta_j) \end{aligned} \tag{2}$$

on the condition that ϖ fulfills the integral equation

$$\varpi(z) = \int_1^\infty H(z,s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s}, \quad z \in [1,\infty),$$

where

$$H(z,s) = \aleph_\ell(z,s) + \frac{(\log z)^{\ell-1}}{\Delta}\left[\sum_{i=1}^m a_i\aleph_{\ell-r}(\xi_i,s) + \sum_{j=1}^k b_j\aleph_{\ell+c}(\eta_j,s)\right], \tag{3}$$

$$\aleph_\ell(z,s) = \frac{1}{\Gamma(\ell)}\begin{cases}(\log z)^{\ell-1} - (\log(\frac{z}{s}))^{\ell-1}, & 1 \leqslant s \leqslant z < \infty,\\ (\log z)^{\ell-1}, & 1 \leqslant z \leqslant s < \infty,\end{cases}$$

$$\aleph_{\ell-r}(\xi_i,s) = \frac{1}{\Gamma(\ell-r)}\begin{cases}(\log \xi_i)^{\ell-r-1} - (\log \frac{\xi_i}{s})^{\ell-r-1}, & 1 \leqslant s \leqslant \xi_i < \infty,\\ (\log \xi_i)^{\ell-r-1}, & 1 \leqslant \xi_i \leqslant s < \infty,\end{cases}$$

$$\aleph_{\ell+c}(\eta_j,s) = \frac{1}{\Gamma(\ell+c)}\begin{cases}(\log \eta_j)^{\ell+c-1} - (\log \frac{\eta_j}{s})^{\ell+c-1}, & 1 \leqslant s \leqslant \eta_j < \infty,\\ (\log \eta_j)^{\ell+c-1}, & 1 \leqslant \eta_j \leqslant s < \infty.\end{cases}$$

Proof. Considering the definitions and characteristics linked to fractional integrals and derivatives, the equation will be formulated as follows:

$$\varpi(z) = c_1(\log z)^{\ell-1} - \frac{1}{\Gamma(\ell)}\int_1^z (z-s)^{\ell-1}\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s}$$

for some $c_1 \in \mathbb{R}$. By implementing the operator ${}^H\mathfrak{D}_{1^+}^{\ell-1}$ to both sides of the fractional integral equation

$${}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(z) = c_1\Gamma(\ell) - \int_1^z \varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s},$$

and noting that

$${}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(\infty) = \sum_{i=1}^m a_i {}^H\mathfrak{D}_{1^+}^{r}\varpi(\xi_i) + \sum_{j=1}^k b_j {}^H I_{1^+}^{c}\varpi(\eta_j),$$

we acquire

$$\begin{aligned} c_1 = \frac{1}{\Delta}\Bigg(&\int_1^\infty \varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s} - \sum_{i=1}^m \frac{a_i}{\Gamma(\ell-r)}\int_1^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\ell-r-1}\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s} \\ &- \sum_{j=1}^k \frac{b_j}{\Gamma(\ell+c)}\int_1^{\eta_j}\left(\log\frac{\eta_j}{s}\right)^{\ell+c-1}\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s}\Bigg). \end{aligned} \tag{4}$$

Plugging c_1 expressed by (4), we derive

$$
\begin{aligned}
\varpi(z) ={}& -\frac{1}{\Gamma(\ell)}\int_1^z\left(\log\frac{z}{s}\right)^{\ell-1}\varphi_q\bigl(h(s)\bigr)\frac{\mathrm{d}s}{s}+\frac{(\log z)^{\ell-1}}{\Delta}\int_1^\infty\varphi_q\bigl(h(s)\bigr)\frac{\mathrm{d}s}{s}\\
&-\frac{(\log z)^{\ell-1}}{\Delta}\sum_{i=1}^m\frac{a_i}{\Gamma(\ell-r)}\int_1^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\ell-r-1}\varphi_q\bigl(h(s)\bigr)\frac{\mathrm{d}s}{s}\\
&-\frac{(\log z)^{\ell-1}}{\Delta}\sum_{j=1}^k\frac{b_j}{\Gamma(\ell+c)}\int_1^{\eta_j}\left(\log\frac{\eta_j}{s}\right)^{\ell+c-1}\varphi_q\bigl(h(s)\bigr)\frac{\mathrm{d}s}{s}\\
={}&\frac{1}{\Gamma(\ell)}\int_1^z\left[(\log z)^{\ell-1}-\left(\log\frac{z}{s}\right)^{\ell-1}\right]h(s)\frac{\mathrm{d}s}{s}+\frac{1}{\Gamma(\ell)}\int_z^\infty(\log z)^{\ell-1}\varphi_q\bigl(h(s)\bigr)\frac{\mathrm{d}s}{s}\\
&-\frac{(\log z)^{\ell-1}}{\Delta}\sum_{i=1}^m\frac{a_i}{\Gamma(\ell-r)}\int_1^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\ell-r-1}\varphi_q\bigl(h(s)\bigr)\frac{\mathrm{d}s}{s}\\
&-\frac{(\log z)^{\ell-1}}{\Delta}\sum_{j=1}^k\frac{b_j}{\Gamma(\ell+c)}\int_1^{\eta_j}\left(\log\frac{\eta_j}{s}\right)^{\ell+c-1}\varphi_q\bigl(h(s)\bigr)\frac{\mathrm{d}s}{s}\\
&+\sum_{i=1}^m\frac{a_i(\log\xi_i)^{\ell-r-1}}{\Delta\Gamma(\ell-r)}\int_1^\infty(\log z)^{\ell-1}\varphi_q\bigl(h(s)\bigr)\frac{\mathrm{d}s}{s}\\
&+\sum_{j=1}^k\frac{b_j(\log\eta_j)^{\ell+c-1}}{\Delta\Gamma(\ell+c)}\int_1^\infty(\log z)^{\ell-1}\varphi_q\bigl(h(s)\bigr)\frac{\mathrm{d}s}{s}\\
={}&\int_1^\infty\aleph_\ell(z,s)\varphi_q\bigl(h(s)\bigr)\frac{\mathrm{d}s}{s}+\sum_{i=1}^m\frac{a_i}{\Delta}(\log z)^{\ell-1}\int_1^\infty\aleph_{\ell-r}(\xi_i,s)\varphi_q\bigl(h(s)\bigr)\frac{\mathrm{d}s}{s}\\
&+\sum_{j=1}^k\frac{b_j}{\Delta}(\log z)^{\ell-1}\int_1^\infty\aleph_{\ell+c}(\eta_j,s)\varphi_q\bigl(h(s)\bigr)\frac{\mathrm{d}s}{s}\\
={}&\int_1^\infty H(z,s)\varphi_q\bigl(h(s)\bigr)\frac{\mathrm{d}s}{s}.
\end{aligned}
$$
□

Lemma 4. *By using Lemma* 3*, we have*

$$
{}^H\mathfrak{D}_{1^+}^{\ell-i}\varpi(z)=\int_1^\infty H_i^*(z,s)\varphi_q\bigl(h(s)\bigr)\frac{\mathrm{d}s}{s},\quad i=1,2,\dots,n-1,\tag{5}
$$

where

$$H_i^*(z,s) = \aleph_{\ell_i}^*(z,s) + k(s)\frac{(\log z)^{i-1}}{\Gamma(i)}\Gamma(\ell), \tag{6}$$

$$\aleph_{\ell_i}^*(z,s) = \frac{1}{\Gamma(i)}\begin{cases}(\log z)^{i-1} - (\log \frac{z}{s})^{i-1}, & 1 \leqslant s \leqslant z < \infty,\\ (\log z)^{i-1}, & 1 \leqslant z \leqslant s < \infty,\end{cases}$$

and

$$k(s) = \frac{1}{\Delta}\left[\sum_{i=1}^{m} a_i \aleph_{\ell-r}(\xi_i,s) + \sum_{j=1}^{k} b_j \aleph_{\ell+c}(\eta_j,s)\right].$$

Proof. From Lemma 3 the solution of problem (2) can be expressed as

$$\begin{aligned}\varpi(z) = {}& \frac{1}{\Gamma(\ell)}\int_1^z \left[(\log z)^{\ell-1} - \left(\log\frac{z}{s}\right)^{\ell-1}\right]\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s}\\ &+ \frac{1}{\Gamma(\ell)}\int_z^\infty (\log z)^{\ell-1}\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s} + (\log z)^{\ell-1}\int_1^\infty k(s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s}.\end{aligned}$$

Then we can get

$$\begin{aligned}{}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(z) ={}& {}^H\mathfrak{D}_{1^+}^{\ell-1}\left[-{}^HI_{1^+}^{\ell}\varphi_q\big(h(z)\big) + \frac{1}{\Gamma(\ell)}\int_1^\infty (\log z)^{\ell-1}\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s}\right.\\ &\left.+ (\log z)^{\ell-1}\int_1^\infty k(s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s}\right]\\ ={}& -\int_1^z \varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s} + \int_1^\infty \varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s} + \Gamma(\ell)\int_1^\infty k(s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s}\\ ={}& \int_1^\infty \aleph_{\ell_1}^*(z,s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s} + \Gamma(\ell)\int_1^\infty k(s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s}\\ ={}& \int_1^\infty H_1^*(z,s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s},\end{aligned}$$

$$\begin{aligned}{}^H\mathfrak{D}_{1^+}^{\ell-2}\varpi(z) ={}& {}^H\mathfrak{D}_{1^+}^{\ell-2}\left[-{}^HI_{1^+}^{\ell}\varphi_q\big(h(z)\big) + \frac{1}{\Gamma(\ell)}\int_1^\infty (\log z)^{\ell-1}\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s}\right.\\ &\left.+ (\log z)^{\ell-1}\int_1^\infty k(s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s}\right]\end{aligned}$$

$$
\begin{aligned}
&= -{}^{H}I_{1+}^{2}\varphi_q\big(h(z)\big) + \frac{\log z}{\Gamma(2)}\int_1^\infty \varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s} \\
&\quad + \frac{\log z}{\Gamma(2)}\Gamma(\ell)\int_1^\infty k(s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s} \\
&= \int_1^\infty \aleph_{\ell_2}^{*}(z,s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s} + \frac{\log z}{\Gamma(2)}\Gamma(\ell)\int_1^\infty k(s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s} \\
&= \int_1^\infty H_2^{*}(z,s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

By the same steps, organizing the above two HFDs of ϖ, we get

$$
\begin{aligned}
{}^{H}\mathfrak{D}_{1+}^{\ell-i}\varpi(z) &= -{}^{H}I_{1+}^{i}\varphi_q\big(h(z)\big) + \frac{(\log z)^{i-1}}{\Gamma(i)}\int_1^\infty \varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s} \\
&\quad + \frac{(\log z)^{i-1}}{\Gamma(i)}\Gamma(\ell)\int_1^\infty k(s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s} \\
&= \int_1^\infty \aleph_{\ell_i}^{*}(z,s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s} + \frac{(\log z)^{i-1}}{\Gamma(i)}\Gamma(\ell)\int_1^\infty k(s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s} \\
&= \int_1^\infty H_i^{*}(z,s)\varphi_q\big(h(s)\big)\frac{\mathrm{d}s}{s}, \quad i = 1,2,\dots,n-1.
\end{aligned}
$$

□

The procedure for achieving the following lemma is quite straightforward.

Lemma 5. *$H(z,s)$ and $H_i^{*}(z,s)$ expressed by* (3), (6) *fulfill the conditions stated below*:

(i) $0 \leqslant H(z,s) \leqslant (\log z)^{\ell-1}\dfrac{1}{\varDelta}, \quad 0 \leqslant \dfrac{H(z,s)}{1+(\log z)^{\ell-1}} \leqslant \dfrac{1}{\varDelta}, \quad z,s \in [1,\infty),$

(ii) $0 \leqslant H_i^{*}(z,s) \leqslant \dfrac{(\log z)^{i-1}}{\Gamma(i)}\dfrac{\Gamma(\ell)}{\varDelta}, \quad i = 1,2,\dots,n-1,$

$0 \leqslant \dfrac{H_i^{*}(z,s)}{1+(\log z)^{i-1}} \leqslant \dfrac{1}{\Gamma(i)}\dfrac{\Gamma(\ell)}{\varDelta}, \quad i = 2,\dots,n-1,\ z,s \in [1,\infty).$

Proof. (i) For $z,s \in [1,\infty)$, use

$$
\sum_{i=1}^{m} a_i\frac{(\log \xi_i)^{\ell-r-1}}{\Gamma(\ell-r)} + \sum_{j=1}^{k} b_j\frac{(\log \eta_j)^{\ell+c-1}}{\Gamma(\ell+c)} = \frac{\Gamma(\ell)-\varDelta}{\Gamma(\ell)}.
$$

From Lemma 3 we get

$$\begin{aligned} 0 \leqslant H(z,s) &= \aleph_\ell(z,s) + \frac{(\log z)^{\ell-1}}{\Delta}\left[\sum_{i=1}^{k} a_i \aleph_{\ell-r}(\xi_i,s) + \sum_{j=1}^{m} b_j \aleph_{\ell+c}(\eta_j,s)\right] \\ &\leqslant \frac{(\log z)^{\ell-1}}{\Gamma(\ell)} + \frac{(\log z)^{\ell-1}}{\Delta}\left[\sum_{i=1}^{m} a_i \frac{(\log \xi_i)^{\ell-r-1}}{\Gamma(\ell-r)} + \sum_{j=1}^{k} b_j \frac{(\log \eta_j)^{\ell+c-1}}{\Gamma(\ell+c)}\right] \\ &\leqslant (\log z)^{\ell-1}\frac{1}{\Delta}. \end{aligned}$$

Also, for $z, s \in [1,\infty)$,

$$0 \leqslant \frac{H(z,s)}{1+(\log z)^{\ell-1}} \leqslant \frac{1}{\Delta}.$$

(ii) For $z, s \in [1,\infty)$ and $i = 1,2,\dots,n-1$, from Lemma 4 we ensure

$$\begin{aligned} 0 &\leqslant H_i^*(z,s) = \aleph_{\ell_i}^*(z,s) + k(s)\frac{(\log z)^{i-1}}{\Gamma(i)}\Gamma(\ell) \\ &\leqslant \frac{(\log z)^{i-1}}{\Gamma(i)} + \frac{1}{\Delta}\left[\sum_{i=1}^{m} a_i \frac{(\log \xi_i)^{\ell-r-1}}{\Gamma(\ell-r)} + \sum_{j=1}^{k} b_j \frac{(\log \eta_j)^{\ell+c-1}}{\Gamma(\ell+c)}\right]\frac{(\log z)^{i-1}}{\Gamma(i)}\Gamma(\ell) \\ &= \frac{(\log z)^{i-1}}{\Gamma(i)}\frac{\Gamma(\ell)}{\Delta}. \end{aligned}$$

Then, for $i = 2,\dots,n-1$,

$$0 \leqslant \frac{H_i^*(z,s)}{1+(\log z)^{i-1}} \leqslant \frac{1}{\Gamma(i)}\frac{\Gamma(\ell)}{\Delta}. \qquad \square$$

Lemma 6. *The space $(X, \|\cdot\|_X)$ is a Banach space.*

Proof. Clearly, $(X, \|\cdot\|_X)$ is a normed space. Let $\{u_k\}_{k=1}^\infty$ be a Cauchy sequence in the space $(X, \|\cdot\|_X)$. Then the sequences

$$\left\{\frac{u_k}{1+(\log z)^{\ell-1}}\right\}_{k=1}^{\infty}, \qquad \left\{{}^H\mathfrak{D}_{1^+}^{\ell-1} u_k\right\}_{k=1}^{\infty}, \qquad \left\{\frac{{}^H\mathfrak{D}_{1^+}^{\ell-i} u_k}{1+(\log z)^{i-1}}\right\}_{k=1}^{\infty}, \qquad i = 2,\dots,n-1,$$

are Cauchy sequences in the space $(C[1,\infty), \|\cdot\|_\infty)$. Assume that they converge to

$$\frac{u}{1+(\log z)^{\ell-1}}, \qquad \varpi, \qquad \frac{\varpi_i}{1+(\log z)^{i-1}}, \qquad i = 2,\dots,n-1,$$

respectively. Obviously,

$$\lim_{k\to\infty} u_k(z) = u(z), \qquad \lim_{k\to\infty} {}^H\mathfrak{D}_{1^+}^{\ell-1} u_k(z) = \varpi(z), \qquad \lim_{k\to\infty} {}^H\mathfrak{D}_{1^+}^{\ell-i} u_k(z) = \varpi_i(z)$$

for $z \in [1, \infty)$, $i = 2, \dots, n-1$. At the moment, we ensure that

$$\varpi_i = {}^H\mathfrak{D}_{1^+}^{\ell-i} u, \quad i = 2, \dots, n-1, \qquad \varpi = {}^H\mathfrak{D}_{1^+}^{\ell-1} u.$$

Lemma 1 implies that

$$\begin{aligned}
&{}^HI_{1^+}^{\ell-1}\, {}^H\mathfrak{D}_{1^+}^{\ell-1} u_k \\
&\quad = u_k + d_1^{(1)} (\log z)^{\ell-2} + d_2^{(1)} (\log z)^{\ell-3} + \dots + d_{n-1}^{(1)} (\log z)^{\ell-n}, \\
&{}^HI_{1^+}^{\ell-2}\, {}^H\mathfrak{D}_{1^+}^{\ell-2} u_k \\
&\quad = u_k + d_1^{(2)} (\log z)^{\ell-3} + d_2^{(2)} (\log z)^{\ell-4} + \dots + d_{n-2}^{(2)} (\log z)^{\ell-n}, \\
&\dots, \\
&{}^HI_{1^+}^{\ell-(n-1)}\, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)} u_k = u_k + d_1^{(n)} (\log z)^{\ell-n}.
\end{aligned}$$

By Definition 2, we get

$$\begin{aligned}
{}^HI_{1^+}^{\ell-1}\, {}^H\mathfrak{D}_{1^+}^{\ell-1} u_k &= \frac{1}{\Gamma(\ell-1)} \int_1^z \left(\log \frac{z}{s}\right)^{\ell-2} {}^H\mathfrak{D}_{1^+}^{\ell-1} u_k(s) \frac{\mathrm{d}s}{s}, \\
{}^HI_{1^+}^{\ell-2}\, {}^H\mathfrak{D}_{1^+}^{\ell-2} u_k &= \frac{1}{\Gamma(\ell-2)} \int_1^z \left(\log \frac{z}{s}\right)^{\ell-3} {}^H\mathfrak{D}_{1^+}^{\ell-2} u_k(s) \frac{\mathrm{d}s}{s}, \\
&\dots, \\
{}^HI_{1^+}^{\ell-(n-1)}\, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)} u_k &= \frac{1}{\Gamma(\ell-n+1)} \int_1^z \left(\log \frac{z}{s}\right)^{\ell-n} {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)} u_k(s) \frac{\mathrm{d}s}{s}.
\end{aligned}$$

Hence,

$$\begin{aligned}
u_k + \sum_{j=1}^{n-1} d_j^{(1)} (\log z)^{\ell-j-1} &= \frac{1}{\Gamma(\ell-1)} \int_1^z \left(\log \frac{z}{s}\right)^{\ell-2} {}^H\mathfrak{D}_{1^+}^{\ell-1} u_k(s) \frac{\mathrm{d}s}{s}, \\
u_k + \sum_{j=1}^{n-2} d_j^{(2)} (\log z)^{\ell-j-2} &= \frac{1}{\Gamma(\ell-2)} \int_1^z \left(\log \frac{z}{s}\right)^{\ell-3} {}^H\mathfrak{D}_{1^+}^{\ell-2} u_k(s) \frac{\mathrm{d}s}{s}, \\
&\dots, \\
u_k + d_1^{(n)} (\log z)^{\ell-n} &= \frac{1}{\Gamma(\ell-n+1)} \int_1^z \left(\log \frac{z}{s}\right)^{\ell-n} {}^H\mathfrak{D}_{1^+}^{\ell-n+1} u_k(s) \frac{\mathrm{d}s}{s}.
\end{aligned}$$

For $i = 2, \dots, n-1$,

$$\left\{\frac{u_k}{1 + (\log z)^{\ell-1}}\right\}_{k=1}^{\infty}, \quad \left\{{}^H\mathfrak{D}_{1^+}^{\ell-1} u_k\right\}_{k=1}^{\infty}, \quad \left\{\frac{{}^H\mathfrak{D}_{1^+}^{\ell-i} u_k}{1 + (\log z)^{i-1}}\right\}_{k=1}^{\infty},$$

are bounded in space $(C[1,\infty), \|\cdot\|_\infty)$. Moreover,

$$\frac{1}{\Gamma(\ell-1)}\int_1^z \left(\log\frac{z}{s}\right)^{\ell-2}\frac{\mathrm{d}s}{s}, \quad \int_1^z \left(\log\frac{z}{s}\right)^{\ell-i-1}\left(1+(\log s)^{i-1}\right)\frac{\mathrm{d}s}{s}$$
$$\leqslant \left(1+(\log z)^{i-1}\right)\int_1^z \left(\log\frac{z}{s}\right)^{\ell-i-1}\frac{\mathrm{d}s}{s},$$

$i=2,\dots,n-1$. By convergence and the Lebesgue dominated theorem, we have

$$u+\sum_{j=1}^{n-1} d_j^{(1)}(\log z)^{\ell-j-1} = \frac{1}{\Gamma(\ell-1)}\int_1^z \left(\log\frac{z}{s}\right)^{\ell-2}\varpi(s)\frac{\mathrm{d}s}{s},$$
$$u+\sum_{j=1}^{n-2} d_j^{(2)}(\log z)^{\ell-j-2} = \frac{1}{\Gamma(\ell-2)}\int_1^z \left(\log\frac{z}{s}\right)^{\ell-3}\varpi_2(s)\frac{\mathrm{d}s}{s},$$
$$\dots,$$
$$u+d_1^{(n)}(\log z)^{\ell-n} = \frac{1}{\Gamma(\ell-n+1)}\int_1^z \left(\log\frac{z}{s}\right)^{\ell-n}\varpi_{n-1}(s)\frac{\mathrm{d}s}{s},$$

that is, $\varpi_i = {}^H\mathfrak{D}_{1^+}^{\ell-i}u$, $i=2,\dots,n-1$, and $\varpi = {}^H\mathfrak{D}_{1^+}^{\ell-1}u$. From this we infer that $(X, \|\cdot\|_X)$ is a Banach space, which wraps up the proof of Lemma 6. □

Observe that the Arzelà–Ascoli theorem is not valid in X. Hence, to continue our analysis, we rely on the following compactness criterion.

Lemma 7. *Let $V \subset X$ be a bounded set. Then V is relatively compact in X if the following conditions hold*:

(i) *For $i=2,\dots,n-1$,*

$$\left\{\frac{\varpi}{1+(\log z)^{\ell-1}},\ \varpi\in V\right\},\quad \left\{\frac{{}^H\mathfrak{D}_{1^+}^{\ell-i}\varpi}{1+(\log z)^{i-1}},\ \varpi\in V\right\},\quad \left\{{}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi,\ \varpi\in V\right\}$$

are equicontinuous on any compact interval of $[1,\infty)$.

(ii) *For every $\varepsilon>0$, there exists $T=T(\varepsilon)>0$ such that for any $z_1, z_2 \geqslant T(\varepsilon)$ and $\varpi\in V$, the following inequalities hold*:

$$\left\|\frac{\varpi(z_2)}{1+(\log z_2)^{\ell-1}} - \frac{\varpi(z_1)}{1+(\log z_1)^{\ell-1}}\right\| < \varepsilon,$$
$$\left\|\frac{{}^H\mathfrak{D}_{1^+}^{\ell-i}\varpi(z_2)}{1+(\log z_2)^{i-1}} - \frac{{}^H\mathfrak{D}_{1^+}^{\ell-i}\varpi(z_1)}{1+(\log z_1)^{i-1}}\right\| < \varepsilon, \quad i=2,\dots,n-1,$$

and

$$\left\|{}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(z_2) - \mathfrak{D}_q^{\ell-1}\varpi(z_1)\right\| < \varepsilon.$$

Proof. The method closely follows the proof of Lemma 10 in [8]; therefore, the details are omitted. □

We define a cone $P \subset X$ by

$$P = \{\varpi \in X\colon \varpi(z) \geqslant 0,\ {}^{H}\mathfrak{D}_{1^+}^{\ell-i}\varpi(z) \geqslant 0,\ z \in [1,\infty),\ i = 1,2,\dots,n-1\},$$

and define the operator $T : P \to P$ as below:

$$T\varpi(z) = \int_1^\infty H(z,s)\varphi_q\bigl(\tau(s,\varpi(s),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(s),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s))\bigr)\frac{\mathrm{d}s}{s}$$

for $z \in [1,\infty)$. Also, according to (5),

$${}^{H}\mathfrak{D}_{1^+}^{\ell-i}T\varpi(z) = \int_1^\infty H_i^*(z,s)\varphi_q\bigl(\tau\bigl(s,\varpi(s),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(s),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s}$$

for $i = 1,2,\dots,n-1$, $z \in [1,\infty)$. From Lemma 5 one can easily observe that $\varpi \in P$, then $T\varpi(z) \geqslant 0$, ${}^{H}\mathfrak{D}_{1^+}^{\ell-i}T\varpi(z) \geqslant 0$ for all $z \in [1,\infty)$.

Lemma 8. *Suppose that* (K_1) *holds, then for all* $\varpi \in X$,

$$\begin{aligned}&\int_1^\infty \varphi_q\bigl(\tau(z,\varpi(z),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(z),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(z))\bigr)\frac{\mathrm{d}z}{z}\\ &\quad\leqslant \max\bigl\{1,2^{1/(p-1)}\bigr\}\left[r_0 + \sum_{i=1}^n r_i\|\varpi\|_X^{\lambda_i}\right].\end{aligned}$$

Proof. By (K_1), for any $\varpi \in X$, we ensure

$$\begin{aligned}&\int_1^\infty \varphi_q\bigl(\tau\bigl(z,\varpi(z),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(z),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(z)\bigr)\bigr)\frac{\mathrm{d}z}{z}\\ &\quad\leqslant \int_1^\infty \varphi_q\bigl(r_0(z) + r_1(z)\varphi_p(\varpi(z)^{\lambda_1}) + r_2(z)\varphi_p\bigl(\bigl[{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(z)\bigr]^{\lambda_2}\bigr) + \dots + r_n(z)\\ &\qquad\times\varphi_p\bigl(\bigl[{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(z)\bigr]^{\lambda_n}\bigr)\bigr)\frac{\mathrm{d}z}{z}\\ &\quad= \int_1^\infty \varphi_q\biggl(r_0(z) + r_1(z)\varphi_p\biggl(\bigl[1+(\log z)^{\ell-1}\bigr]^{\lambda_1}\frac{\varpi(z)^{\lambda_1}}{[1+(\log z)^{\ell-1}]^{\lambda_1}}\biggr)\\ &\qquad+ r_2(z)\varphi_p\bigl(\bigl[{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(z)\bigr]^{\lambda_2}\bigr)\\ &\qquad+ \dots + r_n(z)\varphi_p\biggl(\bigl[1+(\log z)^{n-2}\bigr]^{\lambda_n}\frac{[{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(z)]^{\lambda_n}}{[1+(\log z)^{n-2}]^{\lambda_n}}\biggr)\biggr)\frac{\mathrm{d}z}{z}\end{aligned}$$

$$\begin{aligned}
&\leqslant \max\{1, 2^{1/(p-1)}\} \\
&\quad\times \Biggl[\int_1^\infty \varphi_q\bigl(r_0(z)\bigr)\frac{\mathrm{d}z}{z} + \int_1^\infty \varphi_q\bigl(r_1(z)\bigr)\bigl[1+(\log z)^{\ell-1}\bigr]^{\lambda_1}\frac{\mathrm{d}z}{z}\|\varpi\|_X^{\lambda_1} \\
&\quad+ \int_1^\infty \varphi_q\bigl(r_2(z)\bigr)\frac{\mathrm{d}z}{z}\|\varpi\|_X^{\lambda_2} + \cdots + \int_1^\infty \varphi_q\bigl(r_n(z)\bigr)\bigl[1+(\log z)^{n-2}\bigr]^{\lambda_n}\frac{\mathrm{d}z}{z}\|\varpi\|_X^{\lambda_n} \Biggr] \\
&= \max\{1, 2^{1/(p-1)}\}\bigl[r_0 + r_1\|\varpi\|_X^{\lambda_1} + \cdots + r_n\|\varpi\|_X^{\lambda_n}\bigr] \\
&= \max\{1, 2^{1/(p-1)}\}\Biggl[r_0 + \sum_{i=1}^n r_i\|\varpi\|_X^{\lambda_i}\Biggr]. \qquad \square
\end{aligned}$$

Lemma 9. *Suppose that* (K_0) *and* (K_1) *hold. Then* $T : P \to P$ *is a completely continuous operator.*

Proof. It is obvious that $T(P) \subset P$. Hence, $T : P \to P$. Let us outline the proof in four distinct stages.

(i) We show that $T : P \to P$ is totally bounded. Let $U \subseteq P$ be a bounded set. Then there exists $N > 0$ such that $\|\varpi\|_X \leqslant N$ for any $\varpi \in P$. For any $\varpi \in U$,

$$\begin{aligned}
&\biggl\|\frac{T\varpi}{1+(\log z)^{\ell-1}}\biggr\|_\infty \\
&\quad= \sup_{z\in[1,\infty)} \int_1^\infty \frac{H(z,s)}{1+(\log z)^{\ell-1}}\varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad\leqslant \frac{1}{\Delta}\int_1^\infty \varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad\leqslant \frac{1}{\Delta}\max\{1, 2^{1/(p-1)}\}\Biggl[r_0 + \sum_{i=1}^n r_i\|\varpi\|_X^{\lambda_i}\Biggr] < \infty,
\end{aligned}$$

$$\begin{aligned}
&\bigl\|{}^H\mathfrak{D}_{1^+}^{\ell-1}T\varpi\bigr\|_\infty \\
&\quad= \sup_{z\in[1,\infty)} \int_1^\infty H^*(z,s)\varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad\leqslant \frac{\Gamma(\ell)}{\Delta}\int_1^\infty \varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad\leqslant \frac{\Gamma(\ell)}{\Delta}\max\{1, 2^{1/(p-1)}\}\Biggl[r_0 + \sum_{i=1}^n r_i\|\varpi\|_X^{\lambda_i}\Biggr] < \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \frac{{}^{H}\mathfrak{D}_{1^+}^{\ell-i} T\varpi}{1+(\log z)^{i-1}} \right\|_\infty \\
&\quad = \sup_{z\in[1,\infty)} \int_1^\infty \frac{{H_i}^*(z,s)}{1+(\log z)^{\ell-1}} \varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad \leqslant \frac{\Gamma(\ell)}{\Gamma(i)\Delta} \int_1^\infty \varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad \leqslant \frac{\Gamma(\ell)}{\Gamma(i)\Delta} \max\bigl\{1, 2^{1/(p-1)}\bigr\}\left[r_0 + \sum_{i=1}^n r_i \|\varpi\|_X^{\lambda_i}\right] < \infty, \quad i = 2,3,\dots,n-1, \\
&\|T\varpi\|_X \\
&\quad = \max\left\{\frac{1}{\Delta}, \frac{\Gamma(\ell)}{\Delta}, \frac{\Gamma(\ell)}{\Gamma(i)\Delta},\ i=2,3,\dots,n-1\right\} \max\bigl\{1, 2^{1/(p-1)}\bigr\} \\
&\qquad \times \left[r_0 + \sum_{i=1}^n r_i \|\varpi\|_X^{\lambda_i}\right] \\
&\quad < \infty.
\end{aligned}$$

This demonstrates that $T : P \to P$ is totally bounded.

(ii) We establish that the mapping $T : P \to P$ exhibits equicontinuity for all $\varpi \in U$ over any compact interval I in the range of $[1,\infty)$. For any $L > 0$, $z_1, z_2 \in I = [0, L]$, $z_2 > z_1$,

$$\begin{aligned}
&\left| \frac{T\varpi(z_2)}{1+(\log z_2)^{\ell-1}} - \frac{T\varpi(z_1)}{1+(\log z_1)^{\ell-1}} \right| \\
&\quad \leqslant \int_1^\infty \left| \frac{H(z_2,s)}{1+(\log z_2)^{\ell-1}} - \frac{H(z_1,s)}{1+(\log z_1)^{\ell-1}} \right| \\
&\qquad \times \varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad \leqslant \int_1^{z_2} \left| \frac{H(z_2,s)}{1+(\log z_2)^{\ell-1}} - \frac{H(z_1,s)}{1+(\log z_1)^{\ell-1}} \right| \\
&\qquad \times \varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\qquad + \int_{z_2}^\infty \left| \frac{H(z_2,s)}{1+(\log z_2)^{\ell-1}} - \frac{H(z_1,s)}{1+(\log z_1)^{\ell-1}} \right| \\
&\qquad\quad \times \varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_1^{z_2} \left| \frac{H(z_2,s)}{1+(\log z_2)^{\ell-1}} - \frac{H(z_1,s)}{1+(\log z_1)^{\ell-1}} \right| \\
&\qquad \times \varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad + \frac{1}{\Delta}\int_{z_2}^{\infty} \left| \frac{(\log(z_2))^{\ell-1}}{1+(\log z_2)^{\ell-1}} - \frac{(\log(z_1))^{\ell-1}}{1+(\log z_1)^{\ell-1}} \right| \\
&\qquad\quad \times \varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

It can be inferred that

$$
\left| \frac{T\varpi(z_2)}{1+(\log z_2)^{\ell-1}} - \frac{T\varpi(z_1)}{1+(\log z_1)^{\ell-1}} \right| \to 0
$$

uniformly as $z_1 \to z_2$. So, $T\varpi(z)/(1+(\log z)^{\ell-1})$ is equicontinuous on I. Note that

$$
{}^H\mathfrak{D}_{1^+}^{\ell-1}T\varpi(z) = \int_1^{\infty} H_1^*(z,s)\varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s}.
$$

In this case, the function $H_1^*(z,s) \in C([1,\infty) \times [1,\infty))$ is independent of z. This indicates that ${}^H\mathfrak{D}_{1^+}^{\ell-1}T\varpi(z)$ exhibits equicontinuity over I.

$$
\begin{aligned}
&\left| \frac{{}^H\mathfrak{D}_{1^+}^{\ell-i}T\varpi(z_2)}{1+(\log z_2)^{i-1}} - \frac{{}^H\mathfrak{D}_{1^+}^{\ell-i}T\varpi(z_1)}{1+(\log z_1)^{i-1}} \right| \\
&\quad \leqslant \int_1^{\infty} \left| \frac{{H_i}^*(z_2,s)}{1+(\log z_2)^{i-1}} - \frac{{H_i}^*(z_1,s)}{1+(\log z_1)^{i-1}} \right| \\
&\qquad\quad \times \varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad \leqslant \int_1^{z_2} \left| \frac{{H_i}^*(z_2,s)}{1+(\log z_2)^{i-1}} - \frac{{H_i}^*(z_1,s)}{1+(\log z_1)^{i-1}} \right| \\
&\qquad\quad \times \varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\qquad + \int_{z_2}^{\infty} \left| \frac{{H_i}^*(z_2,s)}{1+(\log z_2)^{i-1}} - \frac{{H_i}^*(z_1,s)}{1+(\log z_1)^{i-1}} \right| \\
&\qquad\qquad \times \varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad \leqslant \int_1^{z_2} \left| \frac{{H_i}^*(z_2,s)}{1+(\log z_2)^{i-1}} - \frac{{H_i}^*(z_1,s)}{1+(\log z_1)^{i-1}} \right| \\
&\qquad\quad \times \varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s}
\end{aligned}
$$

$$+\frac{\Gamma(\ell)}{\Gamma(i)\Delta}\int_{z_2}^{\infty}\left|\frac{(\log z_2)^{i-1}}{1+(\log z_2)^{i-1}}-\frac{(\log z_1)^{i-1}}{1+(\log z_1)^{i-1}}\right|$$
$$\times\varphi_q\bigl(\tau\bigl(s,\varpi(s),{}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s),\dots,{}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s}.$$

Thus, ${}^H\mathfrak{D}_{1^+}^{\ell-i}T\varpi(z)/(1+(\log z)^{i-1})$ exhibits equicontinuity over I. As a result, the operator T demonstrates equicontinuity for $\varpi\in U$ over any compact interval I contained in $[1,\infty)$.

(iii) We establish that T exhibits equiconvergence in the limit as z approaches infinity. Through the use of Lemma 5, we have

$$\begin{aligned}
&\lim_{z\to\infty}\frac{T\varpi}{1+(\log z)^{\ell-1}}\\
&\quad\leqslant\frac{1}{\Delta}\int_1^{\infty}\varphi_q(\tau(s,\varpi(s),{}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s),\dots,{}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)))\frac{\mathrm{d}s}{s}<\infty,\\
&\lim_{z\to\infty}{}^H\mathfrak{D}_{1^+}^{\ell-1}(T\varpi)\\
&\quad\leqslant\frac{\Gamma(\ell)}{\Delta}\int_1^{\infty}\varphi_q(\tau(s,\varpi(s),{}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s),\dots,{}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s))\frac{\mathrm{d}s}{s}<\infty,\\
&\lim_{z\to\infty}\frac{{}^H\mathfrak{D}_{1^+}^{\ell-i}T\varpi(z)}{1+(\log z)^{i-1}}\\
&\quad\leqslant\frac{\Gamma(\ell)}{\Gamma(i)\Delta}\int_1^{\infty}\varphi_q(\tau(s,\varpi(s),{}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s),\dots,{}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s))\frac{\mathrm{d}s}{s}<\infty
\end{aligned}$$

for $i=2,3,\dots,n-1$, which implies that T is equiconvergent at ∞.

(iv) We show that T is continuous. Let $\varpi_n\to\varpi$ as $n\to\infty$ in P, then there exists a constant $r>0$ such that $\|\varpi\|_X\leqslant r$, $\|\varpi_n\|_X\leqslant r$. We ensure

$$\begin{aligned}
&\frac{T\varpi_n(z)}{1+(\log z)^{\ell-1}}\\
&\quad=\int_1^{\infty}\frac{H(z,s)}{1+(\log z)^{\ell-1}}\varphi_q\bigl(\tau\bigl(s,\varpi_n(s),{}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi_n(s),\dots,{}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi_n(s)\bigr)\bigr)\frac{\mathrm{d}s}{s}\\
&\quad\leqslant\frac{1}{\Delta}\int_1^{\infty}\varphi_q\bigl(\tau\bigl(s,\varpi_n(s),{}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi_n(s),\dots,{}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi_n(s)\bigr)\bigr)\frac{\mathrm{d}s}{s}\\
&\quad\leqslant\frac{1}{\Delta}\max\bigl\{1,2^{1/(p-1)}\bigr\}\left[r_0+\sum_{i=1}^{n}r_i\|\varpi_n\|_X^{\lambda_i}\right]<\infty,
\end{aligned}$$

$$
\begin{aligned}
&{}^{H}\mathfrak{D}_{1^+}^{\ell-1}T\varpi_n(z) \\
&\quad= \int_1^\infty H_1{}^*(z,s)\varphi_q\bigl(\tau\bigl(s,\varpi_n(s),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi_n(s),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi_n(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad\leqslant \frac{\Gamma(\ell)}{\Delta}\int_1^\infty \varphi_q\bigl(\tau\bigl(s,\varpi_n(s),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi_n(s),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi_n(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad\leqslant \frac{\Gamma(\ell)}{\Delta}\max\bigl\{1,2^{1/(p-1)}\bigr\}\left[r_0+\sum_{i=1}^n r_i\|\varpi_n\|_X^{\lambda_i}\right]<\infty, \\
&\frac{{}^{H}\mathfrak{D}_{1^+}^{\ell-i}T\varpi_n(z)}{1+(\log z)^{i-1}} \\
&\quad= \int_1^\infty \frac{H_i{}^*(t,s)}{1+(\log z)^{i-1}}\varphi_q\bigl(\tau\bigl(s,\varpi_n(s),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi_n(s),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi_n(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad\leqslant \frac{\Gamma(\ell)}{\Delta\Gamma(i)}\int_1^\infty \varphi_q\bigl(\tau\bigl(s,\varpi_n(s),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi_n(s),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi_n(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad\leqslant \frac{\Gamma(\ell)}{\Delta\Gamma(i)}\max\{1,2^{1/(p-1)}\}\left[r_0+\sum_{i=1}^n r_i\|\varpi_n\|_X^{\lambda_i}\right]<\infty,\quad i=2,3,\dots,n-1.
\end{aligned}
$$

The continuity of τ and the Lebesgue dominated convergence theorem together assure that

$$
\begin{aligned}
&\lim_{n\to\infty}\int_1^\infty \frac{H(z,s)}{1+(\log z)^{\ell-1}}\varphi_q\bigl(\tau\bigl(s,\varpi_n(s),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi_n(s),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi_n(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad= \int_1^\infty \frac{H(z,s)}{1+(\log z)^{\ell-1}}\varphi_q\bigl(\tau\bigl(s,\varpi(s),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(s),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s}, \\
&\lim_{n\to\infty}\int_1^\infty H_1^*(z,s)\varphi_q\bigl(\tau\bigl(s,\varpi_n(s),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi_n(s),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi_n(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad= \int_1^\infty H_1^*(z,s)\varphi_q\bigl(\tau\bigl(s,\varpi(s),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi(s),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s},
\end{aligned}
$$

and

$$
\lim_{n\to\infty}\int_1^\infty \frac{H_i{}^*(z,s)}{1+(\log z)^{i-1}}\varphi_q\bigl(\tau\bigl(s,\varpi_n(s),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi_n(s),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi_n(s)\bigr)\bigr)\frac{\mathrm{d}s}{s}
$$

$$= \int_1^\infty \frac{H_i{}^*(z,s)}{1+(\log z)^{\ell-1}} \varphi_q\bigl(\tau\bigl(s, \varpi(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s)\bigr)\bigr)\frac{\mathrm{d}s}{s},$$
$$i = 2, 3, \dots, n-1.$$

Hence,

$$\left\| \frac{T\varpi_n}{1+(\log z)^{\ell-1}} - \frac{T\varpi}{1+(\log z)^{\ell-1}} \right\|_\infty \to 0, \quad n \to \infty,$$
$$\left\| {}^H\mathfrak{D}_{1^+}^{\ell-1} T\varpi_n - {}^H\mathfrak{D}_{1^+}^{\ell-1} T\varpi \right\|_\infty \to 0, \quad n \to \infty,$$

and

$$\left\| \frac{{}^H\mathfrak{D}_{1^+}^{i-1} T\varpi_n}{1+(\log z)^{i-1}} - \frac{{}^H\mathfrak{D}_{1^+}^{i-1} T\varpi}{1+(\log z)^{i-1}} \right\|_\infty \to 0, \quad n \to \infty,\ i = 2, 3, \dots, n-1.$$

Thus,

$$\|T\varpi_n - T\varpi\|_X \to 0, \quad n \to \infty.$$

From the above steps it is obvious that $T : P \to P$ is completely continuous. The proof is completed. □

3 Existence and uniqueness theorems

For notational convenience, we introduce

$$\begin{aligned}
\kappa_i &= \max\bigl\{1, 2^{1/(p-1)}\bigr\}\frac{1}{\Gamma(i)\varDelta}, \quad i = 1, 2, \dots, n-1, \\
L &= \max\bigl\{\kappa_1, \kappa_1\Gamma(\ell), \kappa_2\Gamma(\ell), \dots, \kappa_{n-1}\Gamma(\ell)\bigr\}, \\
R &\geqslant \max\bigl\{(n+1)Lr_0, \bigl((n+1)Lr_1\bigr)^{(1/1-\lambda_1)}, \dots, \bigl((n+1)Lr_n\bigr)^{(1/1-\lambda_n)}\bigr\}.
\end{aligned} \tag{7}$$

We possess the theorem listed below.

Theorem 1. *Assume that* (K_0)–(K_2) *are ensured. Then problem* (1) *has at least two positive solutions* ϖ^* *and* w^* *satisfying* $0 < \|\varpi^*\|_X \leqslant R$, $\lim_{n\to\infty} \varpi_n = \lim_{n\to\infty} T^n\varpi_0 = \varpi^*$, *and*

$$\varpi_n = T\varpi_{n-1} \quad \textit{with } \varpi_0 = R(\log z)^{\ell-1},\ n = 1, 2, \dots;$$

$0 < \|w^*\|_X \leqslant R$, $\lim_{n\to\infty} w_n = \lim_{n\to\infty} T^n w_0 = w^*$, *and*

$$w_n = Tw_{n-1} \quad \textit{with } w_0 = 0,\ n = 1, 2, \dots .$$

Proof. By Lemma 9, we see that $T : P \to P$ is completely continuous. Then, for each $\varpi \in P$, $z \in [1, \infty)$, it results in the fact that $T(P) \subset P$. Let $0 \leqslant \lambda_k < 1$, $k = 1, 2, \dots, n$.

For the case $\lambda_k \geqslant 1$, the proof is similar to that of $0 \leqslant \lambda_k < 1$. Denote $P_R = \{\varpi \in P\colon \|\varpi\|_X \leqslant R\}$, where

$$R \geqslant \max\bigl\{(n+1)Lr_0,\ \bigl((n+1)Lr_1\bigr)^{(1/1-\lambda_1)}, \dots, \bigl((n+1)Lr_n\bigr)^{(1/1-\lambda_n)}\bigr\}.$$

First, we show that $T : P_R \to P_R$. If $\varpi \in P_R$, then $\|\varpi\|_X \leqslant R$. We get

$$\begin{aligned}
\left\|\frac{T\varpi}{1+(\log z)^{\ell-1}}\right\|_\infty &\leqslant \frac{1}{\varDelta}\max\bigl\{1, 2^{1/(p-1)}\bigr\}\left[r_0 + \sum_{i=1}^n r_i R^{\lambda_i}\right] \\
&= \kappa_1\left[r_0 + \sum_{i=1}^n r_i R^{\lambda_i}\right] \leqslant L\left[r_0 + \sum_{i=1}^n r_i R^{\lambda_i}\right] \leqslant R, \\
\bigl\|{}^H\mathfrak{D}_{1^+}^{\ell-1}(T\varpi)\bigr\|_\infty &\leqslant \frac{\Gamma(\ell)}{\varDelta}\max\bigl\{1, 2^{1/(p-1)}\bigr\}\left[r_0 + \sum_{i=1}^n r_i R^{\lambda_i}\right] \\
&= \kappa_1\Gamma(\ell)\left[r_0 + \sum_{i=1}^n r_i R^{\lambda_i}\right] \leqslant L\left[r_0 + \sum_{i=1}^n r_i R^{\lambda_i}\right] \leqslant R, \\
\left\|\frac{{}^H\mathfrak{D}_{1^+}^{\ell-i}(T\varpi)(z)}{1+(\log z)^{i-1}}\right\|_\infty &\leqslant \frac{\Gamma(\ell)}{\Gamma(i)\varDelta}\max\bigl\{1, 2^{1/(p-1)}\bigr\}\left[r_0 + \sum_{i=1}^n r_i R^{\lambda_i}\right] \\
&= \kappa_i\Gamma(\ell)\left[r_0 + \sum_{i=1}^n r_i R^{\lambda_i}\right] \leqslant L\left[r_0 + \sum_{i=1}^n r_i R^{\lambda_i}\right] \leqslant R
\end{aligned}$$

for $i = 2, 3, \dots, n-1$. Hence,

$$\begin{aligned}
\|T\varpi\|_X = \max\biggl\{&\left\|\frac{T\varpi}{1+(\log z)^{\ell-1}}\right\|_\infty, \bigl\|{}^HD_{1^+}^{\ell-1}T\varpi\bigr\|_\infty, \left\|\frac{{}^HD_{1^+}^{\ell-i}T\varpi}{1+(\log z)^{i-1}}\right\|_\infty, \\
&i = 2, 3, \dots, n-1\biggr\} \\
\leqslant R&,
\end{aligned}$$

and so $T : P_R \to P_R$.

It is apparent that $\varpi_0, w_0 \in P_R$. By utilizing the completely continuous operator T, for $n \in \{1, 2, \dots,\}$, we define ϖ_n, w_n by $\varpi_n = T\varpi_{n-1}$, $w_n = Tw_{n-1}$. Since $T(P_R) \subset P_R$ for $n \in \{1, 2, \dots,\}$, it is observable that $u_n, w_n \in P_R$. For this reason, we need to establish that there exist ϖ^*, w^* such that $\lim_{n\to\infty}\varpi_n = \varpi^*$, $\lim_{n\to\infty} w_n = w^*$, which are two monotone schemes for approximating positive solutions of problem (1). For $z \in [1, \infty)$, we get

$$\begin{aligned}
\varpi_1(z) &= T\varpi_0(z) \\
&= \int_1^\infty H(z,s)\varphi_q\bigl(\tau\bigl(s, \varpi_0(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi_0(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi_0(s)\bigr)\bigr)\frac{\mathrm{d}s}{s}
\end{aligned}$$

$$\begin{aligned}&\leqslant \frac{(\log z)^{\ell-1}}{\Delta}\max\{1,2^{1/(p-1)}\}\left[r_0+\sum_{i=1}^{n} r_i\|\varpi\|_X^{\lambda_i}\right]\\ &\leqslant (\log z)^{\ell-1}L\left[r_0+\sum_{i=1}^{n} r_i R^{\lambda_i}\right]\leqslant R(\log z)^{\ell-1}=\varpi_0(z).\end{aligned}$$

Condition (K_2) on the monotonicity of the functions τ

$$\varpi_2(z)=T\varpi_1(z)\leqslant T\varpi_0(z)=\varpi_1(z)$$

is obtained. By recursion, for $z\in[1,\infty)$ and $n=0,1,2,\dots$, we obtain

$$\varpi_{n+1}(z)\leqslant\varpi_n(z).$$

Now, we discuss the monotonicity of fractional derivative of ϖ. From Lemmas 4 and 5

$$\begin{aligned}&{}^{H}\mathfrak{D}_{1^+}^{\ell-i}\varpi_1(z)\\ &={}^{H}\mathfrak{D}_{1^+}^{\ell-i}T\varpi_0(z)\\ &=\int_1^\infty H_i^*(z,s)\varphi_q\bigl(\tau\bigl(s,\varpi_0(s),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi_0(s),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi_0(s)\bigr)\bigr)\frac{\mathrm{d}s}{s}\\ &\leqslant\frac{(\log z)^{i-1}}{\Gamma(i)}\frac{\Gamma(\ell)}{\Delta}\int_1^\infty\varphi_q\bigl(\tau\bigl(s,\varpi_0(s),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi_0(s),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi_0(s)\bigr)\bigr)\frac{\mathrm{d}s}{s}\\ &=\frac{1}{\Delta}{}^{H}\mathfrak{D}_{1^+}^{\ell-i}(\log z)^{\ell-1}\int_1^\infty\varphi_q\bigl(\tau\bigl(s,\varpi_0(s),{}^{H}\mathfrak{D}_{1^+}^{\ell-1}\varpi_0(s),\dots,{}^{H}\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi_0(s)\bigr)\bigr)\frac{\mathrm{d}s}{s}\\ &\leqslant\frac{1}{\Delta}\max\{1,2^{1/(p-1)}\}\left[r_0+\sum_{i=1}^{n}r_iR^{\lambda_i}\right]{}^{H}\mathfrak{D}_{1^+}^{\ell-i}(\log z)^{\ell-1}\\ &\leqslant L\left[r_0+\sum_{i=1}^{n}r_iR^{\lambda_i}\right]{}^{H}\mathfrak{D}_{1^+}^{\ell-i}(\log z)^{\ell-1}\leqslant R\,{}^{H}\mathfrak{D}_{1^+}^{\ell-i}(\log z)^{\ell-1}\\ &={}^{H}\mathfrak{D}_{1^+}^{\ell-i}\varpi_0(z),\quad i=1,\dots,n-1,\end{aligned}$$

$${}^{H}\mathfrak{D}_{1^+}^{\ell-i}\varpi_2(z)={}^{H}\mathfrak{D}_{1^+}^{\ell-i}T\varpi_1(z)\leqslant{}^{H}\mathfrak{D}_{1^+}^{\ell-i}T\varpi_0(z)={}^{H}\mathfrak{D}_{1^+}^{\ell-i}\varpi_1(z)$$

for $i=1,2,\dots,n-1$. By induction, for $z\in[1,\infty)$, $n=0,1,2,\dots$, one gets

$${}^{H}\mathfrak{D}_{1^+}^{\ell-i}\varpi_{n+1}(z)\leqslant{}^{H}\mathfrak{D}_{1^+}^{\ell-i}\varpi_n(z),\quad i=1,2,\dots,n-1.$$

As a consequence of the complete continuity of the operator T, $\{\varpi_n\}_{n=0}^\infty$ has a convergent subsequence $\{\varpi_{n_k}\}_{k=1}^\infty$, and there exists a $\varpi^*\in P_R$ such that $\varpi_{n_k}\to\varpi^*$ as $k\to\infty$. This, together with iterative scheme, implies that $\lim_{n\to\infty}\varpi_n=\varpi^*$.

Thus, it follows from the complete continuity of T and by the aid of the iterative scheme $\varpi_n = T\varpi_{n-1}$, we have $T\varpi^* = \varpi^*$. For the sequence (w_n) and $z \in [1, \infty)$, we rely on a parallel argument, we reach

$$\begin{aligned} w_1(z) &= Tw_0(z) \\ &= \int_1^\infty H(z,s)\varphi_q\bigl(\tau\bigl(s, w_0(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}w_0(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}w_0(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\ &\geqslant 0 = w_0(z). \end{aligned}$$

Via the monotonicity condition τ, for $z \in [1, \infty)$, we achieve

$$w_2(z) = Tw_1(z) \geqslant Tw_0(z) = w_1(z).$$

By recursion, for $z \in [1, \infty)$ and $n = 0, 1, 2, \dots$, we obtain

$$w_{n+1}(z) \geqslant w_n(z),$$

$$\begin{aligned} {}^H\mathfrak{D}_{1^+}^{\ell-i}w_1(z) &= {}^H\mathfrak{D}_{1^+}^{\ell-i}Tw_0(z) \\ &= \int_1^\infty H_i^*(z,s)\varphi_q\bigl(\tau\bigl(s, w_0(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}w_0(s), \dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}w_0(s)\bigr)\bigr)\frac{\mathrm{d}s}{s} \\ &\geqslant 0 = {}^H\mathfrak{D}_{1^+}^{\ell-i}w_0(z), \quad i = 1, 2, \dots, n-1. \end{aligned}$$

By induction, for $z \in [1, \infty)$, $n = 0, 1, 2, \dots$, one gets

$${}^H\mathfrak{D}_{1^+}^{\ell-i}w_{n+1}(z) \geqslant {}^H\mathfrak{D}_{1^+}^{\ell-i}w_n(z), \quad i = 1, 2, \dots, n-1.$$

It follows from the complete continuity of T and $w_{n+1} = Tw_n$ that $w_n \to w^*$ and $Tw^* = w^*$. If $\tau(z, 0, \dots, 0)$ is nonzero on any subinterval of $[1, \infty)$, then the zero function cannot be a solution to problem (1).

Furthermore, we can prove that ϖ^* and w^* represent the minimal and maximal positive solutions of problem (1). Let x be any positive solution of problem (1), then $Tx(z) = x(z)$ for $[1, \infty)$, and

$$\begin{gathered} w_0(z) = 0 \leqslant x(z) \leqslant \varpi_0(z) = R(\log z)^{\ell-1}, \\ {}^H\mathfrak{D}_{1^+}^{\ell-i}w_0(z) \leqslant {}^H\mathfrak{D}_{1^+}^{\ell-i}x(z) \leqslant {}^H\mathfrak{D}_{1^+}^{\ell-i}\varpi_0(z) \end{gathered}$$

for $z \in [1, \infty)$, $i = 1, 2, \dots, n-1$. Utilizing the operator's monotone property, we get

$$w_1(z) = Tw_0(z) \leqslant Tx(z) = x(z) \leqslant T\varpi_0(z) = \varpi_1(z)$$

for $z \in [1, \infty)$. By following the same procedure, we ensure

$$\begin{gathered} w_n(z) \leqslant x(z) \leqslant \varpi_n(z), \\ {}^H\mathfrak{D}_{1^+}^{\ell-i}w_n(z) \leqslant {}^H\mathfrak{D}_{1^+}^{\ell-i}x(z) \leqslant {}^H\mathfrak{D}_{1^+}^{\ell-i}\varpi_n(z) \end{gathered}$$

for $z \in [1, \infty)$, $i = 1, 2, \dots, n-1$. Since $\lim_{n\to\infty} \varpi_n = \varpi^*$, $\lim_{n\to\infty} w_n = w^*$, then w^* and ϖ^* are the maximal and minimal positive solutions of problem (1). □

Now, we establish a sequence that approaches the unique positive solution of the given nonlocal Hadamard fractional boundary value problem (1). We need the following assumption:

(K_3) There exist nonnegative functions $p_i(z) \in L^1[1,\infty)$ such that

$$\begin{aligned}&\left|\varphi_q\big(\tau(z,\varpi_1,\varpi_2,\dots,\varpi_n)\big) - \varphi_q\big(\tau(z,\overline{\varpi}_1,\overline{\varpi}_2,\dots,\overline{\varpi}_n)\big)\right|\\&\quad\leqslant \sum_{i=1}^{n} p_i(z)|\varpi_i - \overline{\varpi}_i|\end{aligned}$$

for all $z \in [1,\infty)$, $\varpi_i, \overline{\varpi}_i \in [0,\infty)$, $i = 1,2,\dots,n$,

$$\begin{aligned}p_1 &= \int_1^\infty \big(1 + (\log z)^{\ell-1}\big) p_1(z) \frac{\mathrm{d}z}{z} < \infty,\\ p_2 &= \int_1^\infty p_2(z) \frac{\mathrm{d}z}{z} < \infty,\\ p_i &= \int_1^\infty \big(1 + (\log z)^{i-2}\big) p_i(z) \frac{\mathrm{d}z}{z} < \infty, \quad i = 3,\dots,n.\end{aligned}$$

Theorem 2. *If conditions* (K_0)–(K_3) *hold, then problem* (1) *has a unique positive solution* $\varpi^* \in P$*. Also, for any* $\varpi_0 \in P$*, there is an iterative scheme* $\{\varpi_n\}_{n=1}^\infty$ *such that* $\varpi_n \to \varpi^*$ *as* $n \to \infty$ *uniformly in* P*, where*

$$\varpi_n(z) = \int_1^\infty H(z,s)\varphi_q\big(\tau\big(s, \varpi_{n-1}(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi_{n-1}(s),\dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi_{n-1}(s)\big)\big)\frac{\mathrm{d}s}{s}$$

for $z \in [1,\infty)$*,* $n = 1,2,\dots$ *. Moreover, the error estimate can be introduced as*

$$\|\varpi_n - \varpi^*\|_X \leqslant \frac{\overline{L}^n}{1-\overline{L}}\|\varpi_1 - \varpi_0\|_X, \quad n = 1,2,\dots,$$

where L *is defined by* (7)*, and* $\overline{L} = L\sum_{k=1}^n p_k < 1$.

Proof. By Lemma 9, we know $T : P \to P$. We show that T is a contraction. Based on condition (K_3), for any $\varpi, \overline{\varpi} \in P$ and $z \in [1,\infty)$, by Lemma 5, we get

$$\begin{aligned}&\left\|\frac{T\varpi}{1+(\log z)^{\ell-1}} - \frac{T\overline{\varpi}}{1+(\log z)^{\ell-1}}\right\|_\infty\\&\quad= \sup_{z\in[1,\infty)}\left|\int_1^\infty \frac{H(z,s)}{1+(\log z)^{\ell-1}}\big(\varphi_q\big(\tau(s,\varpi(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s),\dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s))\big)\right.\\&\qquad\left. - \varphi_q\big(\tau(s,\overline{\varpi}(s), {}^H\mathfrak{D}_{1^+}^{\ell-1}\overline{\varpi}(s),\dots, {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\overline{\varpi}(s))\big)\big)\frac{\mathrm{d}s}{s}\right|\end{aligned}$$

$$\begin{aligned}
&\leqslant L\int_1^\infty \biggl(p_1(s)\bigl(1+(\log s)^{\ell-1}\bigr)\frac{|\varpi(s)-\overline{\varpi}(s)|}{1+(\log s)^{\ell-1}} + p_2(s)\bigl|{}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi(s) - {}^H\mathfrak{D}_{1^+}^{\ell-1}\overline{\varpi}(s)\bigr| \\
&\qquad + p_3(s)(1+\log s)\frac{|{}^H\mathfrak{D}_{1^+}^{\ell-2}\varpi(s) - {}^H\mathfrak{D}_{1^+}^{\ell-2}\overline{\varpi}(s)|}{1+\log s} + \cdots \\
&\qquad + p_n(s)\bigl(1+(\log s)^{n-2}\bigr)\frac{|{}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi(s) - {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\overline{\varpi}(s)|}{1+(\log s)^{n-2}}\biggr)\frac{\mathrm{d}s}{s} \\
&\leqslant L\biggl(p_1\biggl\|\frac{\varpi-\overline{\varpi}}{1+(\log s)^{\ell-1}}\biggr\|_\infty + p_2\bigl\|{}^H\mathfrak{D}_{1^+}^{\ell-1}\varpi - {}^H\mathfrak{D}_{1^+}^{\ell-1}\overline{\varpi}\bigr\|_\infty + \cdots \\
&\quad + p_n\biggl\|\frac{{}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\varpi - {}^H\mathfrak{D}_{1^+}^{\ell-(n-1)}\overline{\varpi}}{1+(\log s)^{n-2}}\biggr\|_\infty\biggr) \\
&\leqslant L\sum_{k=1}^n p_k\|\varpi-\overline{\varpi}\|_X
\end{aligned}$$

and

$$\biggl\|\frac{{}^H\mathfrak{D}_{1^+}^{\ell-i}T\varpi}{1+(\log z)^{i-1}} - \frac{{}^H\mathfrak{D}_{1^+}^{\ell-i}T\overline{\varpi}}{1+(\log z)^{i-1}}\biggr\|_\infty \leqslant L\sum_{k=1}^n p_k\|\varpi-\overline{\varpi}\|_X, \quad i=2,\dots,n-1,$$
$$\bigl\|{}^H\mathfrak{D}_{1^+}^{\ell-1}T\varpi - {}^H\mathfrak{D}_{1^+}^{\ell-1}T\overline{\varpi}\bigr\|_\infty \leqslant L\sum_{k=1}^n p_k\|\varpi-\overline{\varpi}\|_X.$$

That is, for all $\varpi, \overline{\varpi} \in P$,

$$\|T\varpi - T\overline{\varpi}\|_X \leqslant L\sum_{k=1}^n p_k\|\varpi-\overline{\varpi}\|_X = \overline{L}\|\varpi-\overline{\varpi}\|_X. \tag{8}$$

The condition $\overline{L} < 1$ implies that T is a contraction mapping with the Lipschitz constant $\overline{L} < 1$. The Banach fixed point theorem applies, yielding a unique solution $\varpi^* \in P$. From the definition of $\{\varpi_n\}_{n=1}^\infty$ and (8) we ensure

$$\|\varpi_{n+1}-\varpi_n\|_X \leqslant \overline{L}\|\varpi_n - \varpi_{n-1}\|_X \leqslant \cdots \leqslant \overline{L}^n\|\varpi_1-\varpi_0\|_X.$$

Then, for any $m \in \mathbb{N}$,

$$\begin{aligned}
\|\varpi_{n+m}-\varpi_n\|_X &\leqslant \|\varpi_{n+1}-\varpi_n\|_X + \cdots + \|\varpi_{n+m}-\varpi_{n+m-1}\|_X \\
&\leqslant \sum_{k=0}^{m-1}\overline{L}^{n+k}\|\varpi_1-\varpi_0\|_X = \overline{L}^n\frac{1-\overline{L}^m}{1-\overline{L}}\|\varpi_1-\varpi_0\|_X.
\end{aligned} \tag{9}$$

Then we get $\|\varpi_{n+m}-\varpi_n\|_X \to 0$ as $n\to\infty$. That is, $\{\varpi_n\}_{n=1}^\infty \subset P$ is a Cauchy sequence. The sequence $\{\varpi_n\}_{n=1}^\infty$ satisfies $\|\varpi_n - \varpi^\natural\|_X \to 0$ as $n\to\infty$, where $\varpi_n = T\varpi_{n-1}$. As a result, $T\varpi_{n-1} = \varpi_n \to \varpi^\natural = T\varpi^\natural$ as $n\to\infty$. This yields $\varpi^\natural$ is a fixed point of T. By uniqueness, $\varpi^\natural = \varpi^*$. Also, from (9), for $m\to\infty$, we ensure

$$\|\varpi^* - \varpi_n\|_X \leqslant \frac{\overline{L}^n}{1-\overline{L}}\|\varpi_1-\varpi_0\|_X. \qquad \square$$

Example. We examine the following problem:

$$
\begin{aligned}
&\varphi_2\big({}^H\mathfrak{D}_{1^+}^{5/2}\varpi(z)\big)+\tau\big(z,\varpi(z),{}^H\mathfrak{D}_{1^+}^{3/2}\varpi(z),{}^H\mathfrak{D}_{1^+}^{1/2}\varpi(z)\big)=0,\quad z\in[1,\infty),\\
&\varpi^{(k)}(1)=0,\quad 0\leqslant k\leqslant 1,\\
&{}^H\mathfrak{D}_{1^+}^{3/2}\varpi(\infty)=\frac{1}{2}{}^H\mathfrak{D}_{1^+}^{3/2}\varpi\big(\mathrm{e}^{1/4}\big)+\frac{1}{4}{}^H\mathfrak{D}_{1^+}^{3/2}\varpi\big(\mathrm{e}^{1/2}\big)+\frac{1}{5}{}^HI_{1^+}^{1/2}\varpi\big(\mathrm{e}^{1/4}\big)\\
&\qquad\qquad\qquad+\frac{1}{10}{}^HI_{1^+}^{1/2}\varpi\big(\mathrm{e}^{1/2}\big),
\end{aligned}\tag{10}
$$

where $m=k=2$, $n=3$, $\ell=5/2$, $r=3/2$, $c=1/2$, $p=q=2$, $a_1=1/2$, $a_2=1/4$, $b_1=1/5$, $b_2=1/10$, $\xi_1=\eta_1=\mathrm{e}^{1/4}$, $\xi_2=\eta_2=\mathrm{e}^{1/2}$, $\lambda_1=0.2$, $\lambda_2=0.4$, $\lambda_3=0.6$, and the function τ is as follows:

$$
\begin{aligned}
&\tau\big(z,\varpi(z),{}^H\mathfrak{D}_{1^+}^{3/2}\varpi(z),{}^H\mathfrak{D}_{1^+}^{1/2}\varpi(z)\big)\\
&\quad=\frac{\mathrm{e}^{-z}\log(z+1)}{5}+\frac{\mathrm{e}^{-z}\log(z+1)|\varpi(z)|^{0.2}}{10(1+(\log z)^{3/2})^{0.2}}+\frac{\log(2z+1)|{}^H\mathfrak{D}_{1^+}^{3/2}\varpi(z)|^{0.4}}{(z+1)^2}\\
&\qquad+\frac{\mathrm{e}^{-3z}\log(z+1)|{}^H\mathfrak{D}_{1^+}^{1/2}\varpi(z)|^{0.6}}{(1+\log z)^{0.6}}\\
&\quad\leqslant\frac{\mathrm{e}^{-z}z}{5}+\frac{\mathrm{e}^{-z}z|\varpi(z)|^{0.2}}{10(1+(\log z)^{3/2})^{0.2}}+\frac{2z|{}^H\mathfrak{D}_{1^+}^{3/2}\varpi(z)|^{0.4}}{(z+1)^2}+\frac{\mathrm{e}^{-3z}z|{}^H\mathfrak{D}_{1^+}^{1/2}\varpi(z)|^{0.6}}{(1+\log z)^{0.6}}\\
&\quad=r_0(z)+r_1(z)|\varpi(z)|^{0.2}+r_2(z)\big|{}^H\mathfrak{D}_{1^+}^{3/2}\varpi(z)\big|^{0.4}+r_3(z)\big|{}^H\mathfrak{D}_{1^+}^{1/2}\varpi(z)\big|^{0.6},
\end{aligned}
$$

$$
\begin{aligned}
r_0&=\int_1^\infty\frac{\mathrm{e}^{-s}s}{5}\frac{\mathrm{d}s}{s}=\frac{1}{5\mathrm{e}}<\infty, &\qquad r_1&=\int_1^\infty\frac{\mathrm{e}^{-s}s}{10}\frac{\mathrm{d}s}{s}=\frac{1}{10\mathrm{e}}<\infty,\\
r_2&=\int_1^\infty\frac{2s}{(s+1)^2}\frac{\mathrm{d}s}{s}=1<\infty, &\qquad r_3&=\int_1^\infty\frac{\mathrm{e}^{-3s}s}{1}\frac{\mathrm{d}s}{s}=\frac{1}{3\mathrm{e}^3}<\infty.
\end{aligned}
$$

This indicates that hypothesis (K_1) is fulfilled. Also, $\tau\in\mathcal{C}([1,\infty)\times[0,\infty)^3,[0,\infty))$, $\tau(z,0,\dots,0)$ is not equal to 0 for $z\in[1,\infty)$, and it can be observed that τ is increasing with respect to the second, third, and last variables, which imply that conditions (K_0), (K_2) are satisfied. Moreover, $\Delta=51\sqrt{\pi}/320>0$, $\kappa_1=\kappa_2=640/(51\sqrt{\pi})$, $L\approx 9.41$, $R\geqslant 200.32$. Hence, by Theorem 1, problem (10) has at least two positive solutions.

4 Conclusion

This study investigates the existence of positive solutions for a Hadamard fractional boundary value problem with derivative and integral boundary conditions with the p-Laplacian operator on an infinite interval.

By using certain properties related to the p-Laplacian operator, selecting an appropriate Banach space, and employing the monotone iterative method, a new result regarding the existence of positive solutions has been obtained. The iterative schemes are initialized with a basic function, which is advantageous for computational purposes. As far as we are aware, no studies have been dedicated to the investigation of positive solutions for Hadamard fractional boundary value problems involving the p-Laplacian operator on an infinite intervals. Moreover, the nonlinear part τ contains lower order HFD operators, which creates additional complexity in verifying the existence of positive solutions. These represent the main contributions of our study.

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