Nonlinear Analysis: Modelling and Control, Vol. 31, No. 1, 160–177
https://doi.org/10.15388/namc.2026.31.44253

Fundamental contractions in suprametric spaces: Analysis and applications*

Saad Jamhan Aldosari[a], **Haroon Ahmad**[b], **Mudasir Younis**[c,d], **Mahpeyker Öztürk**[c], **Farwa Asmat**[c]

[a]Department of Mathematics,
College of Science and Humanities in Alkharj,
Prince Sattam Bin Abdulaziz University,
Alkharj 11942, Saudi Arabia

[b]Abdus Salam School of Mathematical Sciences,
Government College University Lahore
Lahore 54600, Pakistan

[c]Department of Mathematics, Faculty of Sciences,
Sakarya University,
Sakarya, 54050, Turkey
mudasiryouniscuk@gmail.com; myounis@sakarya.edu.tr

[d]Department of Mathematics, Saveetha School of Engineering,
Saveetha Institute of Medical and Technical Sciences,
Saveetha University,
Chennai, 602105, Tamil Nadu, India

Received: September 13, 2024 / **Revised:** August 27, 2025 / **Published online:** January 1, 2026

Abstract. In this manuscript, we propose the notion of a strong extended s-suprametric space, a novel extension that outperforms both s-suprametric and extended suprametric spaces. It looks into the aspects of open and closed ball topologies within this structure. It also investigates the concepts of existence and uniqueness using basic contractions viz. Banach and Kannan contractions. Illustrative examples demonstrate how the strong extended s-suprametric space outperforms its extended equivalent. Our examples demonstrate the presence and distinctness of fixed points in this scenario. Furthermore, exploiting these newly launched results, the manuscript investigates the analysis of a boundary value problem, including diffusing chemical material constrained between parallel walls with related concentrations at the boundaries, taking into account supplied raw density and recognized absorbing coefficients. It also applies these insights to a nonlinear boundary value issue involving satellite web coupling in which a thin sheet joins two cylindrical spacecraft. This coupling causes nonlinearity, resulting in a separate boundary value issue influenced by radiation effects within the satellites.

Keywords: strong extended s-suprametric space, Banach contraction principle, diffusing chemical material, satellite web coupling.

*This study is supported via funding from Prince Sattam bin Abdulaziz University, project number (PSAU/2025/R/1446).

1 Introduction

Fixed point theory finds applications in numerous fields, namely variational inequalities, initial and boundary value problems of differential equations, financial mathematics, biology, computer science, physics, and others domains. In recent years, the fixed point theory has witnessed extensive development [22]. After the revolutionary contribution of Banach [5] to fixed point theory, multiple pioneers proceeded to advance the field, extending its applications and theoretical foundations. Among these researchers is Kuratowski [12], who has made substantial contributions to developing the theory and investigating its applications in partnership with Banach. Their teamwork set the framework for subsequent advances in fixed point theory. Furthermore, the seminal works of Saks [17] and Rozenblyum [16] added to the field's richness, providing novel insights and increasing our understanding of fixed point occurrences. Rozenblyum's improvements to the theory of multivalued mappings and Saks' explorations into the topological features of fixed point sets were crucial in modelling the environment of fixed point theory. Together with Banach, these forefathers opened the path for the considerable study and various applications that characterize the current state of fixed point theory [18, 19].

In the realm of metric spaces, significant contributions were made in the late 20th and early 21st centuries, which expanded the foundational understanding of these mathematical structures. Initially, the concept of b-metric spaces was introduced by Bakhtin in 1989 [4], followed by an influential work by Czerwik in 1993 [9], both of whom laid the groundwork for this field of study. Building upon their seminal contributions, Kamran in 2017 [11] advanced this area of research by introducing the notion of extended b-metric spaces, thereby broadening the scope and applicability of b-metric space theory. In 2018, Maliki [14] pioneered the concept of controlled metric-type spaces and unveiled several fixed point theorems within these frameworks. In his seminal work, Maliki articulated the definition of a controlled metric-type space as a set endowed with a control function that quantifies the distance between any two points. This function is mandated to fulfil specific criteria, thereby extending the classical attributes of a metric space. Subsequently, Abdeljawad [1] expanded upon Maliki's foundation by incorporating an additional function into the concept of the triangular inequality, leading to the evolution of double controlled metric type spaces, and delved into the existence and uniqueness of fixed point theorems therein. For foundational insights into fixed point theory, researchers are directed to consult [2, 3, 13, 21, 24].

In 2022, Berzig's research [6] proposed primary concepts for suprametric spaces, building the foundations for subsequent investigation. The research provides preliminary information about the features and applications of suprametric spaces. Looking further, in 2023, Berzig [8] investigated the study of nonlinear contractions inside the context of b-suprametric spaces. The findings, presented in an arXiv preprint demonstrated the complexities of nonlinear contractions in this particular suprametric setting. Concurrently, in 2023, Panda et al. [15] made significant contributions, broadening the breadth of suprametric spaces. Their cooperative determination proposed the notion of extended suprametric spaces and built links with Stone-type theorems, expanding the theoretical framework. In the subsequent year, Berzig [7] kept going to advance his argument by demonstrating fixed point results in generalized suprametric spaces. The research shed new light on the presence of fixed points in the setting of generalized suprametric spaces. This study timeline highlights the dynamic development and ongoing advances in investigating suprametric realms.

Motivated by the literature mentioned above and keeping in view the applications of suprametric spaces, in this article, we launch the notion of a strong extended s-suprametric space; the study intends to overcome limitations in standard suprametric spaces, highlighting open and closed ball topologies. Dealing with crucial features like existence and uniqueness employing the Banach fixed point theorem, this study presents an illustrative example emphasizing the more accurate achievement of the strong extended s-suprametric space. It also applies to real-world problems, such as a boundary value problem with spreading chemical material and nonlinear issues in satellite web coupling. This research advances our understanding of suprametric spaces and provides a flexible structure for various mathematical and scientific contexts.

For the continuation of our research, it is crucial to revisit and underscore a number of fundamental definitions.

Definition 1. Let $\Lambda_s : \Omega \times \Omega \to [0, \infty)$ be a mapping defined on a nonempty set Ω that satisfies:

(i) $\Lambda_s(\hbar, \wp) = 0 \Leftrightarrow \hbar_1 = \hbar_2$,
(ii) $\Lambda_s(\hbar, \wp) = \Lambda_s(\wp, \hbar)$

for all $\hbar, \wp \in \Omega$. Then (Ω, Λ_s) is referred to as a semimetric space.

Definition 2. (See [6].) Let (Ω, Λ_s) be a semimetric space that satisfies

$$\Lambda_s(\hbar, \ell) \leqslant \Lambda_s(\hbar, \wp) + \Lambda_s(\wp, \ell) + \alpha \Lambda_s(\hbar, \wp)\Lambda_s(\wp, \ell),$$

where $\alpha \geqslant 0$. Then the pair (Ω, Λ_s) is called a suprametric space.

Definition 3. (See [8].) If a semimetric space links with

$$\Lambda_s(\hbar, \ell) \leqslant b\big[\Lambda_s(\hbar, \wp) + \Lambda_s(\wp, \ell)\big] + \alpha \Lambda_s(\hbar, \wp)\Lambda_s(\wp, \ell),$$

where $\alpha \geqslant 0$ and $b \geqslant 1$, then pair (Ω, Λ_s) is known as a b-suprametric space.

Definition 4. (See [15].) If for all $\hbar, \ell, \wp \in \Omega$, the inequality

$$\Lambda_s(\hbar, \ell) \leqslant \Lambda_s(\hbar, \wp) + \Lambda_s(\wp, \ell) + \alpha(\hbar, \ell)\Lambda_s(\hbar, \wp)\Lambda_s(\wp, \ell)$$

holds, then the pair (Ω, Λ_s) is called an extended suprametric space.

2 Strong extended s-uprametric spaces

This section discusses the idea of strong extended s-suprametric spaces.

Definition 5. Let $\alpha : \Omega \times \Omega \to [1, +\infty)$ and $\Lambda_s : \Omega \times \Omega \to [0, +\infty)$ be a pair of mappings on a nonempty set Ω with $s \geqslant 1$ that satisfies

(i) $\Lambda_s(\hbar, \wp) = 0$ if and only if $\hbar = \wp$,
(ii) $\Lambda_s(\hbar, \wp) = \Lambda_s(\wp, \hbar)$,
(iii) $\Lambda_s(\hbar, \ell) \leqslant s\Lambda_s(\hbar, \wp) + \Lambda_s(\wp, \ell) + \alpha(\hbar, \ell)\Lambda_s(\hbar, \wp)\Lambda_s(\wp, \ell)$,

third property can also be used as

(iv) $\Lambda_s(\hbar, \ell) \leqslant \Lambda_s(\hbar, \wp) + s\Lambda_s(\wp, \ell) + \alpha(\hbar, \ell)\Lambda_s(\hbar, \wp)\Lambda_s(\wp, \ell)$

for all $\hbar, \wp, \ell \in \Omega$. Then (Ω, Λ_s) is called a strong extended s-suprametric space.

Example 1. Assume that $\alpha : \Omega \times \Omega \to [1, \infty)$ is defined by $\alpha(\hbar, \wp) = \mathrm{e}^{\hbar + \wp}$ and also $\Lambda_s : \Omega \times \Omega \to [0, \infty)$ is defined by $\Lambda_s(\hbar, \wp) = (\hbar - \wp)^2$, where Ω consists set of natural numbers. Then (Ω, Λ_s) is said to be a strong extended s-suprametric space with $s \geqslant 1$.

Proof. Properties (i) and (ii) are obvious. For (iii), we have

$$\begin{aligned} \Lambda_s(\hbar, \ell) &= (\hbar - \ell)^2 = (\hbar - \wp + \wp - \ell)^2 \\ &= (\hbar - \wp)^2 + (\wp - \ell)^2 + 2(\hbar - \ell)(\hbar - \ell) \\ &\leqslant (\hbar - \wp)^2 + (\wp - \ell)^2 + 2(\hbar - \wp)^2(\wp - \ell)^2 \\ &< s(\hbar - \wp)^2 + (\wp - \ell)^2 + \mathrm{e}^{\hbar + \ell}(\hbar - \wp)^2(\wp - \ell)^2. \end{aligned}$$

This can also be expressed as follows:

$$\Lambda_s(\hbar, \ell) < (\hbar - \wp)^2 + s(\wp - \ell)^2 + \mathrm{e}^{\hbar + \ell}(\hbar - \wp)^2(\wp - \ell)^2.$$

Therefore, we obtain the resulting expression:

$$\Lambda_s(\hbar, \ell) \leqslant \Lambda_s(\hbar, \wp) + s\Lambda_s(\wp, \ell) + \alpha(\hbar, \ell)\Lambda_s(\hbar, \wp)\Lambda_s(\wp, \ell)$$

for all $\hbar, \wp, \ell \in \Omega$, so the pair (Ω, Λ_s) is a strong extended s-suprametric space. □

Remark 1. If we set $s = 1$, the strong extended s-suprametric space reduces to an extended s-suprametric space, as defined in [15]. Furthermore, if $s = 1$ and $\alpha(\hbar, \ell) = \alpha \geqslant 0$, the strong extended s-suprametric space simplifies to a standard suprametric space, as discussed in [6].

Definition 6. Let (Ω, Λ_s) be a strong extended s-suprametric space.

(i) $\{\hbar_n\}_{n \in \mathbb{N}}$ converges to some $\hbar$ in Ω if for each positive ϵ, there is some positive N_ϵ such that $\Lambda_s(\hbar_n, \hbar) < \epsilon$ for each $n \geqslant N_\epsilon$, which can be expressed as $\lim_{n \to \infty} \hbar_n = \hbar$.
(ii) The sequence $\{\hbar_n\}_{n \in \mathbb{N}}$ is said to be a Cauchy sequence if for every $\epsilon > 0$, $\Lambda_s(\hbar_n, \hbar_m) < \epsilon$ for all $m, n \geqslant N_\epsilon$, where $N_\epsilon \in \mathbb{N}$.
(iii) (Ω, Λ_s) is known as complete if every Cauchy sequence is convergent in Ω.

Remark 2. Let (Ω, Λ_s) be a strong extended s-suprametric space.

(i) Every convergent sequence has a unique limit if Λ_s is continuous.
(ii) If a complete strong extended s-suprametric space with $\{\hbar_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, then there exists

$$\hbar^* \in \Omega \quad \Longrightarrow \quad \lim_{n\to\infty} \Lambda_s(\hbar_n, \hbar) = 0,$$

and every subsequence $\{\hbar_{n_k}\}_{k\in\mathbb{N}}$ converges to $\hbar^*$.

Definition 7. Let (Ω, Λ_s) be a strong extended s-suprametric space with center at $a \in \Omega$ and radius $r > 0$. Then the open ball is defined as

$$\mathbb{B}(a, r) = \{\hbar \in \Omega\colon\ \Lambda_s(a, \hbar) < r\},$$

and closed ball as

$$\mathbb{B}[a, r] = \{\hbar \in \Omega\colon\ \Lambda_s(a, \hbar) \leqslant r\}.$$

Proposition 1. *Let (Ω, Λ_s) be a strong extended s-suprametric space. Then*:

(i) *every open ball is an open set,*
(ii) *every closed ball is a closed set.*

Proof. For (i), let $a \in \Omega$ and $r > 0$. For $\wp \in \mathbb{B}(a, r)$, assume that

$$r_1 = \frac{r - \Lambda_s(\wp, a)}{s + \alpha(\hbar, a)\Lambda_s(\wp, a)},$$

and if $\hbar \in \mathbb{B}(\wp, r_1)$,

$$\begin{aligned}\Lambda_s(\hbar, a) &\leqslant s\Lambda_s(\hbar, \wp) + \Lambda_s(\wp, a) + \alpha(\hbar, a)\Lambda_s(\hbar, \wp)\Lambda_s(\wp, a)\\ &< sr_1 + \Lambda_s(\wp, a) + \alpha(\hbar, a)r_1\Lambda_s(\wp, a) < r.\end{aligned}$$

Therefore, $\mathbb{B}(\wp, r_1) \subseteq \mathbb{B}(a, r)$, and $\mathbb{B}(a, r)$ is open.

For (ii), assume that $a \in \Omega$, $r > 0$, and choose a sequence $\{\hbar_n\}_{n\in\mathbb{N}}$ in $\mathbb{B}[a, r]$ converging to some $\hbar$ with respect to Λ_s such that

$$\begin{aligned}\Lambda_s(a, \hbar) &\leqslant \Lambda_s(a, \hbar_n) + s\Lambda_s(\hbar_n, \hbar) + \alpha(a, \hbar)\Lambda_s(a, \hbar_n)\Lambda_s(\hbar_n, \hbar)\\ &\leqslant r + \big(s + \alpha(a, \hbar)r\big)\Lambda_s(\hbar_n, \hbar).\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain $\hbar \in \mathbb{B}[a, r]$, which means that $\mathbb{B}[a, r]$ is closed. □

Proposition 2. *For a strong extended s-suprametric space (Ω, Λ_s), the family of open balls forms a base of topology on Ω.*

Proof. Suppose $u \in \mathbb{B}(\hbar, \epsilon), \mathbb{B}(\wp, \epsilon')$, and also determine $r > 0$ such that

$$r + \big(s + r\alpha(\hbar, v)\big)\Lambda_s(\hbar, u) < \epsilon$$

and

$$r + \big(s + r\alpha(\wp, v)\big)\Lambda_s(\wp, u) < \epsilon'.$$

Therefore, by identifying a point $v \in \mathbb{B}(u, r)$, we achieve

$$\begin{aligned} \Lambda_s(\hbar, v) &\leqslant s\Lambda_s(\hbar, u) + \Lambda_s(u, v) + \alpha(\hbar, v)\Lambda_s(\hbar, u)\Lambda_s(u, v) \\ &< \bigl(s + r\alpha(\hbar, v)\bigr)\Lambda_s(\hbar, u) + r < \epsilon. \end{aligned}$$

Likewise, we receive

$$\begin{aligned} \Lambda_s(\wp, v) &\leqslant s\Lambda_s(\wp, u) + \Lambda_s(u, v) + \alpha(\wp, v)\Lambda_s(\wp, u)\Lambda_s(u, v) \\ &< \bigl(s + r\alpha(\wp, v)\bigr)\Lambda_s(\wp, u) + r < \epsilon', \end{aligned}$$

which implies $\mathbb{B}(u, r) \subset \mathbb{B}(\hbar, \epsilon) \cap \mathbb{B}(\wp, \epsilon')$. By means of [10, Lemma 1.4.7], we note that every $\hbar \in \Omega$ is also in $\mathbb{B}(\hbar, \tau)$ for some $\tau > 0$. □

3 Convergence results

This section discusses the existence and uniqueness of fixed points in a strongly extended s-suprametric space through Banach and Kannan contraction.

Theorem 1. *Let $\mathcal{P} : \Omega \to \Omega$ be a continuous mapping on a complete strong extended s-suprametric space (Ω, Λ_s), then*:

(i) *There exists $\beta \in [0, 1)$ such that*

$$\Lambda_s(\mathcal{P}\hbar, \mathcal{P}\wp) \leqslant \beta\Lambda_s(\hbar, \wp) \tag{1}$$

for all $\hbar, \wp \in \Omega$ with the iterative sequence $\hbar_n = \mathcal{P}\hbar_{n-1}$, $n \in \mathbb{N}$, where $\hbar_0 \in \Omega$.

(ii) *Ω has a unique fixed point in (Ω, Λ_s).*

Proof. Construct a sequence $\hbar_n = \mathcal{P}\hbar_{n-1}$ for all $n \in \mathbb{N}$, and let $\hbar_0 \in \Omega$ be arbitrary. In case $\beta = 0$, then it is trivial that $\mathcal{P}$ has a fixed point because $\mathcal{P}$ is a continuous mapping. For $0 < \beta < 1$, we define $\Lambda_s = \{\Lambda_s(\hbar, \mathcal{P}\hbar),\ \hbar \in \Omega\}$ and put $r = \inf \Lambda_s$. Assuming that $r > 0$ and $0 < \beta < 1$, we have $r/\beta > r$ which means that $\hbar_p \in \Omega$ is such that

$$\Lambda_s(\hbar_p, \mathcal{P}\hbar_p) < \frac{r}{\beta}.$$

Then we obtain

$$\Lambda_s\bigl(\mathcal{P}\hbar_p, \mathcal{P}^2\hbar_p\bigr) \leqslant \beta\Lambda_s(\hbar_p, \mathcal{P}\hbar_p) < r,$$

which is a contradiction and implies $r = 0$. Then there exists a sequence $\{\hbar_n\}_{n\in\mathbb{N}} \in \Omega$ such that $\lim_{n\to\infty} \Lambda_s(\hbar_n, \mathcal{P}\hbar_n) = 0$. Therefore, for all $\epsilon > 0$, there exists $\lambda \in \mathbb{N}$ such that for all $n \geqslant \lambda$, we obtain

$$\Lambda_s(\hbar_n, \hbar_{n+1}) < \epsilon. \tag{2}$$

Now, it is necessary to present that $\{\hbar_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Using (2) and for a, b such that $a > b > \lambda$, we get

$$\begin{aligned} \Lambda_s(\hbar_a, \hbar_b) \leqslant{}& \Lambda_s(\hbar_a, \hbar_{a+1}) + s\Lambda_s(\hbar_{a+1}, \hbar_b) \\ &+ \alpha(\hbar_a, \hbar_b)\Lambda_s(\hbar_a, \hbar_{a+1})\Lambda_s(\hbar_{a+1}, \hbar_b) \end{aligned}$$

$$\begin{aligned}
&\leqslant \beta^{a-\lambda}\varLambda_s(\hbar_\lambda,\hbar_{\lambda+1}) + s\varLambda_s(\hbar_{a+1},\hbar_b)\\
&\quad + \alpha(\hbar_a,\hbar_b)\beta^{a-\lambda}\varLambda_s(\hbar_\lambda,\hbar_{\lambda+1})\varLambda_s(\hbar_{a+1},\hbar_b)\\
&\leqslant \beta^{a-\lambda}\epsilon + s\varLambda_s(\hbar_{a+1},\hbar_b) + \alpha(\hbar_a,\hbar_b)\beta^{a-\lambda}\epsilon\varLambda_s(\hbar_{a+1},\hbar_b)\\
&= \beta^{a-\lambda}\epsilon + \bigl[s + \alpha(\hbar_a,\hbar_b)\beta^{a-\lambda}\epsilon\bigr]\varLambda_s(\hbar_{a+1},\hbar_b). \qquad (3)
\end{aligned}$$

Similarly, we achieve

$$\begin{aligned}
\varLambda_s(\hbar_{a+1},\hbar_b) &\leqslant \varLambda_s(\hbar_{a+1},\hbar_{a+2}) + s\varLambda_s(\hbar_{a+2},\hbar_b)\\
&\quad + \alpha(\hbar_{a+1},\hbar_b)\varLambda_s(\hbar_{a+1},\hbar_{a+2})\varLambda_s(\hbar_{a+2},\hbar_b)\\
&\leqslant \beta^{a-\lambda+1}\varLambda_s(\hbar_{\lambda+1},\hbar_{\lambda+2}) + s\varLambda_s(\hbar_{a+2},\hbar_b)\\
&\quad + \alpha(\hbar_{a+1},\hbar_b)\beta^{a-\lambda+1}\varLambda_s(\hbar_{\lambda+1},\hbar_{\lambda+2})\varLambda_s(\hbar_{a+2},\hbar_b)\\
&\leqslant \beta^{a-\lambda+1}\epsilon + s\varLambda_s(\hbar_{a+2},\hbar_b) + \alpha(\hbar_{a+1},\hbar_b)\beta^{a-\lambda+1}\epsilon\varLambda_s(\hbar_{a+2},\hbar_b)\\
&= \beta^{a-\lambda+1}\epsilon + \bigl[s + \alpha(\hbar_{a+1},\hbar_b)\beta^{a-\lambda+1}\epsilon\bigr]\varLambda_s(\hbar_{a+2},\hbar_b). \qquad (4)
\end{aligned}$$

From (3) and (4) we obtain

$$\begin{aligned}
\varLambda_s(\hbar_a,\hbar_b) &\leqslant \beta^{a-\lambda}\epsilon + \bigl[s + \alpha(\hbar_a,\hbar_b)\beta^{a-\lambda}\epsilon\bigr]\varLambda_s(\hbar_{a+1},\hbar_b)\\
&\leqslant \beta^{a-\lambda}\epsilon + \bigl[s + \alpha(\hbar_a,\hbar_b)\beta^{a-\lambda}\epsilon\bigr]\\
&\quad \times \bigl(\beta^{a-\lambda+1}\epsilon + \bigl[s + \alpha(\hbar_{a+1},\hbar_b)\beta^{a-\lambda+1}\epsilon\bigr]\varLambda_s(\hbar_{a+2},\hbar_b)\bigr)\\
&\leqslant \beta^{a-\lambda}\epsilon + \bigl[s + \alpha(\hbar_a,\hbar_b)\beta^{a-\lambda}\epsilon\bigr]\\
&\quad \times \bigl(\beta^{a-\lambda+1}\epsilon + s\varLambda_s(\hbar_{a+2},\hbar_b) + \beta^{a-\lambda+1}\epsilon\alpha(\hbar_{a+1},\hbar_b)\varLambda_s(\hbar_{a+2},\hbar_b)\bigr)\\
&= \epsilon\beta^{a-\lambda} + s\epsilon\beta^{a-\lambda+1} + s^2\varLambda_s(\hbar_{a+2},\hbar_b) + s\epsilon\beta^{a-\lambda+1}\alpha(\hbar_{a+1},\hbar_b)\varLambda_s(\hbar_{a+2},\hbar_b)\\
&\quad + \epsilon^2\beta^{a-\lambda}\beta^{a-\lambda+1}\alpha(\hbar_a,\hbar_b) + s\epsilon\beta^{a-\lambda}\alpha(\hbar_a,\hbar_b)\varLambda_s(\hbar_{a+2},\hbar_b)\\
&\quad + \epsilon^2\beta^{a-\lambda}\beta^{a-\lambda+1}\alpha(\hbar_a,\hbar_b)\alpha(\hbar_{a+1},\hbar_b)\varLambda_s(\hbar_{a+2},\hbar_b)\\
&= \epsilon\beta^{a-\lambda} + \epsilon\beta^{a-\lambda+1}\bigl(s + \epsilon\beta^{a-\lambda}\alpha(\hbar_a,\hbar_b)\bigr) + s^2\varLambda_s(\hbar_{a+2},\hbar_b)\\
&\quad + s\epsilon\beta^{a-\lambda+1}\alpha(\hbar_{a+1},\hbar_b)\varLambda_s(\hbar_{a+2},\hbar_b) + s\epsilon\beta^{a-\lambda}\alpha(\hbar_a,\hbar_b)\varLambda_s(\hbar_{a+2},\hbar_b)\\
&\quad + \epsilon^2\beta^{a-\lambda}\beta^{a-\lambda+1}\alpha(\hbar_a,\hbar_b)\alpha(\hbar_{a+1},\hbar_b)\varLambda_s(\hbar_{a+2},\hbar_b)\\
&= \epsilon\beta^{a-\lambda} + \epsilon\beta^{a-\lambda+1}\bigl(s + \epsilon\beta^{a-\lambda}\alpha(\hbar_a,\hbar_b)\bigr)\\
&\quad + \bigl[\bigl(s + \epsilon\beta^{a-\lambda}\alpha(\hbar_a,\hbar_b)\bigr)\bigl(s + \epsilon\beta^{a-\lambda+1}\alpha(\hbar_{a+1},\hbar_b)\bigr)\bigr]\varLambda_s(\hbar_{a+2},\hbar_b).
\end{aligned}$$

Continue the similar process using (3) and (4) until we achieve

$$\varLambda_s(\hbar_a,\hbar_b) \leqslant \epsilon\beta^{a-\lambda}\sum_{i=0}^{b-a-1}\beta^i\prod_{j=0}^{i-1}\bigl[s + \epsilon\beta^{a-\lambda+j}\alpha(\hbar_{a+j},\hbar_b)\bigr],$$

where $\beta \in [0,1)$, so

$$\varLambda_s(\hbar_a,\hbar_b) \leqslant \epsilon\beta^{a-\lambda}\sum_{i=0}^{b-a-1}\beta^i\prod_{j=0}^{i-1}\bigl[s + \epsilon\beta^{j}\alpha(\hbar_{a+j},\hbar_b)\bigr]. \qquad (5)$$

Assume that

$$Q_i = \beta^i \prod_{j=0}^{i-1} \left[s + \epsilon\beta^j \alpha(\hbar_{a+j}, \hbar_b)\right]. \tag{6}$$

Through the ratio test (6), we obtain

$$\lim_{i\to\infty}\left|\frac{Q_{i+1}}{Q_i}\right| < 1$$

due to $\beta \in [0, 1)$. As a and b approach infinity, inequality (5) becomes

$$\lim_{a,b\to\infty} \Lambda_s(\hbar_a, \hbar_b) = 0,$$

which implies that the sequence $\{\hbar_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since the strong extended s-suprametric space is complete, by the completeness of Ω, it follows that $\hbar_n \to \hbar \in \Omega$. Using Remark 2 and inequality (1), we obtain

$$\Lambda_s(\mathcal{P}\hbar_{n_k}, \mathcal{P}\hbar) \leqslant \beta\Lambda_s(\hbar_{n_k}, \hbar),$$

also taking $k \to \infty$, we achieve that $\hbar = \mathcal{P}\hbar$.

Uniqueness. Assume that there are two distinct fixed points of the mapping $\mathcal{P}$ such that $\hbar_a \neq \hbar_b$. Then we have

$$\begin{aligned} \Lambda_s(\hbar_a, \hbar_b) &= \Lambda_s(\mathcal{P}\hbar_a, \mathcal{P}\hbar_b) \\ &\leqslant \beta\Lambda_s(\hbar_a, \hbar_b), \\ 1 &\leqslant \beta, \end{aligned}$$

which is a contradiction. Hence, $\mathcal{P}$ admits a unique fixed point in (Ω, Λ_s). □

Example 2. Let $\Omega = \{1/2, 1/3, 1/4, 1/5\}$ with $\Lambda_s : \Omega \times \Omega \to \mathbb{R}$ defined as

$$\Lambda_s(\hbar, \wp) = \begin{cases} 0 & \text{if } \hbar = \wp, \\ |\hbar - \wp|^2 & \text{if } \hbar \neq \wp, \end{cases}$$

and $\alpha : \Omega \times \Omega \to [1, \infty)$ defined as $\alpha(\hbar, \wp) = \mathrm{e}^{\hbar+\wp}$. Now, we will find the value of s for strong extended s-suprametric space:

$$\begin{aligned} \Lambda_s\left(\frac{1}{2}, \frac{1}{5}\right) &\leqslant s\Lambda_s\left(\frac{1}{2}, \frac{1}{3}\right) + \Lambda_s\left(\frac{1}{3}, \frac{1}{5}\right) + \mathrm{e}^{1/2+1/5}\Lambda_s\left(\frac{1}{2}, \frac{1}{3}\right), \Lambda_s\left(\frac{1}{3}, \frac{1}{5}\right), \\ 0.09 &\leqslant s \cdot 0.028 + 0.0178 + 2.01 \cdot 0.028 \cdot 0.0178, \\ 2.54 &\leqslant s. \end{aligned}$$

Evidently, (Ω, Λ_s) is a strong extended s-suprametric space for $s \geqslant 2.54$. Define a mapping $\mathcal{P} : \Omega \to \Omega$ as

$$\mathcal{P}\hbar = \begin{cases} \frac{1}{4} & \text{if } x \in \{\frac{1}{4}, \frac{1}{5}\}, \\ \frac{1}{5} & \text{if } x \in \{\frac{1}{2}, \frac{1}{3}\}. \end{cases}$$

Case 1. For $\hbar = 1/5$ and $\wp = 1/2$, we achieve

$$\Lambda_s\left(\mathcal{P}\frac{1}{5}, \mathcal{P}\frac{1}{2}\right) \leqslant \beta\Lambda_s\left(\frac{1}{5}, \frac{1}{2}\right),$$
$$0.0025 \leqslant \beta \cdot 0.09.$$

Case 2. If $\hbar = 1/5$ and $\wp = 1/3$, we achieve

$$\Lambda_s\left(\mathcal{P}\frac{1}{5}, \mathcal{P}\frac{1}{3}\right) \leqslant \beta\Lambda_s\left(\frac{1}{5}, \frac{1}{3}\right),$$
$$0.0025 \leqslant \beta \cdot 0.0178.$$

For $\beta \geqslant 0.14$, both cases are satisfied. Therefore, all the criteria of Theorem 1 are provided, and $1/4$ is the unique fixed point of the mapping $\mathcal{P}$.

Theorem 2. *Let $\mathcal{P} : \Omega \to \Omega$ be a continuous mapping on a complete strong extended s-suprametric space (Ω, Λ_s), then*:

(i) *There exist $\beta \in [0, 1/2)$ and $s \geqslant 1$ such that*

$$\Lambda_s(\mathcal{P}\hbar, \mathcal{P}\wp) \leqslant \frac{\beta}{s}\left[\Lambda_s(\hbar, \mathcal{P}\hbar) + \Lambda_s(\wp, \mathcal{P}\wp)\right] \tag{7}$$

for all $\hbar, \wp \in \Omega$ with iterative sequence $\hbar_n = \mathcal{P}\hbar_{n-1}$, $n \in \mathbb{N}$, where $\hbar_0 \in \Omega$.

(ii) *Ω has a unique fixed point.*

Proof. Choose arbitrary $\hbar_0 \in \mathfrak{X}$ and set up a sequence $\{\hbar_n\}_{n\in\mathbb{N}}$ in $\mathfrak{X}$ such that $\hbar_{n+1} = \Gamma\hbar_n = \Gamma_0^n\hbar$ for all $n \in \mathbb{N} \cup \{0\}$. If $\hbar_{n_0} = \hbar_{n_0+1}$ for some n_0, then we have $\hbar_{n_0} = \Gamma\hbar_{n_0}$, hence $\hbar_{n_0}$ is a fixed point of Γ. Suppose that $\hbar_n \neq \hbar_{n+1}$ for all $n \geqslant 0$. By using inequality (7), we obtain that

$$\begin{aligned}\Lambda_s(\hbar_n, \hbar_{n+1}) &= \Lambda_s(\mathcal{P}\hbar_{n-1}, \mathcal{P}\hbar_n)\\ &\leqslant \frac{\beta}{s}\left[\Lambda_s(\hbar_{n-1}, \mathcal{P}\hbar_{n-1}) + \Lambda_s(\hbar_n, \mathcal{P}\hbar_n)\right]\\ &\leqslant \frac{\beta}{s-\beta}\Lambda_s(\hbar_n, \hbar_{n-1}).\end{aligned}$$

If we set $\tau = \beta/(s-\beta)$, where $\tau \in [0, 1/s)$, then above inequality reduces to

$$\Lambda_s(\hbar_n, \hbar_{n+1}) \leqslant \tau\Lambda_s(\hbar_n, \hbar_{n-1}).$$

Continue this scheme until we obtain

$$\Lambda_s(\hbar_n, \hbar_{n+1}) \leqslant \tau^n\Lambda_s(\hbar_0, \hbar_1). \tag{8}$$

We need to demonstrate that $\{\hbar_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Using (8) and for a, b such that $a > b > \lambda$, we derive that

$$\begin{aligned}\Lambda_s(\hbar_a, \hbar_b) \leqslant{}& \Lambda_s(\hbar_a, \hbar_{a+1}) + s\Lambda_s(\hbar_{a+1}, \hbar_b)\\ &+ \alpha(\hbar_a, \hbar_b)\Lambda_s(\hbar_a, \hbar_{a+1})\Lambda_s(\hbar_{a+1}, \hbar_b)\end{aligned}$$

$$\begin{aligned}&\leqslant \tau^{a-\lambda}\Lambda_s(\hbar_\lambda,\hbar_{\lambda+1}) + s\Lambda_s(\hbar_{a+1},\hbar_b)\\&\quad + \alpha(\hbar_a,\hbar_b)\tau^{a-\lambda}\Lambda_s(\hbar_\lambda,\hbar_{\lambda+1})\Lambda_s(\hbar_{a+1},\hbar_b)\\&\leqslant \tau^{a-\lambda}\Lambda_s(\hbar_0,\hbar_1) + s\Lambda_s(\hbar_{a+1},\hbar_b)\\&\quad + \alpha(\hbar_a,\hbar_b)\tau^{a-\lambda}\Lambda_s(\hbar_0,\hbar_1)\Lambda_s(\hbar_{a+1},\hbar_b)\\&= \tau^{a-\lambda}\Lambda_s(\hbar_0,\hbar_1) + \left[s + \alpha(\hbar_a,\hbar_b)\tau^{a-\lambda}\Lambda_s(\hbar_0,\hbar_1)\right]\\&\quad \times \Lambda_s(\hbar_{a+1},\hbar_b).\end{aligned} \tag{9}$$

Similarly, we achieve

$$\begin{aligned}\Lambda_s(\hbar_{a+1},\hbar_b) &\leqslant \Lambda_s(\hbar_{a+1},\hbar_{a+2}) + s\Lambda_s(\hbar_{a+2},\hbar_b)\\&\quad + \alpha(\hbar_{a+1},\hbar_b)\Lambda_s(\hbar_{a+1},\hbar_{a+2})\Lambda_s(\hbar_{a+2},\hbar_b)\\&\leqslant \tau^{a-\lambda+1}\Lambda_s(\hbar_{\lambda+1},\hbar_{\lambda+2}) + s\Lambda_s(\hbar_{a+2},\hbar_b)\\&\quad + \alpha(\hbar_{a+1},\hbar_b)\tau^{a-\lambda+1}\Lambda_s(\hbar_{\lambda+1},\hbar_{\lambda+2})\Lambda_s(\hbar_{a+2},\hbar_b)\\&\leqslant \tau^{a-\lambda+1}\Lambda_s(\hbar_0,\hbar_1) + s\Lambda_s(\hbar_{a+2},\hbar_b)\\&\quad + \alpha(\hbar_{a+1},\hbar_b)\tau^{a-\lambda+1}\Lambda_s(\hbar_0,\hbar_1)\Lambda_s(\hbar_{a+2},\hbar_b)\\&= \tau^{a-\lambda+1}\Lambda_s(\hbar_0,\hbar_1) + \left[s + \alpha(\hbar_{a+1},\hbar_b)\tau^{a-\lambda+1}\Lambda_s(\hbar_0,\hbar_1)\right]\\&\quad \times \Lambda_s(\hbar_{a+2},\hbar_b).\end{aligned} \tag{10}$$

From (3) and (4) we obtain

$$\begin{aligned}&\Lambda_s(\hbar_a,\hbar_b)\\&\quad\leqslant \tau^{a-\lambda}\Lambda_s(\hbar_0,\hbar_1) + \left[s + \alpha(\hbar_a,\hbar_b)\tau^{a-\lambda}\Lambda_s(\hbar_0,\hbar_1)\right]\Lambda_s(\hbar_{a+1},\hbar_b)\\&\quad\leqslant \tau^{a-\lambda}\Lambda_s(\hbar_0,\hbar_1) + \left[s + \alpha(\hbar_a,\hbar_b)\tau^{a-\lambda}\Lambda_s(\hbar_0,\hbar_1)\right]\\&\qquad \times \left(\tau^{a-\lambda+1}\Lambda_s(\hbar_0,\hbar_1) + \left[s + \alpha(\hbar_{a+1},\hbar_b)\tau^{a-\lambda+1}\Lambda_s(\hbar_0,\hbar_1)\right]\Lambda_s(\hbar_{a+2},\hbar_b)\right)\\&\quad\leqslant \tau^{a-\lambda}\Lambda_s(\hbar_0,\hbar_1) + \left[s + \alpha(\hbar_a,\hbar_b)\tau^{a-\lambda}\Lambda_s(\hbar_0,\hbar_1)\right]\\&\qquad \times \left(\tau^{a-\lambda+1}\Lambda_s(\hbar_0,\hbar_1) + s\Lambda_s(\hbar_{a+2},\hbar_b)\right.\\&\qquad \left. + \tau^{a-\lambda+1}\Lambda_s(\hbar_0,\hbar_1)\alpha(\hbar_{a+1},\hbar_b)\Lambda_s(\hbar_{a+2},\hbar_b)\right)\\&\quad= \Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda} + s\Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda+1} + s^2\Lambda_s(\hbar_{a+2},\hbar_b)\\&\qquad + s\Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda+1}\alpha(\hbar_{a+1},\hbar_b)\Lambda_s(\hbar_{a+2},\hbar_b)\\&\qquad + \left(\Lambda_s(\hbar_0,\hbar_1)\right)^2\tau^{a-\lambda}\tau^{a-\lambda+1}\alpha(\hbar_a,\hbar_b)\\&\qquad + s\Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda}\alpha(\hbar_a,\hbar_b)\Lambda_s(\hbar_{a+2},\hbar_b)\\&\qquad + \left(\Lambda_s(\hbar_0,\hbar_1)\right)^2\tau^{a-\lambda}\tau^{a-\lambda+1}\alpha(\hbar_a,\hbar_b)\alpha(\hbar_{a+1},\hbar_b)\Lambda_s(\hbar_{a+2},\hbar_b)\\&\quad= \Lambda_s(\hbar_0,\hbar_1)\beta^{a-\lambda} + \Lambda_s(\hbar_0,\hbar_1)\beta^{a-\lambda+1}\left(s + \Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda}\alpha(\hbar_a,\hbar_b)\right)\\&\qquad + s^2\Lambda_s(\hbar_{a+2},\hbar_b) + s\Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda+1}\alpha(\hbar_{a+1},\hbar_b)\Lambda_s(\hbar_{a+2},\hbar_b)\\&\qquad + s\Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda}\alpha(\hbar_a,\hbar_b)\Lambda_s(\hbar_{a+2},\hbar_b)\\&\qquad + \left(\Lambda_s(\hbar_0,\hbar_1)\right)^2\tau^{a-\lambda}\tau^{a-\lambda+1}\alpha(\hbar_a,\hbar_b)\alpha(\hbar_{a+1},\hbar_b)\Lambda_s(\hbar_{a+2},\hbar_b)\end{aligned}$$

$$
\begin{aligned}
&= \Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda} + \Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda+1}\big(s+\Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda}\alpha(\hbar_a,\hbar_b)\big) \\
&\quad + \big[\big(s+\Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda}\alpha(\hbar_a,\hbar_b)\big)\big(s+\Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda+1}\alpha(\hbar_{a+1},\hbar_b)\big)\big] \\
&\quad \times \Lambda_s(\hbar_{a+2},\hbar_b).
\end{aligned}
$$

Continue the same process using (9) and (10) until we achieve

$$
\Lambda_s(\hbar_a,\hbar_b) \leqslant \Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda}\sum_{i=0}^{b-a-1}\tau^i\prod_{j=0}^{i-1}\big[s+\Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda+j}\alpha(\hbar_{a+j},\hbar_b)\big],
$$

where $\tau \in [0, 1/s)$, thus we have

$$
\Lambda_s(\hbar_a,\hbar_b) \leqslant \Lambda_s(\hbar_0,\hbar_1)\tau^{a-\lambda}\sum_{i=0}^{b-a-1}\tau^i\prod_{j=0}^{i-1}\big[s+\Lambda_s(\hbar_0,\hbar_1)\tau^{j}\alpha(\hbar_{a+j},\hbar_b)\big]. \tag{11}
$$

Assume that

$$
Q_i = \tau^i\prod_{j=0}^{i-1}\big[s+\Lambda_s(\hbar_0,\hbar_1)\tau^{j}\alpha(\hbar_{a+j},\hbar_b)\big]. \tag{12}
$$

By using ratio test in (12), we achieve

$$
\lim_{i\to\infty}\left|\frac{Q_{i+1}}{Q_i}\right| < 1 \quad \text{since } \tau \in \left[0, \frac{1}{s}\right).
$$

Taking $\lim_{a,b\to\infty}$, inequality (11) becomes

$$
\lim_{a,b\to\infty}\Lambda_s(\hbar_a,\hbar_b) = 0,
$$

which implies that $\{\hbar_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and given that strong extended s-suprametric space is complete.

Uniqueness. Assume that there are two distinct fixed points of the mapping $\mathcal{P}$ such that $\hbar_a \neq \hbar_b$. Then we have

$$
\Lambda_s(\hbar_a,\hbar_b) = \Lambda_s(\mathcal{P}\hbar_a,\mathcal{P}\hbar_b) \leqslant \frac{\beta}{s}\big[\Lambda_s(\hbar_a,\mathcal{P}\hbar_a)+\Lambda_s(\hbar_b,\mathcal{P}\hbar_b)\big] = 0,
$$

which proves that $\mathcal{P}$ admits a unique fixed point in (Ω, Λ_s). □

Example 3. Let $\Omega = \{1,2,3,4\}$ with $\Lambda_s : \Omega\times\Omega \to \mathbb{R}$ is defined as

$$
\Lambda_s(\hbar,\wp) = \begin{cases} 0 & \text{if } \hbar = \wp, \\ |\hbar-\wp|^3 & \text{if } \hbar \neq \wp, \end{cases}
$$

and $\alpha : \Omega\times\Omega \to [1,\infty)$ as $\alpha(\hbar,\wp) = \mathrm{e}^{1/(\hbar^2+\wp^2)}$. Now, we will find the value of s for strong extended s-suprametric space as

$$
\begin{aligned}
\Lambda_s(1,4) &\leqslant \Lambda_s(1,3) + s\Lambda_s(3,4) + \mathrm{e}^{1/(1+16)}\Lambda_s(1,3)\Lambda_s(3,4), \\
27 &\leqslant 8 + s\cdot 1 + 1.06059\cdot 8\cdot 1, \\
s &\geqslant 10.51528.
\end{aligned}
$$

Clearly, (Ω, Λ_s) is a strong extended s-suprametric space for $s \geqslant 10.52$. Define a mapping $\mathcal{P} : \Omega \to \Omega$ as

$$\mathcal{P}\hbar = \begin{cases} 1 & \text{if } \hbar = 4, \\ 2 & \text{if } \hbar \in \{2, 3\}. \end{cases}$$

Case 1. For $\hbar = 4$ and $\wp = 3$, we achieve

$$\begin{aligned} \Lambda_s(\mathcal{P}4, \mathcal{P}3) &\leqslant \frac{\beta}{s}\big[\Lambda_s(4, \mathcal{P}4) + \Lambda_s(3, \mathcal{P}3)\big], \\ 1 &\leqslant \frac{\beta}{10.52} \cdot 28, \\ 1 &\leqslant \beta \cdot 2.6615. \end{aligned}$$

Case 2. If $\hbar = 4$ and $\wp = 2$, we obtain

$$\begin{aligned} \Lambda_s(\mathcal{P}4, \mathcal{P}2) &\leqslant \frac{\beta}{s}\big[\Lambda_s(4, \mathcal{P}4) + \Lambda_s(2, \mathcal{P}2)\big], \\ 1 &\leqslant \frac{\beta}{10.52} \cdot 27, \\ 1 &\leqslant \beta \cdot 2.5665. \end{aligned}$$

For $\beta = 3959/10000 \in [0, 1/2)$, both cases are satisfied. Therefore, all the criteria of Theorem 2 are provided and 2 is the unique fixed point of the mapping $\mathcal{P}$.

4 Application to web coupling of satellite

Inspired by the effective application of fixed-point approaches in diverse range of real-life problems, we solve a satellite web coupling boundary value problem [20] using (1). A satellite web connection can be visualized as a thin sheet connecting two cylindrical spacecraft. Radiation from web coupling between two satellites results in a special nonlinear boundary value problem

$$\begin{aligned} &-\frac{\mathrm{d}^2\hbar}{\mathrm{d}t^2} = \mu\hbar^4, \quad 0 < t < 1, \\ &\hbar(0) = 0, \qquad \hbar(1) = 1, \end{aligned} \tag{13}$$

where $w(t)$ represents radiations temperature for $t \in [0, 1]$, $\mu = al^2K^3/(\zeta b) > 0$ is a positive dimensionless constant, and K is the absolute temperature of both satellites. While at 0 temperature, the heat is radiated from the surface of the web into space. The distance between the two satellites is l, the positive constant that a expresses the properties of radiation on the surface of the web, ζ represents thermal conductivity, b expresses the thickness and factor 2 due to the radiation from the top and bottom surfaces. The defined problem has a solution $\hbar \in \Omega = C[[0, 1], \mathbb{R}]$ in the form of integral equation as

$$\hbar(t) = 1 - \mu \int_0^1 \Psi(t, \zeta)\hbar^4(\zeta)\,\mathrm{d}\zeta, \quad t \in [0, 1], \tag{14}$$

and $\Psi(t,\zeta) : [0,1] \times \mathbb{R}$ is a continuous Green's function linked with (13) and given as

$$\Psi(t,\zeta) = \begin{cases} t(1-\zeta), & 0 < t < \zeta, \\ \zeta(l-t), & \zeta < t < 1. \end{cases}$$

Define a strong extended s-suprametric space on $\Omega = C[I,\mathbb{R}]$, where $\Lambda_s : \Omega \times \Omega \to [0,\infty)$ as

$$\Lambda_s\big(\hbar(t), \wp(t)\big) = \sup_{t\in[0,1]} \big|\hbar(t) - \wp(t)\big|^2.$$

Let $\alpha : \Omega \times \Omega \to [1,\infty)$ defined as

$$\alpha\big(\hbar(t), \wp(t)\big) = \mathrm{e}^{\hbar+\wp+2}$$

for all $\hbar, \wp \in \Omega$ and $t \in [0,1]$. Clearly, (Ω, Λ_s) is complete strong extended s-suprametric space for $s \geqslant 3$. Let $\Theta : \Omega \to \Omega$ be a mapping defined as follows:

$$\Theta\hbar(t) = \hbar(t) = 1 - \mu \int_0^1 \Psi(t,\zeta)\hbar^4(\zeta)\,\mathrm{d}\zeta$$

for $t \in [0,1]$. Then $\hbar$ is a unique solution of (14) if and only if it is a fixed point of Θ. Next theorem provides us with sufficient condition for the existence and uniqueness of the solution of problem (13).

Theorem 3. *Let $\Theta : \Omega \to \Omega$ be a mapping defined on a complete strong extended s-suprametric space that provides*

$$\big|\hbar(t) - \wp(t)\big|^2 > 0 \quad \Longrightarrow \quad \big|\big(\hbar^2(t) + \wp^2(t)\big)\big(\big(\hbar(t) + \wp(t)\big)\big)\big| \leqslant \frac{k}{\mu^2},$$

where $k \in [0,4)$. Then the boundary value problem (13) *related to satellite web coupling admits a unique solution.*

Proof. Define $\Theta : \Omega \to \Omega$ as

$$\begin{aligned} &\big|\Theta\hbar(t) - \Theta\wp(t)\big|^2 \\ &\quad= \left|1 - \mu \int_0^1 \Psi(t,\zeta)\hbar^4(t)\,\mathrm{d}\zeta - 1 + \mu \int_0^1 \Psi(t,\zeta)\wp^4(t)\,\mathrm{d}\zeta\right|^2 \\ &\quad= \mu^2 \int_0^1 \big|\Psi(t,\zeta)\big(\hbar^4(t) - \wp^4(t)\big)\big|^2\,\mathrm{d}\zeta \\ &\quad= \mu^2 \int_0^1 \big|\Psi(t,\zeta)\big(\big(\hbar^2(t) + \wp^2(t)\big)\big(\hbar(t) + \wp(t)\big)\big)\big(\hbar(t) - \wp(t)\big)\big|^2\,\mathrm{d}\zeta \end{aligned}$$

$$\begin{aligned}&\leqslant \mu^2\big|\big(\big(\hbar^2(t)+\wp^2(t)\big)\big(\hbar(t)+\wp(t)\big)\big)\big|^2\big|\hbar(t)-\wp(t)\big|^2\left(\int_0^1 \Psi(t,\zeta)\,\mathrm{d}\zeta\right)^2\\&\leqslant \mu^2\left(\frac{k}{\mu^2}\right)\big|\hbar(t)-\wp(t)\big|^2\left(\int_0^1 \Psi(t,\zeta)\,\mathrm{d}\zeta\right)^2\\&= k\big|\hbar(t)-\wp(t)\big|^2\left(\int_0^1 \Psi(t,\zeta)\,\mathrm{d}\zeta\right)^2.\end{aligned}$$

Now, we calculate the Green's function

$$\int_0^1 \Psi(t,w)\,\mathrm{d}\zeta = \int_0^t t(1-\zeta)\,\mathrm{d}\zeta + \int_t^1 \zeta(l-t)\,\mathrm{d}\zeta = \frac{t^2-t+1}{2}.$$

Therefore, after taking supremum on both sides, the above expression becomes

$$\begin{gathered}\sup_{t\in[0,1]}\big|\Theta\hbar(t)-\Theta\wp(t)\big|^2 \leqslant k \sup_{t\in[0,1]}\big|\hbar(t)-\wp(t)\big|^2\left(\sup_{t\in[0,1]}\frac{t^2-t+1}{2}\right)^2,\\ \Lambda_s(\Theta\hbar,\Theta\wp)\leqslant \beta\Lambda_s(\hbar,\wp),\end{gathered}$$

where $k \in [0,4)$ and $\beta = k/4 \in [0,1)$. Hence, all the conditions of Theorem 1 are satisfied, and Θ admits a unique fixed point. Therefore problem (13) of satellite web coupling has a unique solution. □

5 Application to chemical sciences

Imagine that a diffusing chemical is positioned in an absorbent material within parallel walls with δ_1, δ_2 being the specified concentrations at the walls. Additionally, let $\Omega(t)$ be the supplied raw density, and let $\Theta(t)$ be the recognized absorbing coefficient. Following the aforementioned hypothesis, the concentration $\hbar(t)$ of the substance governs the following boundary value problem

$$\begin{aligned}-\hbar'' + \Theta(t), &\qquad \hbar = \Omega(t),\\ \hbar(0)=\delta_1, &\qquad \hbar(1)=\delta_2.\end{aligned}\tag{15}$$

The defined problem has a solution $\hbar \in \Omega = C[[0,1],\mathbb{R}]$ in the form of the integral equation as

$$\hbar(t) = \delta_1 + (\delta_2-\delta_1)t + \int_0^1 \Psi(t,\zeta)\big(\Omega(t)-\Theta(t)\hbar(t)\big)\,\mathrm{d}\zeta, \quad t\in[0,1], \tag{16}$$

and $\Psi(t,\zeta):[0,1]\times\mathbb{R}$ is a continuous Green's function given as

$$\Psi(t,\zeta)=\begin{cases}\zeta(1-t), & 0\leqslant\zeta\leqslant t,\\ t(1-\zeta), & t\leqslant\zeta\leqslant 1.\end{cases}$$

Define a strong extended s-suprametric space on $\Omega=C[I,\mathbb{R}]$, where $\Lambda_s:\Omega\times\Omega\to[0,\infty)$, as

$$\Lambda_s\big(\hbar(t),\wp(t)\big)=\sup_{t\in[0,1]}\big|\hbar(t)-\wp(t)\big|^2.$$

Let $\alpha:\Omega\times\Omega\to[1,\infty)$ by defined as

$$\alpha\big(\hbar(t),\wp(t)\big)=\mathrm{e}^{\hbar+\wp+2}$$

for all $\hbar,\wp\in\Omega$ and $t\in[0,1]$. Clearly, (Ω,Λ_s) is a complete strong extended s-suprametric space for $s\geqslant 3$. Let $\Theta:\Omega\to\Omega$ be defined as follows:

$$\Theta\hbar(t)=\delta_1+(\delta_2-\delta_1)t+\int_0^1\Psi(t,\zeta)\big(\Omega(t)-\Theta(t)\hbar(t)\big)\,\mathrm{d}\zeta$$

for $t\in[0,1]$. Then $\hbar$ is a unique solution of (16) iff it is a fixed point of Θ.

The subsequent theorem provides us a sufficient condition for the existence and uniqueness of solution of problem (15).

Theorem 4. *Consider the above mentioned problem* (16) *and suppose that there exists a continuous function $\Theta:I\to\mathbb{R}$ such that*:

(i) $\Theta\in C[[0,1],\mathbb{R}]$ *is lower solution of* (15),

$$\Theta(t)\leqslant\delta_1+(\delta_2-\delta_1)t+\int_0^1\Psi(t,\zeta)\big(\Omega(t)-\Theta(t)\hbar(t)\big)\,\mathrm{d}\zeta.$$

(ii) *For any $k>0$, we have*

$$\inf_{0\leqslant t\leqslant 1}\Psi(t,\zeta)>0,\qquad 0<\sup_{0\leqslant t\leqslant 1}\Psi^2(t,\zeta)<\frac{\beta}{s}\in\left[0,\frac{1}{2}\right),$$

where

$$\Psi(t,\zeta)=\frac{t+t^2-t^3}{2}.$$

(iii) *For each $t\in[0,1]$ and $\hbar,\wp\in\Omega$,*

$$\big|\Theta\hbar(t)-\Theta\wp(t)\big|\leqslant\frac{1}{k}\sqrt{\big|\hbar(t)-\Theta\hbar(t)\big|^2+\big|\wp(t)-\Theta\wp(t)\big|^2}.$$

The concentration of diffusing substance with problem (15) *and integral equation* (16) *provides a unique solution in Ω.*

Proof. Define $\Theta : \Omega \to \Omega$ as

$$\begin{aligned}
&\left|\Theta\hbar(t) - \Theta\wp(t)\right| \\
&\quad = \left|\int_0^1 \Psi(t,\zeta)\big(\Omega(t) - \Theta\hbar(t)\big)\,\mathrm{d}\zeta - \int_0^1 \Psi(t,\zeta)\big(\Omega(t) - \Theta\wp(t)\big)\,\mathrm{d}\zeta\right| \\
&\quad \leqslant \int_0^1 \left|\Psi(t,\zeta)\big(\Omega(t) - \Theta\hbar(t) - \big(\Omega(t) - \Theta\wp(t)\big)\big)\right|\mathrm{d}\zeta \\
&\quad = \int_0^1 \left|\Psi(t,\zeta)\big(\Theta\hbar(t) - \Theta\wp(t)\big)\right|\mathrm{d}\zeta \leqslant \int_0^1 \Psi(t,\zeta)\,\mathrm{d}\zeta\,\left|\Theta\hbar(t) - \Theta\wp(t)\right| \\
&\quad \leqslant \frac{t + t^2 - t^3}{2k}\sqrt{\left|\hbar(t) - \Theta\hbar(t)\right|^2 + \left|\wp(t) - \Theta\wp(t)\right|^2}.
\end{aligned}$$

From here we express

$$\begin{aligned}
&\sup_{t\in[0,1]}\left|\Theta\hbar(t) - \Theta\wp(t)\right|^2 \\
&\quad \leqslant \left(\sup_{t\in[0,1]}\frac{t + t^2 - t^3}{2k}\right)^2\left(\left|\hbar(t) - \Theta\hbar(t)\right|^2 + \left|\wp(t) - \Theta\wp(t)\right|^2\right), \\
&\sup_{t\in[0,1]}\left|\Theta\hbar(t) - \Theta\wp(t)\right|^2 \\
&\quad \leqslant \sup_{0\leqslant t\leqslant 1}\big(\Psi^2(t,\zeta)\big)\left(\left|\hbar(t) - \Theta\hbar(t)\right|^2 + \left|\wp(t) - \Theta\wp(t)\right|^2\right).
\end{aligned}$$

We know that $\sup_{0\leqslant t\leqslant 1}\Psi^2(t,\zeta) < 1/2 \in [0, 1/2)$. Therefore, we achieve

$$\Lambda_s(\Theta\hbar, \Theta\wp) \leqslant \frac{\beta}{s}\big[\Lambda_s(\hbar, \Theta\hbar) + \Lambda_s(\wp, \Theta\wp)\big].$$

Thus, all the conditions of Theorem 2 are satisfied, and Θ has a unique fixed in $\hbar \in \Omega$. Hence, we conclude that the boundary value problem (15) has a unique solution in Ω. □

6 Conclusion

This manuscript makes a substantial contribution to the field of metric space theory by providing the novel concept of a strong extended s-suprametric space that outperforms both s-suprametric and extended suprametric spaces. Investigating open and closed ball topologies using this framework improves our understanding of the features of the space. The analysis of existence and uniqueness, driven by the Banach contraction theorem, provides a solid theoretical framework. The illustration example illustrates the superior properties of the strong extended s-suprametric space over its extended equivalent, offering light on the presence and uniqueness of fixed points. Additionally, the manuscript employs these findings to boundary value problems involving dispersing chemical compounds and nonlinear satellite web coupling, extending the research's relevance to real-world settings. The insights provided here help better understand metric spaces and their applications in various mathematical situations.

Can the results presented in this article be further developed in the case of multivalued mappings? Future work can include the extension of the current framework to achieve generalized results, including on common fixed points and proximal point theorems along the lines of the approaches described in [23].

Author contributions. All authors have read and approved the published version of the manuscript.

Conflicts of interest. The authors declare no conflicts of interest.

References

1. T. Abdeljawad, N. Mlaiki, H. Aydi, N. Souayah, Double controlled metric type spaces and some fixed point results, *Mathematics*, **6**(12):320, 2018, https://doi.org/10.3390/math6120320.
2. H. Ahmad, F.U. Din, M. Younis, A novel Ćirić–Reich–Rus fixed point approach for the existence and uniqueness criterion of a fractional-order Aizawa chaotic system, *Chaos Solitons Fractals*, **200**:116932, 2025, https://doi.org/10.1016/j.chaos.2025.116932.
3. H. Ahmad, F.U. Din, M. Younis, L. Chen, A simple interpolative contraction approach for analyzing existence and uniqueness of solutions in the fractional-order King Cobra model, *Chaos Solitons Fractals*, **199**:116578, 2025, https://doi.org/10.1016/j.chaos.2025.116578.
4. I.A. Bakhtin, The contraction mapping principle in quasi-metric spaces, *Funct. Anal. Unianowsk Gos. Ped. Inst.*, **30**:26–37, 1989.
5. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundam. Math.*, **3**:133–181, 1922, https://doi.org/10.4064/fm-3-1-133-181.
6. M. Berzig, First results in suprametric spaces with applications, *Mediterr. J. Math.*, **19**(5):226, 2022, https://doi.org/10.1007/s00009-022-02148-6.
7. M. Berzig, Fixed point results in generalized suprametric spaces, *Topol. Algebra Appl.*, **11**(1):20230105, 2023, https://doi.org/10.1515/taa-2023-0105.
8. M. Berzig, Nonlinear contraction in b-suprametric spaces, *J. Anal.*, **32**(5):2401–2414, 2024, https://doi.org/10.1007/s41478-024-00732-5.
9. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostrav.*, **1**(1):5–11, 1993, http://eudml.org/doc/23748.
10. N. Dunford, J.T. Schwartz, *Linear Operators, Part 1: General Theory*, John Wiley & Sons, Hoboken, NJ, 1988.

11. T. Kamran, M. Samreen, Q. UL Ain, A generalization of b-metric space and some fixed point theorems, *Mathematics*, **5**(2):19, 2017, https://doi.org/10.3390/math5020019.
12. K. Kuratowski, S. Banach, Sur la mesure de la classe des ensembles parfaits, *Fundam. Math.*, **4**:302–311, 1922.
13. G. Mani, P. Ganesh, K. Kannan, A novel approach to caputo–hadamard pantograph problems in l^p-spaces, *Boundary Value Problems*, **2025**(1):61, 2025, https://doi.org/10.1186/s13661-025-02054-2.
14. N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled metric type spaces and the related contraction principle, *Mathematics*, **6**(10):194, 2018, https://doi.org/10.3390/math6100194.
15. S.K. Panda, R.P. Agarwal, E. Karapınar, Extended suprametric spaces and Stone-type theorem, *AIMS Math.*, **8**(10):23183–23199, 2023, https://doi.org/10.3934/math.20231179.
16. G. Rozenblyum, Sur une classe de suites totalement boréliennes et les propriétés des ensembles boréliens, *Bull. Acad. Pol. Sci.. Sér. Sci. Math., Astron. Phys.*, **5**(7):353–358, 1957.
17. S. Saks, *Theory of the Integral*, Monografje Matematyczne, Warszawa, 1935.
18. B.W. Samuel, G. Mani, S.D. Rajkumar, S.T. Thabet, I. Kedim, New fixed point theorems in orthogonal-multiplicative metric spaces and applications to multiplicative and fractional integral equations, *Res. Math.*, **12**(1):2522507, 2025, https://doi.org/10.1080/27684830.2025.2522507.
19. B.W. Samuel, G. Mani, S.S. Ramulu, S.T. Thabet, I. Kedim, Integral-type contraction on orthogonal $\mathcal{S}$-metric spaces with common fixed-point results and applications to fractional integral equations, *Fixed Point Theory Algorithms Sci. Eng.*, **2025**:10, 2025, https://doi.org/10.1186/s13663-025-00792-7.
20. I. Stakgold, M. Hoist, *Green's Functions and Boundary Value Problems*, John Wiley & Sons, Hoboken, NJ, 2011.
21. M. Younis, H. Ahmad, M. Ozturk, F.U. Din, M. Qasim, Unveiling fractional-order dynamics: A new method for analyzing Rössler chaos, *J. Comput. Appl. Math.*, **468**:116639, 2025, https://doi.org/10.1016/j.cam.2025.116639.
22. M. Younis, L. Chen, D. Singh, *Recent Developments in Fixed Point Theory: Theoretical Foundations and Real-World Applications*, Springer, Singapore, 2025, https://doi.org/10.1007/978-981-99-9546-2.
23. M. Younis, M. Öztürk, Some novel proximal point results and applications, *Univers. J. Math. Appl.*, **8**(1):8–20, 2025, https://doi.org/10.32323/ujma.1597874.
24. M. Younis, D. Singh, A. Goyal, A novel approach of graphical rectangular b-metric spaces with an application to the vibrations of a vertical heavy hanging cable, *J. Fixed Point Theory Appl.*, **21**(1):33, 2019, https://doi.org/10.1007/s11784-019-0673-3.