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Dynamical investigation of the nonlinear Schrödinger equation with second-order spatiotemporal involvement of the time-conformable operator

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Abstract. The article analyzes the application of the extended hyperbolic function technique to a conformable-operator nonlinear Schrödinger equation, incorporating group velocity dispersion coefficients and second-order spatiotemporal components. The primary objective is establishing a spectrum of solutions directly pertinent to optical fibers. The extracted results, which include bright, singular, straddled, dark-bright, and dark solitons, are obtained by hyperbolic and trigonometric function-type solutions. We exhibit contour plots with two-dimensional and three-dimensional visualizations to emphasize the implication of the proposed conformable-operator nonlinear Schrödinger equation and to depict the diverse novel optical solutions. Additionally, we study the impact of the conformable operator on these solutions, employing graphical analysis to demonstrate its implications. The governing model shows potential applications in transmitting ultra-fast pulses via optical fibers.

Keywords: extended hyperbolic function method, second-order spatiotemporal terms, group velocity dispersion coefficients, nonlinear conformable Schrödinger equation, optical fibers.

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1 Introduction

Recently, the nonlinear partial differential equations (NPDEs) have been extensively utilized to model dynamic processes in applied sciences and engineering. Creating nonlinear equations for evolution is crucial for comprehending intricate natural occurrences in disciplines like physics, applied mathematics, and engineering. Soliton theory is fundamental in developing analytical techniques for nonlinear models, offering valuable insights into phenomena such as those observed in optical fibers [3, 10], plasma physics [15, 17], biomedical science [4, 14], and others. Many scholars have devised numerous analytical strategies to obtain the analytic solutions of NPDEs [2, 9, 11]. To analyze the analytic soliton solutions, it is necessary to transfer the NPDEs into ordinary differential equations (ODEs). The effectiveness of these analytical solutions in addressing the issues depends significantly on the parameters and coefficients used in the transformations. These parameters and coefficients are crucial for the practical implementation of the models.

Investigating and utilizing soliton solutions in nonlinear evolution systems (NLESs) have played a crucial role in advancing research and growth across diverse scientific and technological fields, presenting new opportunities for innovation. With their distinct properties and behaviors in NLESs, solitons have captivated researchers seeking to deepen scientific understanding and venture into new technological frontiers, such as long-distance communication systems and optical signal processing. Mastering soliton solutions broadens scientific knowledge and unlocks practical applications, improving the efficiency and reliability of communication networks and offering the potential for high-speed, high-efficiency devices capable of manipulating light in innovative ways. As researchers refine methodologies and delve into the characteristics and dynamics of solitons, this relentless pursuit not only enhances the understanding of solitons but also drives innovation, leading to significant advancements in fundamental science and applied engineering. Furthermore, the interdisciplinary nature of soliton research encourages collaboration among physicists, mathematicians, engineers, and technologists, enriching the scientific community and paving the way for cross-disciplinary innovations. The ongoing research into soliton solutions in NLESs is not just about comprehending these captivating phenomena but also about leveraging their potential to propel technological progress and address complex challenges in today's interconnected world, promising to inspire advancements across a wide array of scientific and engineering disciplines for years to come.

This study examines the conformable-operator nonlinear Schrödinger equation for optical fibers utilizing extended hyperbolic function method. This methodology is a highly effective method that has been extensively utilized in recent times to discover precise solutions to differential equations [19]. Hence, a novel Kudryashov methodology is suggested to obtain several innovative optical solutions for the given nonlinear Schrödinger problem [7, 20]:

$$i\left(\frac{\partial p(x,t)}{\partial x}+\eta_1\frac{\partial^\beta p(x,t)}{\partial t^\beta}\right)+\eta_2\frac{\partial^{2\beta} p(x,t)}{\partial t^{2\beta}}+\eta_3\frac{\partial^2 p(x,t)}{\partial x^2}+\left|p(x,t)\right|^2 p(x,t)=0, \quad (1)$$

where $0 \leqslant \beta \leqslant 1$. Equation (1) represents a conformable-operator nonlinear Schrödinger (NLS) equation, which describes the propagation of optical pulses in nonlinear media, such as optical fibers. The equation incorporates conformable operator derivatives to account for memory effects and anomalous dispersion, which are not captured by classical integer-order models. Here $p(x,t)$ is a complex-valued wave function representing the optical field envelope, and β ($0 \leqslant \beta \leqslant 1$) denotes the order of the conformable operator with respect to time. The term involving η_1 characterizes the first-order temporal evolution (group speed ratio), while the term with η_2 corresponds to the second-order time derivative (group velocity dispersion). The spatial dispersion is modeled by the second spatial derivative multiplied by η_3, and the nonlinear term $|p(x,t)|^2 p(x,t)$ captures the self-phase modulation due to Kerr nonlinearity. This generalized model allows for a more accurate description of pulse dynamics in complex fiber systems.

The current problem is analyzed applying a certain type of direct algebraic method to generate various soliton solutions. Meanwhile, the analytic solitary solutions for the conformable operator version of the nonlinear Schrödinger equation are examined in [13,22]. The Schrödinger equation, which incorporates a local M-derivative, has been examined in [7]. In [27], the present problem is tackled using the auxiliary equation technique to derive many solutions in the form of traveling waves. The model described in reference [6] is regarded as being able to account for pulse events that go beyond the traditional slowly-varying envelope estimation. Furthermore, the research examines the spatial and temporal attributes of the issue, as well as its aspects of transition. The amplitude ansatz approach is used to detect many types of analytic traveling soliton solutions, such as dark-bright, bright, and dark solitons [24]. The influence of the conformable operator on various new soliton solutions of this model is investigated in [28]. In [18], the modulation instability of the existing problem is examined and the authors employed the F-expansion method to extract various traveling solutions. In [21], by utilizing the sine-Gordon expansion approach the authors investigated many new soliton solutions for the present conformable operator nonlinear Schrödinger equation. The new version of direct algebraic technique was used to produce several soliton solutions to the present model [8]. In a recent study, researchers discovered numerous new optical solutions for the current problem with conformable operator using extended simplest equation technique. These solutions include periodic wave, dark-bright, dark, singular, and bright solutions [16].

The conformable operator has found applications in various fields of physics due to its ability to model systems with memory and nonlocal properties, which are not adequately described by classical derivatives. Some key applications involve modeling systems that exhibit nonlocal behavior or long-term memory effects, which can be effectively described using conformable operator. The conformable operator provides a simpler way to model such systems while retaining the familiar rules of calculus. Additionally, conformable operator calculus has been used to generalize quantum mechanical equations, such as the Schrödinger equation, to describe nonlocal quantum systems or systems with fractional dimensions. The conformable operator is introduced as follows.

Definition 1. Let $p : (0,\infty) \to \mathbb{R}$. The conformable derivative of order $\beta \in (0,1]$ can be introduced as [12]

$$L_\beta(p)(x) = \lim_{a\to 0} \frac{p(x + a x^{1-\beta}) - p(x)}{a}, \quad x > 0. \tag{2}$$

In this work, we consistently employ the terminology "conformable operator" rather than "fractional derivative." This distinction is crucial since several studies have shown that the conformable derivative is not a genuine fractional operator but rather a local, integer-order construction. Abdelhakim [1] demonstrated that the conformable derivative is mathematically equivalent to a rescaled classical derivative, while Tarasov [25, 26] provided physical and geometrical interpretations confirming that such operators lack the inherent nonlocality of true fractional calculus. Moreover, Liu [23] presented counterexamples invalidating fundamental formulae in certain modified fractional frameworks. These contributions collectively indicate that the conformable operator should be regarded as a separate local operator, distinct from fractional derivatives in the classical sense, thereby motivating the terminology adopted in the present study.

2 Formulation of the problem

Here we utilize the extended hyperbolic function method to generate several accurate solutions in closed form for the current conformable-operator nonlinear Schrödinger problem. The inquiry begins by employing the following wave transformations:

$$\begin{aligned} &q(x,t) = W(z)\mathrm{e}^{\mathrm{i}\varphi(x,t)}, \\ &z = x + \upsilon\frac{t^\beta}{\beta}, \qquad \varphi(x,t) = -kx + w\frac{t^\beta}{\beta}, \end{aligned} \tag{3}$$

where w represents the number of wave, k represents the frequency of solitons, and υ represents the speed of the moving wave. In Eq. (3), the operator used is the conformable operator, as defined in Eq. (2). This operator provides a framework to handle time evolution while preserving key properties of classical calculus. The transformation includes the term t^β/β, which arises naturally from integration with respect to the conformable operator of order β. This ensures consistency between the conformable operator framework and the traveling wave transformation used in the reduction of the original partial differential equation to an ordinary differential equation.

Here we substitute the aforementioned equation into the conformable model in Eq. (1), resulting in the following equation:

$$\big(\eta_2\upsilon^2 + \eta_3\big)W''(\varsigma) + \big(k - \eta_1 w - \eta_2 w^2 - \eta_3 k^2\big)W(\varsigma) + W^3(\varsigma) = 0 \tag{4}$$

with

$$\upsilon = \frac{1 - 2\eta_3 k}{\eta_1 + 2w\eta_2}. \tag{5}$$

By substituting the value of υ into Eq. (4), we obtain the following equation:

$$\big(\eta_2 A_2 + \eta_3 A_1\big)W''(\varsigma) + A_1 A_3 W(\varsigma) + A_1 W^3(\varsigma) = 0, \tag{6}$$

where

$$A_1 = (\eta_1 + 2w\eta_2)^2, \quad A_2 = (1 - 2\eta_3 k)^2, \quad \text{and} \quad A_3 = (k - \eta_1 w - \eta_2 w^2 - \eta_3 k^2).$$

3 Application of extended hyperbolic function method

In this section, we derive a range of innovative optical solutions for the studied problem, obtained via the application of the extended hyperbolic function approach. We posit that the solution to Eq. (6) can be presented as the following series:

$$W(\varsigma) = a_0 + \sum_{i=1}^{M} a_i F(\varsigma)^i, \tag{7}$$

where $a_0, a_1, \ldots, a_M$ are unknown constants, and M denotes a balancing parameter. By balancing principle for W^3 and W'' in Eq. (8), we obtain $M = 1$. Thus, Eq. (7) reduces to the following series:

$$W(\varsigma) = a_0 + a_1 F(\varsigma). \tag{8}$$

In the first case, the function $F(\varsigma)$ satisfies the following relation:

$$F(\varsigma)' = \sqrt{p_0 + p_1 F^2(\varsigma)}, \quad p_0, p_1 \in \mathbb{R}. \tag{9}$$

Here the solutions of the above equation with real constant l are defined as follows.

If $p_0 > 0$ and $p_1 < 0$, we have

$$F_1(\varsigma) = \frac{\sqrt{p_0}}{\sqrt{-p_1}} \operatorname{sech}\bigl[(l + \varsigma)\sqrt{p_0}\bigr]. \tag{10}$$

If $p_0 > 0$ and $p_1 > 0$, we have

$$F_2(\varsigma) = -\frac{\sqrt{p_0}}{\sqrt{p_1}} \operatorname{csch}\bigl[(l + \varsigma)\sqrt{p_0}\bigr]. \tag{11}$$

If $p_0 < 0$ and $p_1 > 0$, we have

$$F_3(\varsigma) = \frac{\sqrt{-p_0}}{\sqrt{p_1}} \sec\bigl[(l + \varsigma)\sqrt{-p_0}\bigr]. \tag{12}$$

If $p_0 < 0$ and $p_1 > 0$, we have

$$F_4(\varsigma) = \frac{\sqrt{-p_0}}{\sqrt{p_1}} \csc\bigl[(l + \varsigma)\sqrt{-p_0}\bigr]. \tag{13}$$

In the second case, the function $F(\varsigma)$ satisfies the following relation:

$$F(\varsigma)' = p_0 + p_1 F^2(\varsigma), \quad p_0, p_1 \in \mathbb{R}. \tag{14}$$

The solutions of Eq. (14) with real constant l can be defined as follows.

If $p_0 p_1 > 0$, we have

$$F_5(\varsigma) = \frac{\sqrt{p_0}}{\sqrt{-p_1}} \tanh\bigl[(l + \varsigma)\sqrt{-p_0 p_1}\bigr]. \tag{15}$$

If $p_0p_1 < 0$, we have

$$F_6(\varsigma) = \frac{\sqrt{p_0}}{\sqrt{-p_1}} \coth\big[(l+\varsigma)\sqrt{-p_0p_1}\big]. \tag{16}$$

If $p_0p_1 > 0$, we have

$$F_7(\varsigma) = \frac{\sqrt{p_0}}{\sqrt{p_1}} \tan\big[(l+\varsigma)\sqrt{p_0p_1}\big]. \tag{17}$$

If $p_0p_1 > 0$, we have

$$F_8(\varsigma) = \frac{\sqrt{p_0}}{\sqrt{p_1}} \cot\big[(l+\varsigma)\sqrt{p_0p_1}\big]. \tag{18}$$

For the first case (9), substituting Eqs. (8) and (9) into Eq. (6), we obtain a polynomial of powers of $F(\varsigma)$. Next, we arrange the terms according to their respective powers and equate each corresponding coefficient to zero. This procedure results in a system of algebraic equations, and solving the system, one can have the following results.

Result 1.1

$$\begin{aligned} a_0 &= 0; \qquad a_1 = \pm\frac{\sqrt{-2p_1}\sqrt{\eta_2+\eta_1^2\eta_3}}{\sqrt{(\eta_1+2w\eta_2)^2-4p_0\eta_2\eta_3}}, \\ k &= \frac{1-\frac{\sqrt{(\eta_1+2w\eta_2)^2((\eta_1+2w\eta_2)^2-4p_0\eta_2\eta_3)(1-4w(\eta_1+w\eta_2)\eta_3+4p_0\eta_3^2)}}{(\eta_1+2w\eta_2)^2-4p_0\eta_2\eta_3}}{2\eta_3}. \end{aligned} \tag{19}$$

Result 1.2

$$\begin{aligned} a_0 &= 0, \qquad a_1 = \pm\sqrt{-\frac{p_1(-\eta_1^2+4\eta_2(-k+(k^2+p_0)\eta_3)+B_2)}{4p_0\eta_2}}, \\ w &= \frac{-2\eta_1+\sqrt{2}\sqrt{\eta_1^2+4\eta_2(k+(-k^2+p_0)\eta_3)+B_2}}{4\eta_2}. \end{aligned} \tag{20}$$

For second case (14), substituting Eqs. (8) and (14) into Eq. (6), we obtain a polynomial in terms of the powers of $F(\varsigma)$. Next, we organize the terms according to their respective powers and equate each corresponding coefficient to zero. This procedure results in a system of algebraic equations, and solving the system, one can have the following results.

Result 2.1

$$\begin{aligned} a_0 &= 0, \qquad a_1 = \pm\frac{\sqrt{2}p_1\sqrt{\eta_2+\eta_1^2\eta_3}}{\sqrt{-(\eta_1+2w\eta_2)^2+8p_0p_1\eta_2\eta_3}}, \\ k &= \frac{-(\eta_1+2w\eta_2)^2+8p_0p_1\eta_2\eta_3+B_1}{2\eta_3(-(\eta_1+2w\eta_2)^2+8p_0p_1\eta_2\eta_3)}. \end{aligned} \tag{21}$$

Result 2.2

$$\begin{aligned} a_0 &= 0, \qquad a_1 = \pm\sqrt{-\frac{p_1(-\eta_1^2 + 4\eta_2(-k + (k^2 + 2p_0p_1)\eta_3) + B_3)}{8p_0\eta_2}}, \\ w &= \frac{-2\eta_1 + \sqrt{2}\sqrt{\eta_1^2 + 4\eta_2(k - (k^2 - 2p_0p_1)\eta_3) + B_3}}{4\eta_2}. \end{aligned} \tag{22}$$

Now, by using the above results, we introduce several optical soliton solutions. Thus, using Eqs. (3), (5), (8), (10)–(13), and (19), we can have the following optical solutions:

$$p_1(x,t) = \pm\frac{\sqrt{-2p_0p_1(\eta_2 + \eta_1^2\eta_3)}\operatorname{sech}[\sqrt{p_0}\varsigma]}{\sqrt{-p_1}\sqrt{(\eta_1 + 2w\eta_2)^2 - 4p_0\eta_2\eta_3}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

$$p_2(x,t) = \pm\frac{\operatorname{csch}[\sqrt{p_0}\varsigma]\sqrt{-2p_0}\sqrt{\eta_2 + \eta_1^2\eta_3}}{\sqrt{(\eta_1 + 2w\eta_2)^2 - 4p_0\eta_2\eta_3}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

$$p_3(x,t) = \pm\frac{\sec[\sqrt{-p_0}\varsigma]\sqrt{2p_0}\sqrt{\eta_2 + \eta_1^2\eta_3}}{\sqrt{(\eta_1 + 2w\eta_2)^2 - 4p_0\eta_2\eta_3}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

$$p_4(x,t) = \pm\frac{\csc[\sqrt{-p_0}\varsigma]\sqrt{2p_0}\sqrt{\eta_2 + \eta_1^2\eta_3}}{\sqrt{(\eta_1 + 2w\eta_2)^2 - 4p_0\eta_2\eta_3}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

where

$$\begin{aligned} \varsigma &= l + x \\ &\quad - \frac{t^\beta\sqrt{-(\eta_1+2w\eta_2)^2((\eta_1+2w\eta_2)^2 - 4p_0\eta_2\eta_3)(-1 + 4w(\eta_1+w\eta_2)\eta_3 - 4p_0\eta_3^2)}}{\beta(\eta_1+2w\eta_2)((\eta_1+2w\eta_2)^2 - 4p_0\eta_2\eta_3)}. \end{aligned}$$

Using Eqs. (3), (5), (14), (15)–(18), and (21), we can have the following optical solutions:

$$p_5(x,t) = \pm\frac{\sqrt{-2p_0}p_1\sqrt{\eta_2 + \eta_1^2\eta_3}\tanh[\sqrt{-p_0p_1}\varsigma]}{\sqrt{-p_1(\eta_1 + 2w\eta_2)^2 + p_18p_0p_1\eta_2\eta_3}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

$$p_6(x,t) = \pm\frac{\coth\left[\sqrt{-p_0p_1}\varsigma\right]\sqrt{-2p_0}p_1\sqrt{\eta_2 + \eta_1^2\eta_3}}{\sqrt{-p_1}\sqrt{(\eta_1 + 2w\eta_2)^2 - 8p_0p_1\eta_2\eta_3}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

$$p_7(x,t) = \pm\frac{\sqrt{-2p_0p_1}\sqrt{\eta_2 + \eta_1^2\eta_3}\tan[\sqrt{p_0p_1}\varsigma]}{\sqrt{(\eta_1 + 2w\eta_2)^2 - 8p_0p_1\eta_2\eta_3}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

$$p_8(x,t) = \pm\frac{\cot\left[\sqrt{p_0p_1}\varsigma\right]\sqrt{-2p_0p_1}\sqrt{\eta_2 + \eta_1^2\eta_3}}{\sqrt{(\eta_1 + 2w\eta_2)^2 - 8p_0p_1\eta_2\eta_3}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

where

$$B_2 = \bigl(-(\eta_1 + 2w\eta_2)^2\bigl((\eta_1 + 2w\eta_2)^2 - 8p_0p_1\eta_2\eta_3\bigr)\bigl(-1 + 4w(\eta_1 + w\eta_2)\eta_3 - 8p_0p_1\eta_3^2\bigr)\bigr)^{1/2}$$

and

$$\varsigma = l + x - \frac{t^\beta}{\beta(\eta_1 + 2w\eta_2)}\left(1 - \frac{-(\eta_1 + 2w\eta_2)^2 + 8p_0p_1\eta_2\eta_3 + B_2}{-(\eta_1 + 2w\eta_2)^2 + 8p_0p_1\eta_2\eta_3}\right).$$

Using Eqs. (3), (5), (8), (10)–(13), and (20), we can have the following optical solutions:

$$p_9(x,t) = \pm\frac{\sqrt{-\frac{p_1(-\eta_1^2+4\eta_2(-k+(k^2+p_0)\eta_3)+B_1)}{\eta_2}}\,\mathrm{sech}[\sqrt{p_0}\varsigma]}{2\sqrt{-p_1}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

$$p_{10}(x,t) = \mp\frac{\sqrt{-\frac{p_1(-\eta_1^2+4\eta_2(-k+(k^2+p_0)\eta_3)+B_1)}{\eta_2}}\,\mathrm{csch}[\sqrt{p_0}\varsigma]}{2\sqrt{p_1}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

$$p_{11}(x,t) = \pm\frac{\sqrt{\frac{p_1(-\eta_1^2+4\eta_2(-k+(k^2+p_0)\eta_3)+B_1)}{p_0\eta_2}}\,\sec[\sqrt{-p_0}\varsigma]}{2\sqrt{p_0p_1}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

$$p_{12}(x,t) = \pm\frac{\sqrt{\frac{p_1(-\eta_1^2+4\eta_2(-k+(k^2+p_0)\eta_3)+B_1)}{p_0\eta_2}}\,\csc[\sqrt{-p_0}\varsigma]}{2\sqrt{p_0p_1}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

where

$$B_1 = \sqrt{\eta_1^4 + 8\eta_1^2\eta_2(k + (-k^2 + p_0)\eta_3) + 16(k^2 + p_0)\eta_2^2(1 - 2k\eta_3 + (k^2 + p_0)\eta_3^2)}$$

and

$$\varsigma = l + x - \frac{t^\beta(1 - 2k\eta_3)}{\beta(\eta_1 + \frac{1}{2}(-2\eta_1 + \sqrt{2}\sqrt{\eta_1^2 + 4\eta_2(k + (-k^2 + p_0)\eta_3) + B_1}))}.$$

Using Eqs. (3), (5), (14), (15)–(18), and (22), we can have the following optical solutions:

$$p_{13}(x,t) = \pm\frac{\sqrt{\frac{p_1(4\eta_2(-k+(k^2+2p_0p_1)\eta_3)-\eta_1^2+B_3)}{-\eta_2}}\,\tanh[\sqrt{-p_0p_1}\varsigma]}{\sqrt{-8p_1}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

$$p_{14}(x,t) = \pm\frac{\sqrt{\frac{p_1(4\eta_2(-k+(k^2+2p_0p_1)\eta_3)-\eta_1^2+B_3)}{-\eta_2}}\,\coth[\sqrt{-p_0p_1}]}{\sqrt{-8p_1}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

$$p_{15}(x,t) = \pm\frac{\sqrt{\frac{p_1(4\eta_2(-k+(k^2+2p_0p_1)\eta_3)-\eta_1^2+B_3)}{-\eta_2}}\,\tan[\sqrt{p_0p_1}\varsigma]}{\sqrt{8p_1}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

$$p_{16}(x,t) = \pm\frac{\sqrt{\frac{p_1(-\eta_1^2+4\eta_2(-k+(k^2+2p_0p_1)\eta_3)+B_3)}{-\eta_2}}\,\cot[\sqrt{p_0p_1}\varsigma]}{\sqrt{8p_1}}\mathrm{e}^{\mathrm{i}(-kx+wt^\beta/\beta)},$$

where

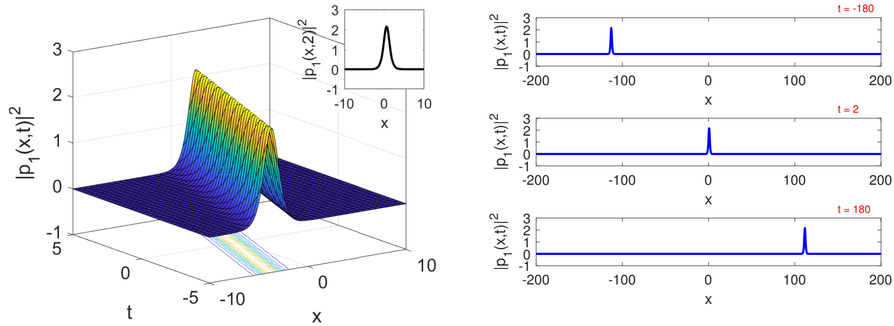
$$B_3 = \big(\eta_1^4 + 8\eta_1^2\eta_2\big(k - \big(k^2 - 2p_0p_1\big)\eta_3\big) + 16\big(k^2 + 2p_0p_1\big)\eta_2^2\big(1 - 2k\eta_3 + \big(k^2 + 2p_0p_1\big)\eta_3^2\big)\big)^{1/2}$$

and

$$\varsigma = l + x - \frac{t^\beta(1 - 2k\eta_3)}{\beta(\eta_1 + \frac{1}{2}(-2\eta_1 + \sqrt{2}\sqrt{\eta_1^2 + 4\eta_2(k - (k^2 - 2p_0p_1)\eta_3) +}))}.$$

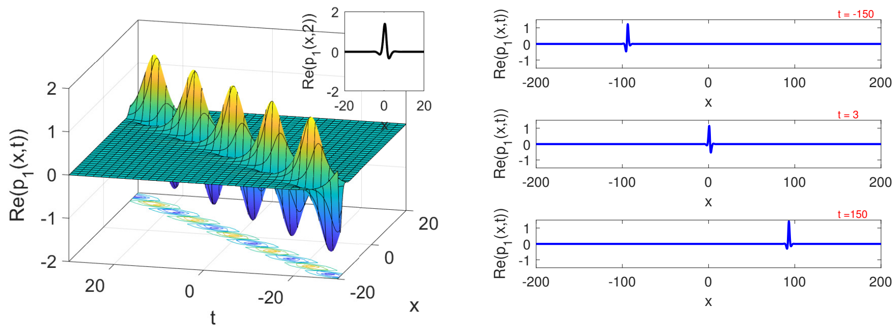
4 Results and discussion

This portion discusses the physical behavior and characteristics of the novel optical soliton solutions. We have chosen suitable values for the physical parameters used in this research to demonstrate the significance of the current conformable operator Schrödinger equation. The impact of the conformable operator and temporal parameter on the present solitary wave solutions is illustrated through various graphical simulations. Both the two-dimensional and three-dimensional graphs show how various temporal values and the conformable operator parameter affect the novel soliton solutions. Figures 1(a) and 1(b) illustrate the bright optical solution $|p_1(x,t)|^2$ and the impact of the temporal parameter on the soliton solution $|p_1(x,t)|^2$, respectively. Bright solitons are explored in optical computing applications, where they can be used as bits in optical logic gates and other computing elements, contributing to the development of ultrafast optical processors. Figures 2(a) and 4(a) illustrate the mixed dark-bright solutions of $\mathrm{Re}(p_1(x,t))$ and $\mathrm{Im}(p_9(x,t))$, while Figs. 2(b) and 4(b) depict the impact of the temporal parameter on the soliton solutions $\mathrm{Re}(p_1(x,t))$ and $\mathrm{Im}(p_9(x,t))$. From Figs. 2(b) and 4(b) one can observe the stability of the current optical soliton solutions, which remain stable over a long distance. The figures illustrate how the topological characteristics of the soliton solutions behave as the time parameter t changes. However, in practical applications, mixed dark-bright optical solutions involve a blend of dark solutions, which have lower intensity, and bright solutions, which have higher intensity, coexisting within the same wave function. This combination results in regions with both decreased and increased amplitude or intensity, leading to complex interactions between the dark and bright components. This indicates the presence of regions with both reduced and elevated amplitude or intensity within the wave, resulting in a complex interaction between the dark and bright components. The comparison of dark soliton solutions of $|p_5(x,t)|^2$ is depicted in Fig. 5(a) and 5(b). Here one can observe the effect of the conformable operator on the soliton solution $|p_5(x,t)|^2$. The value of the conformable operator can affect the speed of the soliton solutions. Figures 6(a) and 6(b) present wave optical soliton solutions corresponding to the imaginary part $\mathrm{Re}(p_5(x,t))$, respectively. The optical soliton solutions in these figures may resemble stable, localized waveforms propagating through a medium. Moreover, the paper employs two-dimensional graphs to depict the influence of the conformable operator and temporal parameters. In Figs. 7 and 9, graphs (a) and (b) show how different values of the conformable operator parameter affect the soliton solutions. These findings suggest that the current optical soliton solutions derived from the conformable-operator



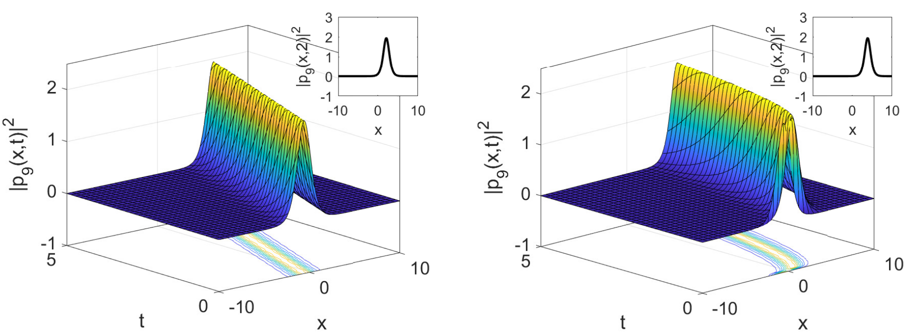
(a) 3D of $|q(x,t)|^2$ (b) Contour view of $|q(x,t)|^2$

Figure 1. The bright plots of $|p_1(x,t)|^2$ with the effect of different temporal parameter, where $\eta_1 = 3$, $\eta_2 = 2$, $\eta_3 = 0.3$, $w = -0.1$, $l = 0.6$, $p_1 = -0.1$, $p_0 = 1$, and $\beta = 1$.



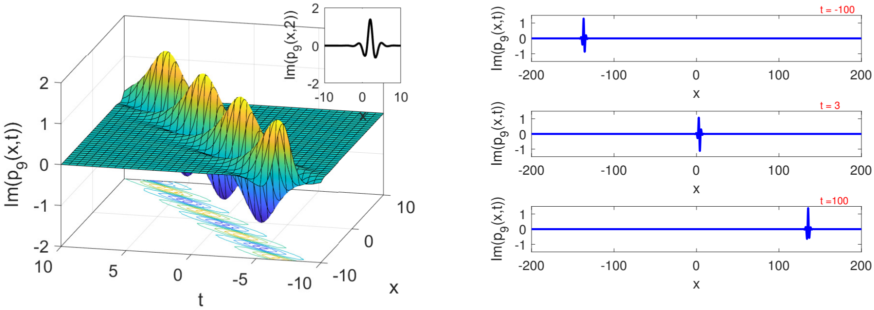
(a) 3D of $|q(x,t)|^2$ (b) Contour view of $|q(x,t)|^2$

Figure 2. The dark-bright plots $\mathrm{Re}(p_1(x,t))$ with the effect of different temporal parameter, where $\eta_1 = 3$, $\eta_2 = 2$, $\eta_3 = 0.3$, $w = -0.1$, $l = 0.6$, $p_1 = -0.1$, $p_0 = 1$, and $\beta = 1$.



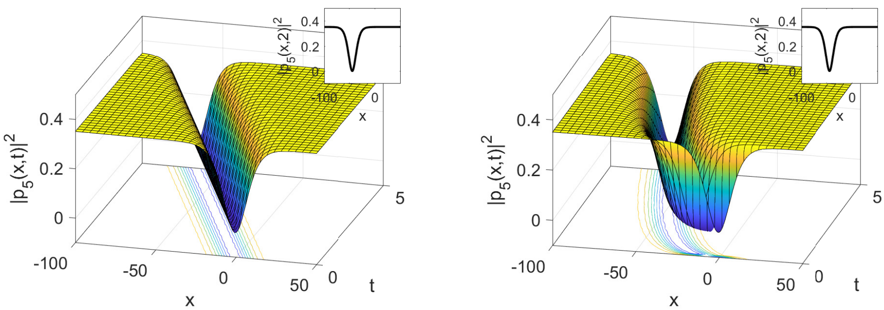
(a) 3D of $|q(x,t)|^2$ (b) Contour view of $|q(x,t)|^2$

Figure 3. The comparison of bright plots $|p_9(x,t)|^2$ for $\beta = 1$ and $\beta = 0.4$, where $\eta_1 = 3$, $\eta_2 = 0.2$, $\eta_3 = 0.6$, $k = -2$, $l = 0.6$, $p_1 = -0.1$, and $p_0 = 1$.



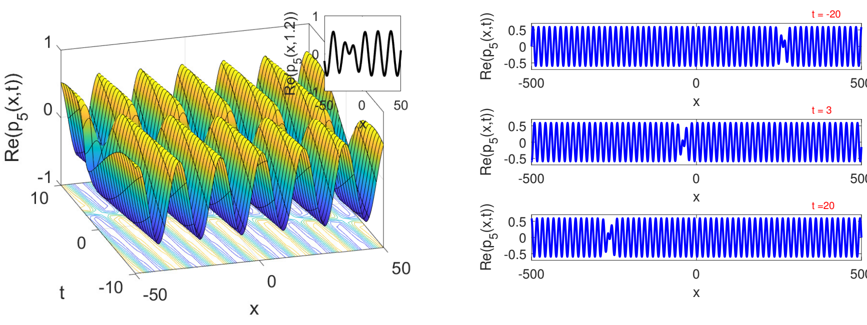
(a) 3D of $|q(x,t)|^2$ (b) Contour view of $|q(x,t)|^2$

Figure 4. The dark-bright plots $\mathrm{Im}(p_9(x,t))$ with the effect of different temporal parameter, where $\eta_1 = 3$, $\eta_2 = 0.2$, $\eta_3 = 0.6$, $k = -2$, $l = 0.6$, $p_1 = -0.1$, $p_0 = 1$, and $\beta = 1$.



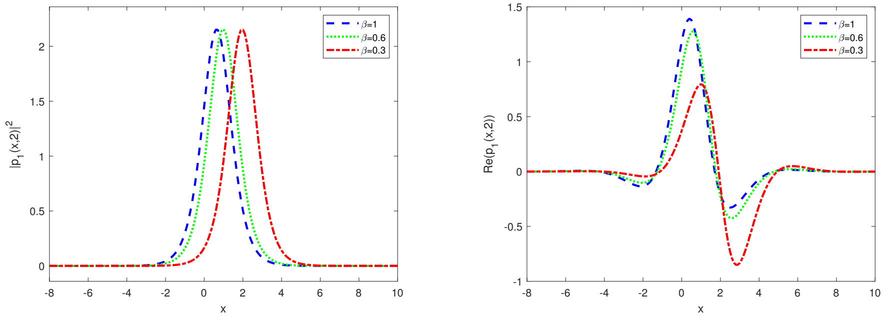
(a) 3D of $|q(x,t)|^2$ (b) Contour view of $|q(x,t)|^2$

Figure 5. The comparison of dark plots $|p_5(x,t)|^2$ for $\beta = 1$ and $\beta = 0.4$, where $\eta_1 = -0.03$, $\eta_2 = 0.1$, $\eta_3 = 0.1$, $w = -0.2$, $l = 0.6$, $p_1 = -0.1$, and $p_0 = 0.1$.



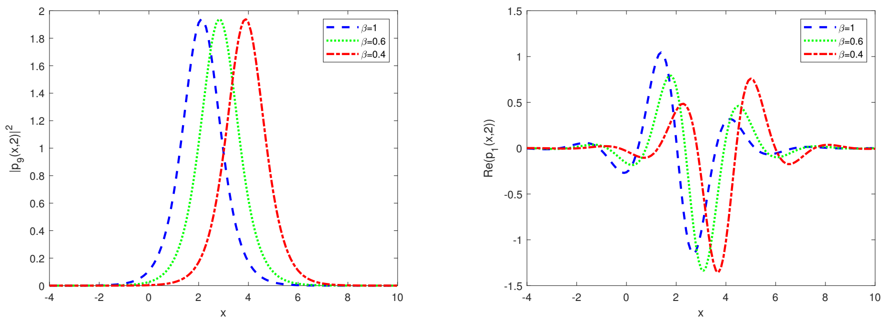
(a) 3D of $|q(x,t)|^2$ (b) Contour view of $|q(x,t)|^2$

Figure 6. The wave plots of $\mathrm{Re}(p_5(x,t))$ with the influence of t, where $c_1 = -0.3$, $c_3 = 0.3$, $k = -1$, $l = 0.6$, $n = 3$, $b_0 = 1$, $\theta = 0.2$, $a = 1$, $p_1 = -1$, $p_0 = 1$, and $\beta = 1$.



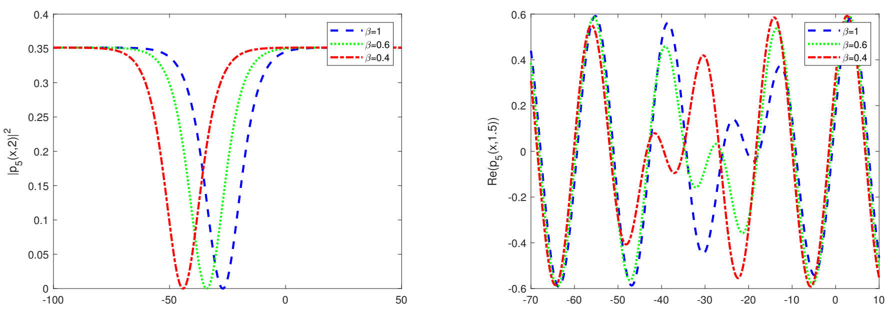
(a) 3D of $|q(x,t)|^2$ (b) Contour view of $|q(x,t)|^2$

Figure 7. The influence of β on $|p_1(x,t)|^2$ and $\mathrm{Re}(p_1(x,t))$, where $\eta_1 = 3$, $\eta_2 = 2$, $\eta_3 = 0.3$, $w = -0.1$, $l = 0.6$, $p_1 = -0.1$, and $p_0 = 1$.



(a) 3D of $|q(x,t)|^2$ (b) Contour view of $|q(x,t)|^2$

Figure 8. The influence of β on $|p_9(x,t)|^2$ and $\mathrm{Re}(p_9(x,t))$, where $\eta_1 = 3$, $\eta_2 = 0.2$, $\eta_3 = 0.6$, $k = -2$, $l = 0.6$, $p_1 = -0.1$, and $p_0 = 1$.



(a) (b)

Figure 9. The influence of β on $|p_5(x,t)|^2$ and $\mathrm{Re}(p_5(x,t))$, where $\eta_1 = -0.03$, $\eta_2 = 0.1$, $\eta_3 = 0.1$, $w = -0.2$, $l = 0.6$, $p_1 = -0.1$, and $p_0 = 0.1$.

nonlinear Schrödinger equation, which includes group velocity dispersion coefficients and second-order spatiotemporal effects, are innovative and have not been documented in previous research.

5 Conclusion

This study investigated the conformable-operator Schrödinger equation in optical fibers. By employing the extended hyperbolic function method, a wide range of novel analytical soliton solutions were derived, including bright, dark, singular, straddled, and dark-bright solitons. These solutions were obtained in closed form and analyzed under various physical and temporal parameters.

Using an appropriate wave transformation, the governing model was reduced to an ordinary differential equation. The impact of the conformable operator order β was thoroughly analyzed through graphical illustrations. It was observed that the conformable operator significantly affects the amplitude, width, and propagation speed of the solitons, offering enhanced control for pulse shaping in nonlinear optical media.

In comparison to previously reported methods – such as the auxiliary equation method [27], sine-Gordon expansion [21], and F-expansion technique [18] – the current approach yields a broader spectrum of analytical solutions, including complex structures like dark-bright solitons. The extended hyperbolic function method provides enhanced flexibility and accuracy, generating both trigonometric and hyperbolic families of solutions that were not previously reported. Moreover, while prior works often focused on classical or M-derivatives, the use of the conformable operator in this study offers a mathematically simpler yet physically meaningful alternative that preserves key properties of integer-order calculus.

A specific comparison is made with the work in [5], where the generalized nonlinear Schrödinger system was studied using a compatible Riccati-type technique. That study was limited to integer-order derivatives and yielded only bright and dark soliton solutions. In contrast, the present work incorporates temporal dynamics using the conformable operator, enabling the modeling of memory and hereditary effects in pulse propagation – phenomena not captured by classical models. Furthermore, the variety and complexity of soliton structures derived here demonstrate the extended capability of the proposed methodology.

Graphical simulations in both 2D and 3D were used to validate the analytical findings, illustrating the evolution and interaction of soliton structures under different conformable operator and temporal effects. These results have potential applications in ultrafast optical communication systems, where fine control of soliton dynamics is crucial.

In summary, this study introduces an effective mathematical framework for analyzing nonlinear Schrödinger equations using the conformable operator and delivers novel soliton structures with strong physical relevance. Future work may involve extending the model to include external potentials, stochastic influences, or other types of operators (e.g., Caputo, Atangana–Baleanu), as well as performing numerical simulations or experimental validations to further explore and support the theoretical findings.

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Conflicts of interest. The authors declare no conflicts of interest.

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