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Nonparametric changed segment detection in functional data

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Abstract. We address the epidemic change point detection problem without parametric assumptions. We propose statistics based on Cramér–von Mises-type statistic and reproducing kernel Hilbert space that iterate through all interval subsets, rescaling them to remain sensitive to both short and long epidemics. We prove limit theorems and provide quantiles for both statistics under the different parametrizations. The simulations show consistent power across a wide range of scenarios, and an application to electricity balancing prices consistently detects a market disturbance.

Keywords: epidemic change, reproducing kernel Hilbert space, Cramér–von Mises-type statistic.

1 Introduction

Let $(\mathbb{S}, \mathcal{S})$ be a sample space and $\mathcal{P}(\mathbb{S})$ a set of all probability distributions on $(\mathbb{S}, \mathcal{S})$. Assume that we are given a random sample $\mathcal{X} = (X_1, X_2, \dots, X_n)$ of independent random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, taking values in the sample space $(\mathbb{S}, \mathcal{S})$ and with respective distributions $P_{X_1}, P_{X_2}, \dots, P_{X_n}$. We wish to test the null hypothesis

$$(\mathrm{H}_0) \qquad P_{X_1} = P_{X_2} = \cdots = P_{X_n} = P$$

versus the alternative

(H_1) there is a segment $I^* := \{k^* + 1, \dots, k^* + \ell^*\} \subset I_n := \{1, 2, \dots, n\}$ such that

$$P_{X_i} = \begin{cases} P & \text{for } i \in I_n \setminus I^*, \\ Q & \text{for } i \in I^*, \end{cases} \quad \text{where } P, Q \in \mathcal{P}(\mathbb{S}) \text{ and } P \neq Q.$$

Under (H_1), the sample $\mathcal{X}_{I^*} = (X_i,\, i \in I^*)$ constitutes a changed segment of length ℓ^* starting at $k^* + 1$, and we have two samples $\mathcal{X}_{I^*}$ and $\mathcal{X}_{I_n \setminus I^*}$ taken from different distributions. If k^* and ℓ^* were known, we would face a two-sample problem that examines whether these two samples are from the same distribution or not. This indicates that test statistics for detecting epidemic change points can be based on two-sample tests, but have to account for the fact that neither the beginning nor the length of a changed segment is known.

Nonparametric change point methods date back to [8,13,17,21,23]; epidemic (changed segment) alternatives were studied in [1, 4, 9, 10, 14, 19, 22, 32, 33] and remain active in modern work [3, 11, 12, 18, 27]. Kernel methods are now standard for two-sample testing [15,16] and have been extended to the epidemic/segment setting by maximizing an RKHS two-sample discrepancy (MMD^2) over all contiguous interval candidates $I(k,\ell)$; see kernel segmentation via RKHS mean embeddings (KCP) and kernel scan statistics [2,30]. In functional time series applications – such as household electricity consumption – combining changed segment testing with functional representations can improve pattern based grouping [20,31].

To construct a test procedure, for any subsample $\mathcal{X}_I = \{X_i,\, i \in I\}$, where $I \subset I_n := \{1, 2, \dots, n\}$, we consider the probability

$$\mathbf{P}_n^I := \frac{1}{|I|} \sum_{i \in I} \delta_{X_i},$$

where $|I|$ denotes the number of elements in the set I, and δ_x is the Dirac probability. We abbreviate $\mathbf{P}_n^{I_n}$ to $\widehat{\mathbf{P}}_n$, which is the empirical distribution of the sample $\mathcal{X}$. To estimate heterogeneity in a sample $\mathcal{X}$, we consider a seminorm $\mathcal{N}$ defined in the space $\mathcal{M}(\mathbb{S}, \mathcal{S})$ of signed measures in $(\mathbb{S}, \mathcal{S})$, which may depend on n and may also be random. Next, as a measure of heterogeneity, consider $\mathcal{N}(\mathbf{P}_n^I - \mathbf{P}_n^{I^c})$, where $I^c = I_n \setminus I$.

For $1 \leqslant \ell \leqslant n$, $k = 0, 1, \dots, n-\ell$, let $I(k,\ell) = \{k+1, \dots, k+\ell\} \subset I_n$. Then the discrepancy between probabilities $\mathbf{P}_n^{I(k,\ell)}$ and $\mathbf{P}_n^{I^c(k,\ell)}$ is estimated by

$$\mathcal{N}\big(\mathbf{P}_n^{I(k,\ell)} - \mathbf{P}_n^{I^c(k,\ell)}\big) = \frac{n}{\ell(n-\ell)} \mathcal{N}\Bigg(\sum_{i=k+1}^{k+\ell} (\delta_{X_i} - \widehat{\mathbf{P}}_n)\Bigg).$$

Based on ideas in [26], the following statistic is suggested:

$$\max_{1 \leqslant \ell < n} \max_{0 \leqslant k \leqslant n-\ell} \frac{\mathcal{N}(\sum_{i=k+1}^{k+\ell} (\delta_{X_i} - \widehat{\mathbf{P}}_n))}{\rho(\ell/n(1-\ell/n))},$$

where $\rho : (0,1) \to \mathbb{R}$ is a weight function. This statistic takes into account both the small change segment, where $\ell^*/n \to 0$ as $n \to \infty$, and the large one, where $\ell^*/n \to 1$ as $n \to \infty$. If $\ell^*/n \to 0$, then the factor $1 - \ell/n$ does not contribute to the critical values of the test statistic; and if $\ell^*/n \to 1$, then the factor ℓ/n does not contribute to the critical values of the test statistic. Observing that

$$\sum_{i \in I(k,\ell)} (\delta_{X_i} - \widehat{\mathbf{P}}_n) = -\sum_{i \in I_n \setminus I(k,\ell)} (\delta_{X_i} - \widehat{\mathbf{P}}_n),$$

we confine ourself to the situation where, under alternative hypothesis, $\ell^*/n \to 0$ as $n \to \infty$, and consider the statistic

$$T_{n,\rho}^{(\mathcal{N})}(\mathcal{X}) = \max_{1 \leqslant \ell < n/2} \max_{0 \leqslant k \leqslant n-\ell} \frac{\mathcal{N}(\sum_{i=k+1}^{k+\ell} (\delta_{X_i} - \widehat{\mathbf{P}}_n))}{\rho(\ell/n)}. \tag{1}$$

A large class of seminorms $\mathcal{N}$ is obtained from integral-type probability semimetrics induced by a class $\mathcal{F}$ of measurable functions $f : \mathbb{S} \to \mathbb{R}$: for $P, Q \in \mathcal{P}(\mathbb{S})$,

$$\zeta_{\mathcal{F}}(P, Q) = \sup_{f \in \mathcal{F}} |Pf - Qf|,$$

where $Pf = \int_{\mathbb{S}} f(s)\, P(\mathrm{d}s)$.

In statistics, integral-type distances are widely employed in nonparametric two-sample testing, most notably in the Kolmogorov–Smirnov [7, 28, 29] and kernel tests [15, 16]. In this paper, we extend tests based on the L_p metric and kernel methods from the two-sample framework to the epidemic change setting.

Taking $\mathcal{N}(P - Q) = \zeta_{\mathcal{F}}(P, Q)$ in (1) leads to the statistic

$$T^{\mathcal{F}}_{n,\rho}(\mathcal{X}) := \max_{1 \leqslant \ell \leqslant n/2} \max_{0 \leqslant k < n - \ell} \frac{\sup_{f \in \mathcal{F}} |\sum_{j=k+1}^{k+\ell} (f(X_j) - \overline{f(X)}_n)|}{\rho(\ell/n)}. \tag{2}$$

Throughout, we write $\overline{X}_n := \sum_{i=1}^{n} X_i/n$ for the sample mean.

Example 1. For a real-valued sample $\mathcal{X} = (X_1, \dots, X_n)$, the L_p-type test is obtained via the seminorm

$$\mathcal{N}(\mu) = \left(\int_{-\infty}^{\infty} \left|\mu\big((-\infty, x]\big)\right|^p F_n(\mathrm{d}x) \right)^{1/p},$$

where $p \geqslant 2$, $\mu \in \mathcal{M}(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, and $F_n(x) = \widehat{\mathbf{P}}_n((-\infty, x])$ is the empirical distribution function. This is a random seminorm depending on n. The L_p-type test is discussed in detail in Section 2.

Example 2. Reproducing kernel distance $\zeta_{\mathcal{F}}$ is obtained when $\mathcal{F} = \{f\colon \|f\|_{\mathcal{H}} \leqslant 1\}$, where $\mathcal{H}$ represents a reproducing kernel Hilbert space (RKHS) with κ as its reproducing kernel. Section 3 is devoted to application of kernel distance for testing epidemic change point.

In Section 4, we conduct experiments in order to analyze the power of defined tests. Finally, in Section 5, we apply defined tests to the electricity balancing prices.

2 L_p-type test

In this section, we consider real-valued sample $X_1, \dots, X_n$ with distribution functions $F_{X_1}, \dots, F_{X_n}$. We wish to test the null hypothesis

(H$_0$) $$F_{X_1} = F_{X_2} = \cdots = F_{X_n} = F$$

versus the alternative

(H$_1$) there is a segment $I^* := \{k^* + 1, \dots, k^* + \ell^*\} \subset I_n := \{1, 2, \dots, n\}$ such that

$$F_{X_i} = \begin{cases} F & \text{for } i \in I_n \setminus I^*, \\ F_{X_i} & \text{for } i \in I^*, \end{cases}$$

where F, F' are distribution functions, and $F \neq F'$.

The L_p-type statistic is defined as

$$T_{n,\rho,p}(\mathcal{X}) = \max_{1\leqslant \ell < n} \max_{0\leqslant k\leqslant n-\ell} \frac{(\int_{-\infty}^{\infty} |\sum_{i=k+1}^{k+\ell}(\mathbf{1}_{\{X_i\leqslant x\}} - F_n(x)|^p \,\mathrm{d}F_n(x))^{1/p})}{\rho(\ell/n)}.$$

If $p = 2$, this statistic is known as Cramér–von Mises-type statistic. In the case where the distribution function F is continuous, we reduce the statistic $T_{n,\rho,p}$ to

$$\widehat{T}_{n,\rho,p} = \max_{1\leqslant \ell\leqslant n/2} \max_{0\leqslant k\leqslant n-\ell} \frac{(\int_0^1 |\sum_{i=k+1}^{k+\ell}(\mathbf{1}_{\{F(X_i)\leqslant t\}} - t)|^p \mathrm{d}t)^{1/p}}{\rho(\ell/n)}.$$

Before we establish asymptotic distribution of the statistic $\widehat{T}_{n,\rho,p}(\mathcal{X})$ under null hypothesis, we need some preparation.

Let $W^{(2)} := (W(s,t),\,(s,t) \in [0,1]^2)$ be a standard Wiener sheet. That is, $W^{(2)}$ is a Gaussian zero-mean random process with covariance

$$\mathbf{E}\bigl(W^{(2)}(s,t)W^{(2)}(s',t')\bigr) = \min\{s,s'\}\min\{t,t'\}, \quad s,s',t,t' \in [0,1].$$

Concerning weighting function ρ, we consider a class $\mathcal{R}$ of functions $\rho : (0,1) \to \mathbb{R}$ satisfying the following conditions:

(i) for some $0 < \alpha \leqslant 1/2$ and some function L, which is normalized slowly varying at infinity, $\rho(s) = s^{\alpha}L(1/s)$, $0 < s \leqslant 1$;
(ii) $\theta(t) = t^{1/2}\rho(1/t)$ is C^1 on $[1,\infty)$;
(iii) there is a $\beta > 1/2$ and some $a > 0$ such that $\theta(t)\log^{-\beta}(t)$ is nondecreasing on $[a,\infty)$.

This class includes functions $\rho_\gamma = (\rho_\gamma(t),\, t \in [0,1])$ with $0 < \gamma < 1/2$ and $\rho_{0.5,\beta} = (\rho_{0.5,\beta}(t), t \in [0,1])$ with $\beta > 1/2$, where

$$\rho_\gamma(t) = t^\gamma \quad \text{and} \quad \rho_{0.5,\beta}(t) = t^{1/2}\log^\beta \frac{c}{t}.$$

For a separable Banach space $\mathbb{B}$ with the norm $\|x\|, x \in \mathbb{B}$, we denote by $C([0,1],\mathbb{B})$ a Banach space of continuous functions $x : [0,1] \to \mathbb{B}$ endowed with the norm $\|x\|_{\mathbb{B}} = \sup_{0\leqslant t\leqslant 1}\|x(t)\|$. Set

$$\mathcal{H}_\rho^o\bigl([0,1],\mathbb{B}\bigr) = \biggl\{x \in C\bigl([0,1],\mathbb{B}\bigr)\colon \lim_{\delta\to 0}\sup_{0<t-s<1}\frac{\|x(t)-x(s)\|}{\rho(t-s)} = 0\biggr\}.$$

With the norm

$$\|x\|_{\rho,\mathbb{B}} = \bigl\|x(0)\bigr\| + \sup_{0<t-s\leqslant 1}\frac{\|x(t)-x(s)\|}{\rho(t-s)}, \quad x \in \mathcal{H}_\rho^o\bigl([0,1],\mathbb{B}\bigr),$$

the set $\mathcal{H}_\rho^o([0,1],\mathbb{B})$ is a separable Banach space.

Theorem 1. *Let $\rho \in \mathcal{R}$ and $p \geqslant 2$. If $X_1, \dots, X_n$ are i.i.d. random variables with continuous distribution function F, then*

$$\lim_{n\to\infty} \mathbf{P}\big(n^{-1/2}\widehat{T}_{n,\rho,p}(\mathcal{X}) \leqslant x\big) = \mathbf{P}(T_{\rho,p} \leqslant x), \quad x \geqslant 0,$$

where

$$T_{\rho,p} = \sup_{0<t-s\leqslant 1/2} \frac{(\int_0^1 \Delta_{t,s}^p(u)\,\mathrm{d}u)^{1/p}}{\rho(t-s)}$$

with

$$\Delta_{t,s}(u) = W(u,t) - W(u,s) - (t-s)W(u,1) \\ - u\big(W(1,t) - W(1,s) - (t-s)W(1,1)\big).$$

Proof. Consider random functions $Z_k := (Z_k(t) = \mathbf{1}_{\{F(X_k)\leqslant t\}} - t,\ t \in [0,1])$, $k = 1, \dots, n$. The random functions $Z_1, \dots, Z_n$ can be identified with a random sample of i.i.d. $L_p := L_p(0,1)$-valued random variables. Clearly, $\mathbf{E}(Z_i) = 0$. Covariance operator $Q = \mathbf{E}(Z_i \otimes Z_i) : L_q \to L_p$, $1/p + 1/q = 1$, is defined by the covariance function

$$q(t,s) = \mathbf{E}Z_i(t)Z_i(s) = \min\{t,s\} - ts, \quad t,s \in [0,1],$$

via

$$Qh(g) = \int_0^1\int_0^1 h(t)g(s)q(t,s)\,\mathrm{d}t\,\mathrm{d}s, \quad h,g \in L_q.$$

With these notations, the statistic $\widehat{T}_{n,\rho,p}$ takes the form

$$\widehat{T}_{n,\rho,p}(\mathcal{X}) = \max_{1\leqslant \ell<n} \frac{1}{\rho(\ell/n)} \max_{0\leqslant k<n-\ell} \left\| \sum_{i=k+1}^{k+\ell} Z_i \right\|_p.$$

Set

$$\zeta_n(t) = \sum_{i=1}^{\lfloor nt\rfloor} Z_i + \big(nt - \lfloor nt\rfloor\big) Z_{\lfloor nt\rfloor+1}, \quad t \in [0,1].$$

Random variables Z_i with values in L_p are bounded, therefore, they satisfy the central limit theorem and, as a result of Theorem 8 in [25],

$$n^{-1/2}\zeta_n \xrightarrow{\mathcal{D}} W_Q \quad \text{in the space } \mathcal{H}_\rho^o\big([0,1], L_p\big),$$

where $W_Q = (W_Q(t),\ t \in [0,1])$ is a Gaussian random process with values in L_p such that $W_Q(t) - W_Q(s)$ has the same distribution as $Y_Q|t-s|$, and Y_Q is zero-mean Gaussian random variable in L_p with covariance Q. By continuous mapping theorem we get

$$n^{-1/2} \sup_{0<t\leqslant 1/2} \frac{\|\zeta_n(t) - \zeta_n(s)\|}{\rho(t-s)} \xrightarrow{\mathcal{D}} \sup_{0<t-s\leqslant 1/2} \frac{\|W_Q(t) - W_Q(s)\|}{\rho(t-s)}.$$

Table 1. Estimated quantiles of $T_{\rho,p}$.

p	γ	2.5%	5%	25%	50%	75%	95%	97.5%
2	0	0.259	0.281	0.338	0.383	0.441	0.544	0.576
	0.25	0.391	0.406	0.477	0.526	0.585	0.683	0.709
	0.45	0.715	0.750	0.818	0.882	0.957	1.082	1.135
4	0	0.321	0.336	0.392	0.453	0.521	0.610	0.649
	0.25	0.465	0.479	0.552	0.618	0.671	0.775	0.796
	0.45	0.845	0.864	0.943	1.008	1.096	1.235	1.255
10	0	0.408	0.423	0.483	0.540	0.608	0.727	0.759
	0.25	0.567	0.600	0.672	0.736	0.821	0.994	1.065
	0.45	1.003	1.018	1.105	1.180	1.271	1.475	1.541

By [24, Thm. 3],

$$\sup_{0<|t-s|\leqslant 1/2} \frac{\|\zeta_n(t)-\zeta_n(s)\|}{\rho(t-s)} = \max_{1\leqslant \ell\leqslant n/2}\max_{0\leqslant k\leqslant n-\ell} \frac{\|\sum_{i=k+1}^{k+\ell} Z_i\|}{\rho(\ell/n)} = \widehat{T}_{n,\rho,p}.$$

Hence,

$$n^{-1/2}\widehat{T}_{n,\rho,p} \xrightarrow{\mathcal{D}} \max_{0<t-s\leqslant 1/2} \frac{\|W_Q(t)-W_Q(s)-(t-s)W_Q(1)\|_p}{\rho(t-s)}.$$

It is easy to see that the random variable $\|W_Q(t)-W_Q(s)-(t-s)W_Q(1)\|_p^p$ has the same distribution as

$$\int_0^1 \bigl|W(u,t)-W(u,s)-(t-s)W(u,1) \\ -u\bigl(W(1,t)-W(1,s)-(t-s)W(1,1)\bigr)\bigr|^p\,\mathrm{d}u.$$

This completes the proof. □

Let $\alpha\in(0,1)$ be the preassigned asymptotic level of significance, which controls the probability of falsely rejecting hypothesis (H_0) when it is true, and define the test

$$\widehat{T}_{n,\rho,p} > \sqrt{n}C_\alpha,$$

where $C_\alpha>0$ satisfies

$$\mathbf{P}(T_{\rho,p}>C_\alpha)=\alpha.$$

The quantiles C_α of $T_{\rho,p}$ are calculated in Table 1.

3 Reproducing kernel distance test

Following [16], we consider (2), where the function class $\mathcal{F}$ is the unit ball in a reproducing kernel Hilbert space defined via symmetric kernel. To be more precise, let $\kappa:\mathbb{S}\times\mathbb{S}\to\mathbb{R}$ be a symmetric positive definite function. According to the Moore–

Aronszajn theorem [6], there is an associated reproducing kernel Hilbert space (RKHS) $H_\kappa(\mathbb{S})$, a separable Hilbert space of real-valued functions defined on $\mathbb{S}$ with the norm $\|x\|_\kappa$ and inner product $\langle x, y\rangle_\kappa$, $x, y \in H_\kappa(\mathbb{S})$, for which the function κ is a reproducing kernel, that is,

(i) for each $s \in \mathbb{S}$, $\kappa(\cdot; s) = (\kappa(t, s), t \in \mathbb{S}) \in H_\kappa(\mathbb{S})$, and
(ii) for each $s \in \mathbb{S}$ and each $f \in H_\kappa(\mathbb{S})$, $\langle f, \kappa(\cdot, s)\rangle_\kappa = f(s)$.

The kernel embedding of a probability P on $(\mathbb{S}, \mathcal{S})$ into $H_\kappa(\mathbb{S})$ is $P \to \mu_\kappa(P) \in H_\kappa(\mathbb{S})$ and is defined by a Bochner integral

$$\mu_\kappa(P) = \int_{\mathbb{S}} \kappa(\cdot, s)\, P(\mathrm{d}s).$$

A sufficient condition for the existence of $\mu_\kappa(P)$ is that κ is Borel measurable and that $\int_{\mathbb{S}} \kappa^{1/2}(s; s)\, P(\mathrm{d}s) < \infty$. If δ_x is a Dirac measure on $(\mathbb{S}, \mathcal{S})$, then

$$\mu_\kappa(\delta_x) = \kappa(\cdot, x) = \bigl(\kappa(s, x),\ s \in \mathbb{S}\bigr).$$

Now consider the pseudodistance $d_{\mathcal{F}}(P, Q)$, where $\mathcal{F} = \{f \in H_\kappa\colon \|f\|_\kappa = 1\}$ is the unit ball in RKHS $H_\kappa(\mathbb{S})$. We have

$$\begin{aligned} d_{\mathcal{F}}(P, Q) &= \sup_{f\in\mathcal{F}} \biggl| \int_{\mathbb{S}} f(s)\, P(\mathrm{d}s) - \int_{\mathbb{S}} f(s)\, Q(\mathrm{d}s) \biggr| \\ &= \sup_{f\in\mathcal{F}} \biggl| \int_{\mathbb{S}} \bigl\langle f, \kappa(\cdot, s)\bigr\rangle\, P(\mathrm{d}s) - \int_{\mathbb{S}} \bigl\langle f, \kappa(\cdot, s)\bigr\rangle\, Q(\mathrm{d}s) \bigr| \\ &= \sup_{f\in\mathcal{F}} \bigl| \bigl\langle f, \mu_\kappa(P)\bigr\rangle - \bigl\langle f, \mu_\kappa(Q)\bigr\rangle \bigr| = \sup_{f\in\mathcal{F}} \bigl| \bigl\langle f, \mu_\kappa(P) - \mu_\kappa(Q)\bigr\rangle \bigr| \\ &= \bigl\| \mu_\kappa(P) - \mu_\kappa(Q) \bigr\|_\kappa. \end{aligned}$$

With this observation, the test statistic $T_{n,\rho}^{(\mathcal{F})}(\mathcal{X})$ becomes

$$T_{n,\rho}^{\kappa}(X_1, \dots, X_n) = \max_{1\leqslant \ell\leqslant n/2}\ \max_{0\leqslant k<n-\ell} \frac{\| \sum_{j=k+1}^{k+\ell} [\mu_\kappa(\delta_{X_j}) - \mu_\kappa(\widehat{P}_n)] \|_\kappa}{\rho(\ell/n)}.$$

If the distribution P is known, we may use instead the statistic

$$\widehat{T}_{n,\rho}^{\kappa}(X_1, \dots, X_n) = \max_{1\leqslant \ell\leqslant n/2}\ \max_{0\leqslant k<n-\ell} \frac{\| \sum_{j=k+1}^{k+\ell} [\mu_\kappa(\delta_{X_j}) - \mu_\kappa(P)] \|_\kappa}{\rho(\ell/n)}.$$

For any two distributions P and Q on $(\mathbb{S}, \mathcal{S})$,

$$\begin{aligned} \bigl\| \mu_\kappa(P) - \mu_\kappa(Q) \bigr\|_\kappa^2 = {}& \int_{\mathbb{S}} \int_{\mathbb{S}} \kappa(s, s')\, P(\mathrm{d}s)\, P(\mathrm{d}s') + \int_{\mathbb{S}} \int_{\mathbb{S}} \kappa(s, s')\, Q(\mathrm{d}s)\, Q(\mathrm{d}s') \\ & - 2 \int_{\mathbb{S}} \int_{\mathbb{S}} \kappa(s, s')\, P(\mathrm{d}s)\, Q(\mathrm{d}s'). \end{aligned}$$

Hence,

$$\begin{aligned}\varDelta^2_{k,\ell}(X_1,\dots,X_n) &:= \Biggl\| \sum_{j=k+1}^{k+\ell} \bigl[\mu_\kappa(\delta_{X_j}) - \mu_\kappa(\widehat{P}_n)\bigr] \Biggr\|_\kappa^2 \\ &= \sum_{i,j=k+1}^{k+\ell} \kappa(X_i, X_j) + \left(\frac{\ell}{n}\right)^2 \sum_{i,j=1}^{n} \kappa(X_i, X_j) \\ &\quad - 2\left(\frac{\ell}{n}\right) \sum_{i=k+1}^{k+\ell} \sum_{j=1}^{n} \kappa(X_i, X_j).\end{aligned}$$

This expression makes statistic $T^{\mathcal{F}}_{n,\rho}(\mathcal{X})$ easy to calculate:

$$T^{(\kappa)}_{n,\rho}(X_1,\dots,X_n) = \max_{1\leqslant \ell\leqslant n/2} \max_{0\leqslant k<n-\ell} \frac{\varDelta_{k,\ell}(X_1,\dots,X_n)}{\rho(\ell/n)}.$$

Under (H_0), $X_i, i = 1,\dots,n$ are independent and identically distributed with marginal distribution P. So $\mu_\kappa(\delta_{X_i}), i = 1,\dots,n$, are also identically distributed and independent. Its mean

$$\mathbf{E}\bigl(\mu_\kappa(\delta_{X_i})\bigr) = \mathbf{E}\kappa(\cdot, X_i) = \int_{\mathbb{S}} \kappa(\cdot, s)\, P(\mathrm{d}s) = \mu_\kappa(P),$$

and covariance $Q : H_\kappa(\mathbb{S}) \to H_\kappa(\mathbb{S})$,

$$\langle Qh, g\rangle_\kappa = \mathbf{E}\bigl(\bigl\langle \mu_\kappa(\delta_{X_i}), h\bigr\rangle\bigl\langle \mu_\kappa(\delta_{X_i}), g\bigr\rangle\bigr) = \mathbf{E}\bigl[f(X_i)g(X_i)\bigr], \quad f, g \in H_\kappa.$$

Similarly,

$$\widehat{T}^{(\kappa)}_{n,\rho}(X_1,\dots,X_n) = \max_{1\leqslant \ell\leqslant n/2} \max_{0\leqslant k<n-\ell} \frac{\widehat{\varDelta}_{k,\ell}(X_1,\dots,X_n)}{\rho(\ell/n)},$$

where

$$\begin{aligned}\widehat{\varDelta}^2_{k,\ell}(X_1,\dots,X_n) &:= \Biggl\| \sum_{j=k+1}^{k+\ell} \bigl[\mu_\kappa(\delta_{X_j}) - \mu_\kappa(P)\bigr] \Biggr\|_\kappa^2 \\ &= \sum_{i,j=k+1}^{k+\ell} \kappa(X_i, X_j) + \ell^2 \int_{\mathbb{S}}\int_{\mathbb{S}} \kappa(s,t)\, P(\mathrm{d}s)\, P(\mathrm{d}t) \\ &\quad - 2\ell \sum_{i=k+1}^{k+\ell} \int_{\mathbb{S}} \kappa(X_i, s)\, P(\mathrm{d}s).\end{aligned}$$

Recall that W_Q is a Gaussian $H_\kappa(\mathbb{S})$-valued random process such that $W_Q(t) - W_Q(s)$ has the same distribution as $Y_Q\sqrt{|t-s|}$, where Y_Q is a zero-mean Gaussian random variable in $H_\kappa(\mathbb{S})$ with covariance Q.

Theorem 2. *Let $\rho \in \mathcal{R}$. Assume that the random variable X with distribution P satisfies*

$$\lim_{t\to\infty} t\mathbf{P}\bigl(\kappa^{1/2}(X,X) > A\theta(t)\bigr) = 0 \quad \textit{for any } A > 0. \tag{3}$$

Then under (H_0),

(i) *If P is known, then*

$$n^{-1/2}\widehat{T}^{(\kappa)}_{n,\rho}(X_1,\dots,X_n) \xrightarrow{\mathcal{D}} \sup_{0<t-s\leqslant 1/2} \frac{\|W_Q(t) - W_Q(s)\|_\kappa}{\rho(t-s)};$$

(ii) *If P is unknown, then*

$$n^{-1/2}T^{(\kappa)}_{n,\rho}(X_1,\dots,X_n) \xrightarrow{\mathcal{D}} \sup_{0<t-s\leqslant 1/2} \frac{\|W_Q(t) - W_Q(s) - (t-s)W_Q(1)\|_\kappa}{\rho(t-s)}.$$

Here $W_Q = (W_Q(t),\, t\in[0,1])$ is a Q-Wiener process in $H_\kappa(\mathbb{S})$.

Proof. Consider $Z_j := \mu_\kappa(\delta_{X_j}) - \mu_\kappa(P)$, $j = 1,\dots,n$, as random variables with values in the space $H_\kappa(\mathbb{S})$. Set

$$\zeta_n(t) = \sum_{j=1}^{\lfloor nt\rfloor} Z_j + \bigl(nt - \lfloor nt\rfloor\bigr) Z_{\lfloor nt\rfloor+1}, \quad t\in[0,1].$$

Under (H_0), $(Z_j,\, j = 1,\dots,n)$ are independent and identically distributed random variables with values in the space $H_\kappa(\mathbb{S})$, and for each $A > 0$,

$$\lim_{t\to\infty} t\mathbf{P}\bigl(\|Z_1\|_\kappa > A\theta(t)\bigr) = 0. \tag{4}$$

Since $\mu_\kappa(P)$ does not contribute to this condition, we have to check that

$$\lim_{t\to\infty} t\mathbf{P}\bigl(\bigl\|\mu_\kappa(\delta_{X_1})\bigr\|_\kappa > A\theta(t)\bigr) = 0. \tag{5}$$

Since

$$\bigl\|\mu_\kappa(\delta_{X_1})\bigr\|_\kappa^2 = \kappa^{1/2}(X_1,X_1),$$

condition 5 follows by (3). By [25, Thm. 8],

$$n^{-1/2}\zeta_n \xrightarrow{\mathcal{D}} W_Q \quad \text{in the space } \mathcal{H}^o_\rho\bigl([0,1], H_\kappa(\mathbb{S})\bigr). \tag{6}$$

Proof of (i). Let the function $\widehat{L} : \mathcal{H}^o_\rho([0,1], H_\kappa(\mathbb{S})) \to \mathbb{R}$ is defined by

$$\widehat{L}f = \sup_{0<t-s\leqslant 1/2} \frac{\|f(s) - f(t)\|_\kappa}{\rho(t-s)}.$$

It is easy to check that $\widehat{L}$ is a continuous function. By the continuous mapping theorem and (6) it follows that

$$\widehat{L}\big(n^{-1/2}\zeta_n\big) \xrightarrow{\mathcal{D}} LW_Q.$$

Finally, notice that $\widehat{L}(n^{-1/2}\zeta_n) = \widehat{T}^{\kappa}_{n,\rho}(X_1, \dots, X_n)$ by [24, Thm. 3].

Proof of (ii). Now let the function $L : \mathcal{H}^o_\rho([0,1], H_\kappa(\mathbb{S})) \to \mathbb{R}$ be defined by

$$Lf = \sup_{0<t-s\leqslant 1/2} \frac{\|f(t) - f(s) - (t-s)f(1)\|_\kappa}{\rho(t-s)}.$$

Again, it is easy to check that L is a continuous function. By the continuous mapping theorem and (6) it follows that

$$L\big(n^{-1/2}\zeta_n\big) \xrightarrow{\mathcal{D}} LW_Q. \tag{7}$$

Notice that $L(\zeta_n)$ is a polygonal process with vertexes at $(0,0)$, $(k/n, \sum_{j=1}^{k}(Z_j - \overline{Z}_n))$, $k = 1, \dots, n$. Hence, by [24, Thm. 3], $L(n^{-1/2}\zeta_n) = T^{\kappa}_{n,\rho}(X_1, \dots, X_n)$ and (7) end the proof of (ii).

Now the proof of theorem is complete. □

Example 3. Let $\mathbb{S} \subset \mathbb{R}$ be any compact set. Suppose κ is a continuous positive semidefinite kernel on $\mathbb{S}$, and the integral operator $T_\kappa : L_2(\mathbb{S}) \to L_2(\mathbb{S})$, defined by

$$(T_\kappa f)(\cdot) = \int_{\mathbb{S}} \kappa(\cdot, t) f(t)\,\mathrm{d}t,$$

is positive semidefinite, that is,

$$\int_{\mathbb{S}} \kappa(u,v) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v \geqslant 0, \quad f \in L_2(\mathbb{S}).$$

Then according to Mercer's theorem, there is an orthonormal basis (e_i) of $L_2(\mathbb{S})$ consisting of eigenfunctions of T_κ such that the corresponding eigenvalues (λ_i) are nonnegative. The eigenfunctions corresponding to nonzero eigenvalues are continuous on $\mathbb{S}$, and $\kappa(u,v)$ has the representation

$$\kappa(u,v) = \sum_{i=1}^{\infty} \lambda_i e_i(u) e_i(v), \quad u, v \in \mathbb{S},$$

where the convergence is absolute and uniform, that is,

$$\lim_{n\to\infty} \sup_{u,v\in\mathbb{S}} \left| \kappa(u,v) - \sum_{i=1}^{n} \lambda_i e_i(u) e_i(v) \right| = 0.$$

Define

$$L_{2,\lambda}(\mathbb{S}) = \left\{ f \in L_2(\mathbb{S})\colon\ \sum_k \lambda_k^{-1} \langle f, e_k\rangle^2 < \infty \right\}$$

Table 2. Estimated quantiles of $T_{n,\rho}^{(\kappa)}$.

$\kappa(x,y)$	γ	2.5%	5%	25%	50%	75%	95%	97.5%
$2-\max(x,y)$	0	1.809	1.858	1.918	1.938	1.946	1.948	1.949
	0.25	1.805	1.843	1.914	1.938	1.946	1.948	1.949
	0.45	1.820	1.860	1.916	1.938	1.947	1.949	1.950
$\exp\{-\frac{(x-y)^2}{2}\}$	0	0.795	0.864	1.031	1.137	1.174	1.193	1.203
	0.25	0.832	0.892	1.069	1.145	1.182	1.231	1.268
	0.45	1.332	1.352	1.439	1.527	1.635	1.857	1.939
$\min(x,y)$	0	0.622	0.641	0.781	0.942	1.126	1.355	1.382
	0.25	0.882	0.911	1.048	1.159	1.303	1.446	1.526
	0.45	2.069	2.110	2.292	2.442	2.577	2.822	2.903

and endow with the inner product

$$\langle f,g\rangle_\kappa=\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\langle f,e_k\rangle\langle g,e_k\rangle.$$

Then $L_{2,\lambda}(\mathbb{S})$ is the RKHS corresponding to the kernel κ. Indeed, since $\langle\kappa(\cdot,s),e_j\rangle=\lambda_j e_j(s)$, we have

$$\begin{aligned}\bigl\langle f,\kappa(\cdot,s)\bigr\rangle&=\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\langle f,e_k\rangle\bigl\langle\kappa(\cdot,s),e_k\bigr\rangle=\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\langle f,e_k\rangle\lambda_k e_k(s)\\&=f(s),\quad s\in[0,1].\end{aligned}$$

The quantiles in Table 2 were obtained by Monte Carlo simulation under the null hypothesis. For each kernel listed below, we generated 300 independent samples, each of size $n=2000$, and estimated the sampling distribution of the test statistic:

- (K_1) $\kappa(x,y)=2-\max(x,y)$, $x,y\in[0,1]$, with data drawn from the $\mathrm{B}(1,9)$ distribution.
- (K_2) $\kappa(x,y)=\exp\{(x-y)^2/2\}$, $x,y\in\mathbb{R}$, with data drawn from the standard normal distribution.
- (K_3) $\kappa(x,y)=\min(x,y)$, $x,y\in\mathbb{R}^+$, with data drawn from a chi-squared distribution with $df=1$.

4 Simulation study

We examined the test statistics defined above in a Monte Carlo simulation study. In the first subsection, we describe the simulated data under consideration. The statistical power analysis of the tests is presented in the following sections.

4.1 Data scenarios

We used the following scenarios:

- (S_1) *Normal*. Under the null hypothesis, we take $\xi_j\sim\mathcal{N}(0,1)$, $j=1,2,\dots,n$. In the alternative, we consider $\xi_j\sim\mathcal{N}(a,1)$, $j\in I^*$.

(S$_2$) *Noncentral chi-squared*. Under the null hypothesis, we take $\xi_j \sim {\chi'}^2_1(0)$, $j = 1, 2, \ldots, n$. In the alternative, we consider $\xi_j \sim {\chi'}^2_1(a)$, $j \in I^*$.

(S$_3$) *Pareto*. Under the null hypothesis, we take $\xi_j \sim \mathrm{Pa}(1, 3)$, $j = 1, 2, \ldots, n$. In the alternative, we consider $\xi_j \sim \mathrm{Pa}(1 + a, 3)$, $j \in I^*$.

(S$_4$) *Beta*. Under the null hypothesis, we take $\xi_j \sim \mathrm{B}(1, 9)$, $j = 1, 2, \ldots, n$. In the alternative, we consider $\xi_j \sim \mathrm{B}(1 + a,\ 9 - a)$, $j \in I^*$.

(S$_5$) *Pareto shape change*. Under the null hypothesis, we take $\xi_j \sim \mathrm{Pa}(1, 3)$, $j = 1, 2, \ldots, n$. In the alternative, we consider $\xi_j \sim \mathrm{Pa}(1,\ 3 + 10a)$, $j \in I^*$.

In all scenarios, we consider sample size $n = 200$, epidemic change length $l \in \{\lfloor \log n \rfloor + 1,\ n/5\}$, and size of change $a \in \{0, 1/4, 1/2, 1, 2, 4, 8\}$ with $a = 0$ representing the null hypothesis – no change.

4.2 Power analysis

Table 3 shows that in all scenarios, the setting $\gamma = 0.45$ provides the most attractive operating characteristics. Under the null hypothesis ($a = 0$) it gives the smallest type I error rate, while for every genuine departure ($a \geqslant 0.25$) and especially for $a \geqslant 2$, it matches or exceeds the power achieved by smaller γ values. The weaker performance observed for $\gamma < 0.45$ is largely explained by a slower convergence of the test statistic with its limit distribution, which inflates the rejection rate under the null hypothesis and, in turn, reduces the effective power. In short, $\gamma = 0.45$ offers the best trade-off: accurate type I error control combined with maximal sensitivity to meaningful changes.

Table 4 reveals a different operating profile for the RKHS test than for its L_p counterpart. In scenarios (S$_1$) and (S$_2$), settings with $\gamma \leqslant 0.25$ achieve the desirable combination of a low type-I error rate and high power once $a \geqslant 1$. In scenario (S$_4$), however, the choice $\gamma = 0.45$ performs best.

Table 3. L_p-type test power.

Scenario	γ	$a = 0$	$a = 0.25$	$a = 0.5$	$a = 1$	$a = 2$	$a = 4$	$a = 8$
(S$_1$)				$l = \lfloor \log n \rfloor + 1$				
	0	0.41	0.45	0.43	0.46	0.75	0.79	0.83
	0.25	0.38	0.32	0.29	0.52	0.83	1.00	1.00
	0.45	0.02	0.09	0.09	0.22	0.97	1.00	1.00
				$l = n/5$				
	0	0.38	0.59	0.87	1.00	1.00	1.00	1.00
	0.25	0.29	0.55	0.80	1.00	1.00	1.00	1.00
	0.45	0.02	0.18	0.67	1.00	1.00	1.00	1.00
(S$_2$)				$l = \lfloor \log n \rfloor + 1$				
	0	0.42	0.42	0.40	0.43	0.68	0.74	0.75
	0.25	0.38	0.36	0.22	0.38	0.68	1.00	1.00
	0.45	0.08	0.08	0.06	0.11	0.57	1.00	1.00
				$l = n/5$				
	0	0.39	0.33	0.41	0.81	1.00	1.00	1.00
	0.25	0.40	0.34	0.31	0.75	1.00	1.00	1.00
	0.45	0.07	0.04	0.12	0.64	1.00	1.00	1.00

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Table 3 (*continued from previous page*)

Scenario	γ	$a=0$	$a=0.25$	$a=0.5$	$a=1$	$a=2$	$a=4$	$a=8$
(S_3)				$l=\lfloor \log n \rfloor+1$				
	0	0.28	0.38	0.59	0.62	0.65	0.71	0.76
	0.25	0.29	0.50	0.67	0.93	1.00	1.00	1.00
	0.45	0.08	0.10	0.49	1.00	1.00	1.00	1.00
				$l=n/5$				
	0	0.28	1.00	1.00	1.00	1.00	1.00	1.00
	0.25	0.27	1.00	1.00	1.00	1.00	1.00	1.00
	0.45	0.02	1.00	1.00	1.00	1.00	1.00	1.00
(S_4)				$l=\lfloor \log n \rfloor+1$				
	0	0.28	0.30	0.35	0.49	0.57	0.75	0.72
	0.25	0.24	0.27	0.31	0.46	0.70	1.00	1.00
	0.45	0.09	0.07	0.05	0.27	0.76	1.00	1.00
				$l=n/5$				
	0	0.28	0.49	0.81	1.00	1.00	1.00	1.00
	0.25	0.27	0.55	0.88	1.00	1.00	1.00	1.00
	0.45	0.05	0.09	0.63	1.00	1.00	1.00	1.00
(S_5)				$l=\lfloor \log n \rfloor+1$				
	0	0.32	0.40	0.45	0.45	0.57	0.67	0.81
	0.25	0.23	0.32	0.45	0.50	0.72	0.89	1.00
	0.45	0.05	0.08	0.11	0.23	0.76	0.99	1.00
				$l=n/5$				
	0	0.31	0.83	0.99	1.00	1.00	1.00	0.00
	0.25	0.31	0.87	1.00	1.00	1.00	1.00	0.00
	0.45	0.07	0.60	0.99	1.00	1.00	1.00	0.00

Table 4. RKHS test power.

Scenario	γ	$a=0$	$a=0.25$	$a=0.5$	$a=1$	$a=2$	$a=4$	$a=8$
(S_1) with $\exp\{-\frac{(x-y)^2}{2}\}$				$l=\lfloor \log n \rfloor+1$				
	0	0.09	0.09	0.07	0.01	0.12	0.02	0.05
	0.25	0.03	0.07	0.08	0.12	0.19	0.97	1.00
	0.45	0.41	0.40	0.27	0.22	0.60	1.00	1.00
				$l=n/5$				
	0	0.07	0.04	0.10	0.46	1.00	1.00	1.00
	0.25	0.02	0.09	0.24	0.94	1.00	1.00	1.00
	0.45	0.35	0.31	0.22	0.83	1.00	1.00	1.00
(S_2) with $\min(x,y)$				$l=\lfloor \log n \rfloor+1$				
	0	0.07	0.08	0.12	0.09	0.15	1.00	1.00
	0.25	0.11	0.10	0.07	0.09	0.76	1.00	1.00
	0.45	0.36	0.50	0.46	0.32	0.82	1.00	1.00
				$l=n/5$				
	0	0.08	0.09	0.07	0.26	1.00	1.00	1.00
	0.25	0.05	0.10	0.14	0.68	1.00	1.00	1.00
	0.45	0.40	0.37	0.37	0.34	1.00	1.00	1.00

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Table 4 (*continued from previous page*)

Scenario	γ	$a=0$	$a=0.25$	$a=0.5$	$a=1$	$a=2$	$a=4$	$a=8$
(S_4) with $2-\max(x,y)$				$l=\lfloor\log n\rfloor+1$				
	0	0.01	0.04	0.00	0.02	0.03	0.04	0.03
	0.25	0.04	0.00	0.04	0.02	0.04	0.02	0.03
	0.45	0.02	0.02	0.04	0.05	0.04	0.03	0.99
				$l=n/5$				
	0	0.02	0.02	0.03	0.04	0.03	0.04	0.70
	0.25	0.03	0.03	0.02	0.02	0.02	0.15	1.00
	0.45	0.02	0.02	0.00	0.01	0.07	1.00	1.00

These results point to complementary use cases. With $\gamma = 0.45$, the L_p test is particularly sensitive to modest mean shifts that occur in short segments (S_1)–(S_4) and also detects broader distributional changes (S_5), provided the baseline distribution is known. The RKHS procedure is more general: critical quantiles can be generated for any kernel and data sample combination, although the requisite Monte Carlo calibration is computationally demanding.

5 Application – Litgrid

To illustrate the practical performance of the proposed tests, we analyze hourly electricity balancing prices for Lithuania from January 2018 to September 2024, released by the national electricity grid operator [1].

We follow [5]: for each week j, we map the hourly observations to a function $X_j : [0,1] \to \mathbb{R}$ using the Faber–Schauder basis truncated at level $K = 4$. This yields a sequence of weekly price curves $\{X_j(t)\colon j = 1,\dots,n;\ t \in [0,1]\}$, where n is the number of weeks in the study period. We then compute L_2 norms of the weekly curves to obtain the univariate inputs.

Figure 1 shows the functional time series, which contained an outlier week that, when removed, yielded a more stable dataset. Next, we performed the proposed tests. In both tests, we apply the test under three weighting parameters, $\gamma \in \{0, 0.25, 0.45\}$. We apply some transformations before applying tests to satisfy necessary conditions:

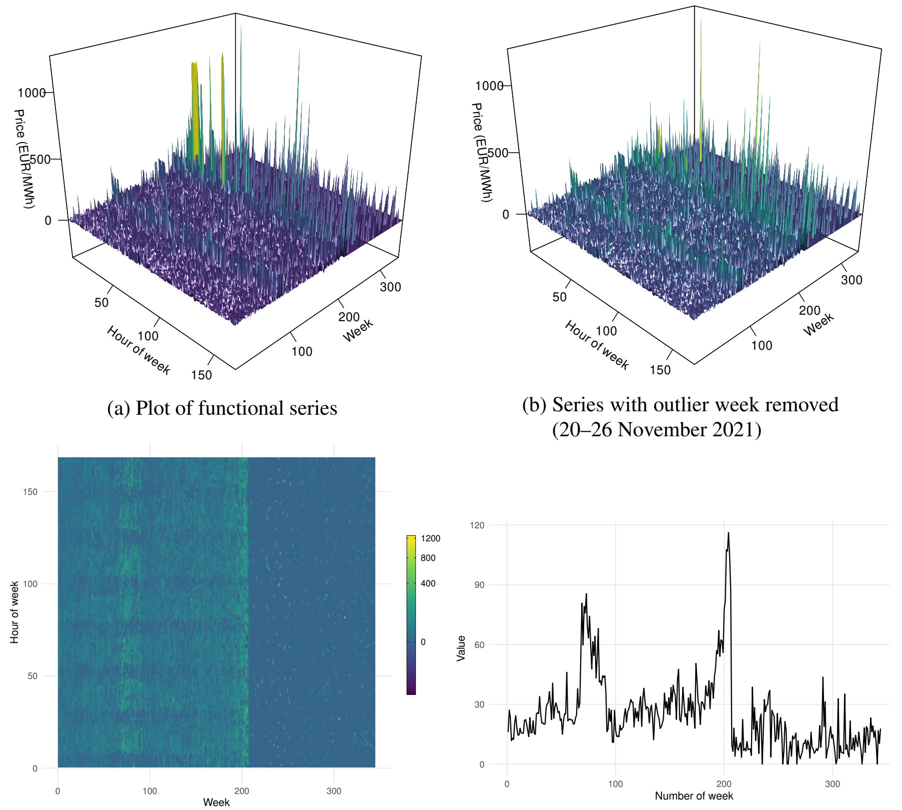
(T_1) *L_p-type*. For the observed sample $\{X_1,\dots,X_n\}$, we define the empirical distribution function

$$\widehat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{X_i \leqslant x\}}.$$

Then each observation X_i is transformed via

$$\widehat{F}_n(X_i) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{\{X_j \leqslant X_i\}}.$$

[1] Data provided by Litgrid: `https://baltic.transparency-dashboard.eu/`



(a) Plot of functional series

(b) Series with outlier week removed (20–26 November 2021)

(c) Contour plot, outlier removed

(d) L_2 norms, outlier removed

Figure 1. Visualisation of the electricity-balancing dataset.

(T$_2$) *RKHS*. Next, we transform $\widehat{F}_n(X_i) \in [0, 1]$ into the chi-square distribution with one degree of freedom through the quantile function $Q_{\chi_1^2}$. Define the transformed values

$$Y_i := Q_{\chi_1^2}\big(\widehat{F}_n(X_i)\big),$$

where $Q_{\chi_1^2}(u) = \inf\{x \in \mathbb{R}\colon F_{\chi_1^2}(x) \geqslant u\}$ denotes the inverse cumulative distribution function (CDF) of the χ_1^2 distribution. This transformation yields a sample $\{Y_1, \dots, Y_n\}$ that is approximately chi-squared distributed and, therefore, can be used in the RKHS type test with the precalculated quantile values.

Table 5 summarizes the results of tests with these transformations. The tests detect a segment of change in all cases. There is some variation in the exact boundaries of the detected segment, that is, when $\gamma = 0.45$. This can be explained by high values of γ outweighing shorter segments. Nevertheless, the tests are robust and identify logical distribution changes.

Table 5. Combined L_p-type and RKHS test results.

Method	γ	T	Interval (low)	Interval (high)	(H_0) rejection
L_p-type	0.00	2.1130	0.0785	0.6017	yes
	0.25	2.4844	0.0785	0.6017	yes
	0.45	2.8834	0.6017	1.0000	yes
RKHS	0.00	3.3728	0.1017	0.5988	yes
	0.25	4.1059	0.1860	0.5988	yes
	0.45	5.3501	0.1948	0.2616	yes

6 Conclusion

We proposed two nonparametric procedures for the changed segment detection in functional data: a Cramér–von Mises (L_p) statistic and an RKHS-based statistic, both rescaled by a weight ρ to remain sensitive to short and long changed segments. We established their limit distributions under (H_0), tabulated quantiles, examined finite sample power across different scenarios, and illustrated the methods on electricity balancing prices, where all variants detected a plausible disturbance. Future work includes extending the theory to weakly dependent data, as well as developing data-driven choices of ρ and kernel/feature sets.

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