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Energetic formulation of the subgroup commutativity degree

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Abstract. Finite groups in which every pair of subgroups (H, K) satisfies $HK = KH$ have been classified by Iwasawa, but only in the last decade it was introduced the notion of subgroup commutativity degree $\mathrm{sd}(G)$ of groups G. From restrictions of numerical nature on $\mathrm{sd}(G)$ one usually derives structural conditions on G; in fact, among groups G with $\mathrm{sd}(G) = 1$, one finds those originally studied by Iwasawa. Here we offer a new perspective of study for $\mathrm{sd}(G)$; we use a recently introduced graph, which is called nonpermutability graph of subgroups $\Gamma_{\mathrm{L}(G)}$ of G, in order to restrict $\mathrm{sd}(G)$ via the notion of energy of $\Gamma_{\mathrm{L}(G)}$ and by means of methods of spectral graph theory. In particular, we find new criteria of nilpotence for G along with new bounds for $\mathrm{sd}(G)$.

Keywords: spectral graph theory, energy of a graph, subgroup commutativity degree, nonpermutability graph of subgroups, adjacency matrix.

1 Introduction

The present paper deals only with finite groups and finite graphs. In a group G, it was largely studied the probability that two randomly chosen subgroups H and K of G commute, that is, the quantity

$$\mathrm{sd}(G) = \frac{1}{|\mathrm{L}(G)|^2}\bigl|\bigl\{(H,K) \in \mathrm{L}(G) \times \mathrm{L}(G) \bigm| HK = KH\bigr\}\bigr|, \tag{1}$$

where $\mathrm{L}(G)$ denotes the subgroups lattice of G. In fact, (1) is called *subgroup commutativity degree* of G by many authors and has been originally studied in [1, 11, 12, 22, 24, 27], but only a few years ago it was found a new approach via the methods of the spectral graph

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theory in [19–21]. A series of open questions remain in the theory of the subgroup commutativity degree [16], especially when one wants to use the perspective of Farrokhi and others [11, 12] who involved the factorization of groups. In the present paper, we continue the line of research, which was proposed in [19–21], focusing on a notion of energy that is inspired by the work of Gutman and others [13–15] in a more specific context.

We start by recalling, from [20], that the undirected simple graph

$$\mathit{\Gamma}_{\mathrm{L}(G)} = \bigl(V(\mathit{\Gamma}_{\mathrm{L}(G)}), E(\mathit{\Gamma}_{\mathrm{L}(G)})\bigr) \tag{2}$$

was called *nonpermutability graph of subgroups* of G when one considers vertices and edges as follows, respectively:

$$V(\mathit{\Gamma}_{\mathrm{L}(G)}) = \mathrm{L}(G) - \mathfrak{C}_{\mathrm{L}(G)}\bigl(\mathrm{L}(G)\bigr), \tag{3}$$
$$E(\mathit{\Gamma}_{\mathrm{L}(G)}) = \bigl\{(H,K) \in V(\mathit{\Gamma}_{\mathrm{L}(G)}) \times V(\mathit{\Gamma}_{\mathrm{L}(G)}) \bigm| H \sim K \iff HK \neq KH\bigr\}.$$

Note that $\mathfrak{C}_{\mathrm{L}(G)}(X)$ is the set of all subgroups of $\mathrm{L}(G)$ commuting with $X \in \mathrm{L}(G)$:

$$\mathfrak{C}_{\mathrm{L}(G)}(X) = \bigl\{Y \in \mathrm{L}(G) \bigm| XY = YX\bigr\},$$

but soon we find a problem: the intersection

$$\bigcap_{X \in \mathrm{L}(G)} \mathfrak{C}_{\mathrm{L}(G)}(X) = \bigl\{Y \in \mathrm{L}(G) \bigm| YX = XY \ \forall X \in \mathrm{L}(G)\bigr\} \tag{4}$$

is not (in general) a sublattice of $\mathrm{L}(G)$. Consequently, we consider the smallest sublattice of $\mathrm{L}(G)$ containing (4). This is the meaning of $\mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))$ in (3).

The adjacency matrix of $\mathit{\Gamma}_{\mathrm{L}(G)}$ is a well-known notion, which can be found in [3, 18] for arbitrary graphs and corresponds to the square matrix

$$A(\mathit{\Gamma}_{\mathrm{L}(G)}) = (a_{X,Y})_{X,Y \in V(\mathit{\Gamma}_{\mathrm{L}(G)})}, \tag{5}$$

where

$$a_{X,Y} = \begin{cases} 1 & \text{if } (X,Y) \in E(\mathit{\Gamma}_{\mathrm{L}(G)}), \\ 0 & \text{if } (X,Y) \notin E(\mathit{\Gamma}_{\mathrm{L}(G)}). \end{cases}$$

The adjacency matrix can also be used to calculate the degree of a vertex of the nonpermutability graph of subgroups as follows:

$$\deg(X) = \sum_{Y \in V(\mathit{\Gamma}_{\mathrm{L}(G)})} a_{X,Y}. \tag{6}$$

We do not really use the notion of degree here, but later, on we will focus on large families of *regular* nonpermutability graphs of subgroups, that is, graphs where all vertices have the same degree. Since $\mathit{\Gamma}_{\mathrm{L}(G)}$ is an undirected graph without loops, the Laplace matrix of $\mathit{\Gamma}_{\mathrm{L}(G)}$ is the matrix

$$\mathcal{L}(\mathit{\Gamma}_{\mathrm{L}(G)}) = D - A(\mathit{\Gamma}_{\mathrm{L}(G)}), \tag{7}$$

where $D = \operatorname{diag}(\deg(X_i))$ for all $X_i \in V(\Gamma_{\mathrm{L}(G)})$ and $i = 1, 2, \dots, m = |V(\Gamma_{\mathrm{L}(G)})|$. Laplacian and adjacency matrices are "well-known notions", which are usually considered in spectral graph theory [3, 5, 8]. On the other hand, the combinatorial formulas found in [19, Thm. 1.3, Prop. 3.2, Cor. 3.3] and [21], illustrate "less-known connections" between spectral graph theory and combinatorial group theory, that is, new relations between (1) and (5), (7). For example, if

$$\begin{aligned} \operatorname{spec}\bigl(A(\Gamma_{\mathrm{L}(G)})\bigr) &= \{\lambda_1, \lambda_2, \dots, \lambda_m\}, \\ \operatorname{spec}\bigl(\mathcal{L}(\Gamma_{\mathrm{L}(G)})\bigr) &= \{\mu_1, \mu_2, \dots, \mu_m\} \end{aligned} \tag{8}$$

are the spectra of the adjacency and Laplacian matrices, then [19, (3.6)] shows that for groups with nontrivial subgroup commutativity degree, one has

$$\operatorname{sd}(G) = 1 - \frac{1}{|\mathrm{L}(G)|^2} \sum_{i=1}^{m} \lambda_i^2 = 1 - \frac{1}{|\mathrm{L}(G)|^2} \sum_{i=1}^{m} \mu_i, \tag{9}$$

that is, we may compute (1) via (8).

Recall also from [26] that a group G is called *modular* if $\mathrm{L}(G)$ satisfies the modular law as lattice, that is, if $\langle H, K \cap T\rangle = \langle H, K\rangle \cap T$ for all subgroups H, K, T of G such that $H \leqslant T$. Note that groups satisfying $\operatorname{sd}(G) = 1$ are exactly the *quasihamiltonian groups*, i.e., nilpotent modular groups studied by Iwasawa; see [26, p. 87, Ex. 3] and [26, Thm. 5.1.1]. If H is a subgroup of a group G, H is said to be *permutable* in G if $HK = KH$ for all $K \in \mathrm{L}(G)$. It is *subnormal* in G whenever we have an ascending (finite) chain $H = H_1 \leqslant H_2 \leqslant \cdots \leqslant H_{n-1} \leqslant H_n = G$ in which H_i is normal in H_{i+1} for $1 \leqslant i \leqslant n$. A celebrated result of Ore shows that permutable subgroups are always subnormal; see [26]. This supports the idea that the study of the subgroup commutativity degree deals with restrictions of structural nature. In particular, a maximal subgroup of a quasihamiltonian group is always a normal subgroup. In 1941, Iwasawa [26] proved also that a p-group G is modular if and only if one of the following two cases happens:

(i) G is a Dedekind group; i.e., every subgroup of G is normal, or
(ii) G contains an abelian normal subgroup N such that the quotient group $G/N = C$ is cyclic, and if c denotes a generator of C, then for all $n \in N$, we have $c^{-1}nc = n^{1+p^s}$, where $s \geqslant 1$ in general, but $s \geqslant 2$ for $p = 2$.

Of course, one can check that

$$\text{abelian groups} \implies \text{Dedekind groups} \implies \text{modular groups},$$

but there are modular nonnilpotent groups [26]. Therefore $\operatorname{sd}(G)$ is a useful tool to determine when G belongs to a specific class, which is between the class of abelian groups, that of nilpotent groups, that of Dedekind groups, or none of these.

In particular, this aspect of structural nature motivates us to determine lower and upper bounds $\operatorname{sd}(G)$, but here we find new lower and upper bounds in terms of a new quantity, called the energy $\Gamma_{\mathrm{L}(G)}$. Matrix energy is a concept, which is well established in mathematics with several practical applications in physics and chemistry [14] and in the field of complex networks [18]. Motivated by [14, Def. 2.1], we introduce the following notion:

Definition 1 [Energy of the nonpermutability graph of subgroups]. Consider the nonpermutability graph of subgroups of G and $\mathit{\Gamma}_{\mathrm{L}(G)}$ with $m = |V(\mathit{\Gamma}_{\mathrm{L}(G)})|$ and the spectrum $\mathrm{spec}(A(\mathit{\Gamma}_{\mathrm{L}(G)})) = \{\lambda_j \mid 1 \leqslant j \leqslant m\}$. The energy of $\mathit{\Gamma}_{\mathrm{L}(G)}$ is given by

$$\mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(G)}) = \sum_{j=1}^{m} |\lambda_j|.$$

After Section 1, which has the character of an introduction, we list the main needed tools in Section 2 in order to prepare our main results, namely Theorems 1–6. Section 3 is devoted to prove Theorems 1 and 2, where we compute $\mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(G)})$ for dihedral groups, an important class of metacyclic groups that are not necessarily nilpotent. This test case helps to understand the behavior of $\mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(G)})$ when G is a solvable group, so we can find a clear description of the general case. Section 4 is devoted to the proof of Theorems 3 and 4, which provide new bounds for the subgroup commutativity degree of arbitrary groups in terms of the energy of the nonpermutability graph of subgroups. In Section 5, we illustrate two new criteria of nilpotence in Theorems 5 and 6. We end in Section 6 by proposing a new direction of study, where general methods of geometric measure theory and nonlinear analysis on graphs may allow us to bound vertices and edges of $\mathit{\Gamma}_{\mathrm{L}(G)}$ and consequently the magnitude of $\mathrm{sd}(G)$.

2 Preliminary results

The following result from [26] is useful to recognize modular p-groups.

Lemma 1. (*See* [26, *Lemma* 2.3.3].) *A p-group is modular if and only if each of its sections of order p^3 is modular. Moreover, if a p-group is nonmodular, then it contains a section isomorphic to D_8 or to $E(p^3)$, the nonabelian group of order p^3, and exponent p for $p > 2$.*

In chemistry, the π-electron energy of a conjugated carbon molecule may be computed via the Hückel theory of the chemical graphs [14], and essentially this notion of energy coincides with what we are going to investigate here, that is, with the Gutman's energy [13] in combinatorics. Therefore the results on graph energy assume a special significance in connection with chemistry and mathematical physics since these two disciplines are historcially at the basis of the concepts.

Remark 1. From [4] we may conclude that Definition 1 implies

$$2\sqrt{\left|E(\mathit{\Gamma}_{\mathrm{L}(G)})\right|} \leqslant \mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(G)}) \leqslant 2\left|E(\mathit{\Gamma}_{\mathrm{L}(G)})\right|. \tag{10}$$

In this situation, if $\mathit{\Gamma}_{\mathrm{L}(G)}$ has no isolated vertices, then the lower bound of (10) holds if and only if $\mathit{\Gamma}_{\mathrm{L}(G)}$ is a complete bipartite graph. Under the same assumption, the upper bound of (10) holds if and only if $\mathit{\Gamma}_{\mathrm{L}(G)}$ is regular of degree 1.

Motivated by a result from [15], we introduce the following notion:

Definition 2. Consider $\mathit{\Gamma}_{\mathrm{L}(G)}$, the nonpermutability graph of subgroups of G with $m = |V(\mathit{\Gamma}_{\mathrm{L}(G)})|$ and $\operatorname{spec}(\mathcal{L}(\mathit{\Gamma}_{\mathrm{L}(G)})) = \{\mu_j \mid 1 \leqslant j \leqslant m\}$. The *Laplacian energy of* $\mathit{\Gamma}_{\mathrm{L}(G)}$ is given by

$$\mathcal{LE}(\mathit{\Gamma}_{\mathrm{L}(G)}) = \sum_{j=1}^{m} \left| \mu_j - \frac{2|E(\mathit{\Gamma}_{\mathrm{L}(G)})|}{m} \right|.$$

By analogy with (10) we note some relevant numerical restrictions. On the basis of the definitions, we find that

$$2\sqrt{\left|M(\mathit{\Gamma}_{\mathrm{L}(G)})\right|} \leqslant \mathcal{LE}(\mathit{\Gamma}_{\mathrm{L}(G)}) \leqslant 2|M(\mathit{\Gamma}_{\mathrm{L}(G)})|,$$

where the number of edges should be recounted by an additional term of adjustment, that is,

$$\left|M(\mathit{\Gamma}_{\mathrm{L}(G)})\right| = \left|E(\mathit{\Gamma}_{\mathrm{L}(G)})\right| + \frac{1}{2}\sum_{j=1}^{m}\left(\mu_j - \frac{2|E(\mathit{\Gamma}_{\mathrm{L}(G)})|}{m}\right)^2.$$

The first relevant restriction shows that the energy of $\mathit{\Gamma}_{\mathrm{L}(G)}$ determines the bounds for $\operatorname{sd}(G)$.

Proposition 1. *Let G be a nonquasihamiltonian group with* $\mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(G)}) > 0$, *where* $\mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(G)})$ *is the energy of the corresponding graph* $\mathit{\Gamma}_{\mathrm{L}(G)}$. *Then*

$$1 - \frac{\mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(G)})^2}{2|\mathrm{L}(G)|^2} \leqslant \operatorname{sd}(G) \leqslant 1 - \frac{\mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(G)})}{|\mathrm{L}(G)|^2}.$$

Proof. Note that if G is quasihamiltonian, then we have the null graph, and so we should necessarily deal with G nonquasihamiltonian in order to avoid trivial situations. Assume that $\operatorname{spec}(A(\mathit{\Gamma}_{\mathrm{L}(G)})) = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$ and that we have m vertices according to Definition 1. Then from (9) we have

$$\operatorname{sd}(G) = 1 - \frac{1}{|\mathrm{L}(G)|^2}\sum_{i=1}^{m}\lambda_i^2,$$

and Definition 1 implies

$$1 - \operatorname{sd}(G) = \frac{1}{|\mathrm{L}(G)|^2}\sum_{i=1}^{m}\lambda_i^2 \geqslant \frac{1}{|\mathrm{L}(G)|^2}\sum_{i=1}^{m}|\lambda_i| = \frac{\mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(G)})}{|\mathrm{L}(G)|^2}. \tag{11}$$

Therefore we get the upper bound for $\operatorname{sd}(G)$, which we are looking for. To show the second part, from [19, Lemma 2.5] we know that a group G has

$$2\left|E(\mathit{\Gamma}_{\mathrm{L}(G)})\right| = \left|\mathrm{L}(G)\right|^2\bigl(1 - \operatorname{sd}(G)\bigr), \tag{12}$$

and from (10) we have

$$4\left|E(\mathit{\Gamma}_{\mathrm{L}(G)})\right| \leqslant \mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(G)})^2.$$

This implies

$$\big|E\big(\varGamma_{\mathrm{L}(G)}\big)\big| \leqslant \frac{\mathcal{E}(\varGamma_{\mathrm{L}(G)})^2}{4} \quad \Longrightarrow \quad 1-\frac{\mathcal{E}(\varGamma_{\mathrm{L}(G)})^2}{2|\mathrm{L}(G)|^2} \leqslant \mathrm{sd}(G).$$

Hence the result follows completely. □

Remark 2. In the argument of Proposition 1, if we have λ_i, which belongs to $\{0,1,-1\}$ in (11), then

$$\sum_{i=1}^{m}\lambda_i^2=\sum_{i=1}^{m}|\lambda_i|.$$

In this situation, if $\lambda_i = 0$ for all i, then $A(\varGamma_{\mathrm{L}(G)})$ is the zero matrix, and $\varGamma_{\mathrm{L}(G)}$ is the empty graph. If $\lambda_i \in \{-1,1\}$, then $(A(\varGamma_{\mathrm{L}(G)}))^2 = A(\varGamma_{\mathrm{L}(G)})$, and we get either the empty graph or the line graph with two elements.

Corollary 1. *Let G be a nonquasihamiltonian group with $\mathcal{E}(\varGamma_{\mathrm{L}(G)}) > 0$ and integral $\mathrm{spec}(A(\varGamma_{\mathrm{L}(G)}))$. Then*

$$\mathrm{sd}(G) < 1-\frac{\mathcal{E}(\varGamma_{\mathrm{L}(G)})}{|\mathrm{L}(G)|^2}.$$

Proof. From the argument of Proposition 1 the condition $\sum_{i=1}^{m}\lambda_i^2=\sum_{i=1}^{m}|\lambda_i|$ is satisfied if and only if $\lambda_i \in \{0,1,-1\}$ in (11) (by definition of graph with integral spectrum, its entries are integers). Therefore we can never get the upper bound since we need to have by definition at least 3 vertices, and so the result follows. □

The following result connects $\mathrm{sd}(G)$ and $\mathcal{E}(\varGamma_{\mathrm{L}(G)})$ for the first time.

Lemma 2. *Let G be a nonquasihamiltonian group with*

$$\mathrm{spec}\big(A(\varGamma_{\mathrm{L}(G)})\big)=\{\lambda_1,\lambda_2,\dots,\lambda_m\}.$$

Then

$$\mathcal{E}(\varGamma_{\mathrm{L}(G)})=\sqrt{\big(1-\mathrm{sd}(G)\big)\big|\mathrm{L}(G)\big|^2+2\sum_{i<j}|\lambda_i||\lambda_j|}.$$

Proof. It is easy to check that if $\mathrm{spec}(A(\varGamma_{\mathrm{L}(G)}))=\{\lambda_1,\lambda_2,\dots,\lambda_m\}$, then

$$\big[\mathcal{E}\big(\varGamma_{\mathrm{L}(G)}\big)\big]^2=\left(\sum_{j=1}^{m}|\lambda_j|\right)^2=\sum_{j=1}^{m}\lambda_j^2+\sum_{j\neq i}|\lambda_j||\lambda_i|. \tag{13}$$

Now, using (9), we obtain

$$\big[\mathcal{E}\big(\varGamma_{\mathrm{L}(G)}\big)\big]^2=(1-\mathrm{sd}(G))\big|\mathrm{L}(G)\big|^2+\sum_{j\neq i}|\lambda_j||\lambda_i|. \tag{14}$$

Since $\sum_{j\neq i}|\lambda_j||\lambda_i|=2\sum_{i<j}|\lambda_i||\lambda_j|$, the result follows by combining (13) and (14). □

Let us recall that

$$D_{2n} = \left\langle a, b \;\middle|\; a^n = b^2 = 1,\ b^{-1}ab = a^{-1} \right\rangle$$

denotes the usual dihedral group of order $2n$, where $n \geqslant 1$. From well-known facts on dihedral groups [25, 26], if n is an odd prime, then $D_{2n} = H \rtimes K$, namely it splits in the semidirect product of a normal cyclic subgroup $H \simeq C_n$ of order n by a cyclic group of order two $K \simeq C_2$ (acting by conjugation on H). This is an example of a *metacyclic group*, that is, of a group $G = AB$ with cyclic normal subgroup A and cyclic subgroup B such that $G/A \simeq B$. In the specific case of $D_{2n} = H \rtimes K$, we also have $H \cap K = 1$ and that K is nonnormal. Note also that for n odd number, the center of D_{2n} becomes trivial, so the group is nonnilpotent; see [25] for terminology and definitions of nilpotent groups, metacyclic groups, and solvable groups. Therefore it is clear that relevant nonabelian examples of dihedral groups begin from $n \geqslant 3$, in fact, $\mathrm{sd}(D_{2n}) < 1$ for all $n \geqslant 3$.

Remark 3. Recall from [19–21] that the subgroups of D_{2n} are of one (and only of one) of the following types:

(i) Cyclic $H_0^r = \langle a^{n/r} \rangle$ of order r, where r is a divisor of n;
(ii) Cyclic $H_i^1 = \langle ba^{i-1} \rangle$ of order 2, where $i = 1, 2, \dots, n$;
(iii) Dihedral $H_i^r = \langle a^{n/r}, ba^{i-1} \rangle$ of order $2r$, where r is a divisor of n, $r \neq 1, n$, and $i = 1, 2, \dots, n/r$.

In particular, one can show that for all $n \geqslant 3$,

$$\left|\mathrm{L}(D_{2n})\right| = \tau(n) + \sigma(n),$$

where $\tau(n)$ is the number of all divisors of n, and $\sigma(n)$ is the sum of all divisors of n. Note also that for two generic divisors r and s of n, the numbers $i \in \{1, 2, \dots, n/r\}$ and $j \in \{1, 2, \dots, n/s\}$ satisfy

$$H_i^r H_j^s = H_j^s H_i^r \quad \Longleftrightarrow \quad a^{2(i-j)} \in \left\langle a^{n/\gcd(r,s)} \right\rangle.$$

Now, we report some counting formulas for permutable subgroups in $\mathrm{L}(D_{2n})$.

Remark 4. Let x_i^r be the number of subgroups in $\mathrm{L}(D_{2n})$ of type (iii) in Remark 3 that commute with H_i^r. Then we may write precisely x_i^r in the following way:

$$x_i^r = \begin{cases} \sum_{s|n} \frac{\gcd(r,s)}{s} = r \sum_{s|n} \frac{1}{(r,s)} & \text{if } n \text{ is odd}, \\ 2^{u+2} - 2u + 2\alpha - 5 \\ \quad \text{if } n = 2^{\alpha-1},\ \alpha \geqslant 3,\ r = 2^u,\ 0 \leqslant u \leqslant \alpha - 1, \\ (2^{\alpha+1} - 1)x_i^{r'} \\ \quad \text{if } n = 2^\alpha n',\ n' \text{ is odd},\ \alpha \geqslant 1,\ r = 2^\beta r',\ r' \,|\, n',\ \beta = \alpha, \\ (2^{\beta+2} - 2\beta + 2\alpha - 3)x_i^{r'} \\ \quad \text{if } n = 2^\alpha n',\ n' \text{ is odd},\ \alpha \geqslant 1,\ r = 2^\beta r',\ r' \,|\, n',\ \beta \neq \alpha. \end{cases}$$

These expressions can be also found in [19–21, 27], so they are not new.

A further useful arithmetic function should be recalled from [20, 24, 27].

Remark 5. From [20, 27] we know that if g is the following natural function,

$$k \in \mathbb{N} \longmapsto g(k) = r \sum_{r|k, s|k} \frac{1}{\gcd(r, s)} \in \mathbb{N},$$

then we have

$$g(n) = \begin{cases} \sum_{r|n} \sum i = 1^{n/r} x_i^r \quad \text{if } n \text{ is odd}, \\ \sum_{r|n} \sum i = 1^{n/r} x_i^r = (\alpha - 2)2^{\alpha+3} + 9 \\ \quad \text{if } n = 2^{\alpha-1},\ \alpha \geqslant 3,\ r = 2^u,\ 0 \leqslant u \leqslant \alpha - 1, \\ \sum_{r|n} \sum i = 1^{n/r} x_i^r = ((\alpha - 1)2^{\alpha+3} + 9) g(n') \\ \quad \text{if } n = 2^\alpha n',\ n' \text{ is odd } \alpha \geqslant 1,\ r = 2^\beta r',\ r' \,|\, n',\ \beta \leqslant \alpha. \end{cases} \tag{15}$$

Thanks to the function $g(n)$, which has been just recalled, we are in the position to formulate new bounds for the Gutman's energy in Definition 1.

Lemma 3. *For $n \geqslant 3$, let $\mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(D_{2n})})$ be the energy of $\mathit{\Gamma}_{\mathrm{L}(D_{2n})}$, and let $g(n)$ denote the arithmetic function in* (15)*.*

(i) *If n is odd, then*

$$2\sqrt{\frac{\sigma(n)^2 - g(n)}{2}} \leqslant \mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(D_{2n})}) \leqslant \sigma(n)^2 - g(n).$$

(ii) *If $n = 2^\alpha$ and $\alpha \geqslant 3$, then*

$$\begin{aligned} &2\sqrt{\frac{(\alpha + 2^\alpha - 1)^2 - 2^{\alpha+2}(\alpha - 2) - 2^{\alpha+1}\alpha - (\alpha - 1)^2 - 8}{2}} \\ &\quad \leqslant \mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(D_{2n})}) \\ &\quad \leqslant \bigl(\alpha + 2^\alpha - 1\bigr)^2 - 2^{\alpha+2}(\alpha - 2) - 2^{\alpha+1}\alpha - (\alpha - 1)^2 - 8. \end{aligned}$$

(iii) *If $n = 2^\alpha n'$ and $\alpha \geqslant 1$ with n' odd, then*

$$\begin{aligned} &2\sqrt{\frac{\sigma(n)^2 - ((\alpha - 1)2^{\alpha+3} + 9)g(n')}{2}} \\ &\quad \leqslant \mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(D_{2n})}) \leqslant \sigma(n)^2 - \bigl((\alpha - 1)2^{\alpha+3} + 9\bigr)g(n'). \end{aligned}$$

Proof. The proof follows from Remark 2 and [20, Cor. 4.5]. □

3 Energy of graphs arising from dihedral groups

It is well known that the spectrum of the adjacency matrix of the complete graph K_m is $m - 1$ with multiplicity 1 and -1 with multiplicity $m - 1$; thus, its energy is

$$\mathcal{E}(K_m) = m - 1 + (m - 1) \cdot 1 = 2m - 2.$$

Now we will compute $\mathcal{E}(\Gamma_{\mathrm{L}(D_{2n})})$ for some n, and we will see that Lemma 3 provides an equality for the energetic bounds. The above expression $\mathcal{E}(K_m)$ will be useful in our computations.

Theorem 1. *For all $n \geqslant 3$, we have*

$$\mathcal{E}(\Gamma_{\mathrm{L}(D_{2n})}) = \begin{cases} 2\sigma(n) - 4 & \text{if } n \text{ is odd prime}, \\ 2\sigma(n) - 12 & \text{if } n = 2p \text{ for odd prime } p. \end{cases}$$

Proof. For all $n \geqslant 3$, we may consider the energy $\mathcal{E}(\Gamma_{\mathrm{L}(D_{2n})})$ with $m = |V(\Gamma_{\mathrm{L}(D_{2n})})|$. From [20, Lemma 4.1] we have

$$\left|V(\Gamma_{\mathrm{L}(D_{2n})})\right| = \begin{cases} \sigma(n) - 1 & \text{if } n \text{ is odd}, \\ \sigma(n) - 3 & \text{otherwise}. \end{cases}$$

Then we need to consider the following cases for n:

(i) *n is odd prime*. From [19, Cor. 2.3], for $n \geqslant 3$, $\Gamma_{\mathrm{L}(D_{2n})}$ is complete if and only if n is odd prime. This implies

$$\mathcal{E}(\Gamma_{\mathrm{L}(D_{2n})}) = 2m - 2 = 2\sigma(n) - 4.$$

(ii) *$n = 2p$ for odd prime p.* Then from [19, Lemma 2.2 and Cor. 4.1] $\Gamma_{\mathrm{L}(D_{2n})}$ is regular, and $A(\Gamma_{\mathrm{L}(D_{2n})})$ is of size

$$\bigl(\sigma(2p) - 3\bigr) \times \bigl(\sigma(2p) - 3\bigr).$$

Therefore the spectrum of $A(\Gamma_{\mathrm{L}(D_{2n})})$ is 0 with multiplicity n, $\sigma(n) - 6$ with multiplicity 1, and -3 with multiplicity $\sigma(n) - n - 4 = \sigma(p) - 4$. Hence

$$\begin{aligned} \mathcal{E}(\Gamma_{\mathrm{L}(D_{2n})}) &= |n \cdot 0| + \bigl|(\sigma(n) - 6)\bigr| + |-3|\bigl(\sigma(n) - n - 4\bigr) \\ &= 4\sigma(n) - 3n - 18 = 2\sigma(n) - 12, \end{aligned}$$

and the result follows completely. □

The situation changes drastically when we look into the case of n, which is a square of an odd prime, or eventually into the case of n, which is the product of two distinct primes. Here we will make use of [28, 29] for some computations.

Theorem 2. *Assume that $n \geqslant 3$.*

(i) *If $n = p^2$ for an odd prime p, then*

$$\begin{aligned} \mathcal{E}(\Gamma_{\mathrm{L}(D_{2n})}) &= \bigl(p^2 - p\bigr) + (p - 1)\bigl(|-1 - \sqrt{p}| + |-1 + \sqrt{p}|\bigr) \\ &\quad + \frac{p-1}{2}\bigl(\bigl|(p+2) - \sqrt{p(p+4)}\bigr| + \bigl|(p+2) + \sqrt{p(p+4)}\bigr|\bigr). \end{aligned}$$

(ii) *If* $n = 3q$ *for a prime* $q > 3$*, then*

$$\mathcal{E}(\Gamma_{\mathrm{L}(D_{2n})}) = 2q - 2 + \frac{q-1}{2}\left(|-1+\sqrt{13}| + |-1-\sqrt{13}|\right) + \left|-1+\sqrt{\sigma(3q)-3}\right| + \left|-1-\sqrt{\sigma(3q)-3}\right| + \left|\frac{3q-1}{2} + \sqrt{\left(\frac{3q-1}{2}\right)\left(\frac{7q-3}{2}\right) + \left(\frac{q+5}{2}\right)}\right| + \left|\frac{3q-1}{2} - \sqrt{\left(\frac{3q-1}{2}\right)\left(\frac{7q-3}{2}\right) + \left(\frac{q+5}{2}\right)}\right|.$$

(iii) *If* $n = 5q$*, then for a prime* $q > 5$*,*

$$\mathcal{E}(\Gamma_{\mathrm{L}(D_{2n})}) = 2q - 2 + \frac{q-1}{2}\left(|-1+\sqrt{21}| + |-1-\sqrt{21}|\right) + 7\left(\left|-1+\sqrt{\sigma(3q)-3}\right| + \left|-1-\sqrt{\sigma(3q)-3}\right|\right) + \left|\frac{5q-1}{2} + \sqrt{\left(\frac{5q-1}{2}\right)\left(\frac{9q+3}{2}\right) + \left(\frac{13-q}{2}\right)}\right| + \left|\frac{5q-1}{2} - \sqrt{\left(\frac{5q-1}{2}\right)\left(\frac{9q+3}{2}\right) + \left(\frac{13-q}{2}\right)}\right|.$$

(iv) *If* $n = pq$ *for two distinct primes* p *and* q *such that* $7 \leqslant p < q$*, then*

$$\mathcal{E}(\Gamma_{\mathrm{L}(D_{2n})}) = 2q - 2 + \frac{q-1}{2}\left(\left|-1+\sqrt{\sigma(3p)-3}\right| + \left|-1-\sqrt{\sigma(3p)-3}\right|\right) + \frac{p-1}{2}\left(\left|-1+\sqrt{\sigma(3q)-3}\right| + \left|-1-\sqrt{\sigma(3q)-3}\right|\right) + \left|\frac{pq-1}{2} + \sqrt{b}\right| + \left|\frac{pq-1}{2} - \sqrt{b}\right|,$$

where

$$b = \frac{1}{2}\left(\sigma(n)^2 - g(n) - \sum_{i=1}^{m-2} \lambda_i^2\right) - \frac{1}{4}(pq-1)^2.$$

Proof. For all $n \geqslant 3$, we consider $\mathcal{E}(\Gamma_{\mathrm{L}(D_{2n})})$ with $m = |V(\Gamma_{\mathrm{L}(D_{2n})})|$. From [20, Lemma 4.1] we have again $|V(\Gamma_{\mathrm{L}(D_{2n})})| = \sigma(n) - 1$ if n is odd. Now let us consider the following two cases for n:

(i) Assume that $n = p^\beta$ for $\beta \geqslant 2$ with p odd prime. We perform some computations using the matrix in [19, Cor. 4.1], which can be written

$$A(\Gamma_{\mathrm{L}(D_{2n})}) = \begin{pmatrix} A_1 & J & \cdots & J \\ J & A_2 & \cdots & J \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \cdots & A_p \end{pmatrix}_{(\sigma(p^\beta)-1)\times(\sigma(p^\beta)-1)},$$

where J is the all-ones matrix of size $(p+1)\times(p+1)$, $A_i = A(\varGamma_{\mathrm{L}(D_{2p^{\beta-1}})} \cup K_1)$ for all $i = 1,\dots,p$ with

$$A(\varGamma_{\mathrm{L}(D_{2p})}) = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}_{p\times p}.$$

For $n = p^2$, the spectrum of $A(\varGamma_{\mathrm{L}(D_{2n})})$ can be computed with [29] and gives -1 with multiplicity p^2-p, then $-1-\sqrt{p}$ with multiplicity $p-1$, then $-1+\sqrt{p}$ with multiplicity $p-1$, then $((p-1)/2)((p+2)-\sqrt{p(p+4)})$ with multiplicity 1, and finally, $((p-1)/2)\times((p+2)+\sqrt{p(p+4)})$ with multiplicity 1. By taking these values of the spectrum and by Definition 1 we obtain, as claimed, that $\mathcal{E}(\varGamma_{\mathrm{L}(D_{2n})})$ is equal to

$$\begin{aligned}&\big(p^2-p\big) + (p-1)\big(|-1-\sqrt{p}| + |-1+\sqrt{p}|\big)\\ &\quad + \frac{p-1}{2}\big(\big|(p+2)-\sqrt{p(p+4)}\big| + \big|(p+2)+\sqrt{p(p+4)}\big|\big).\end{aligned}$$

(ii) Let $n = pq$ for primes $3 \leqslant p < q$. Since $\varGamma_{\mathrm{L}(D_{2n})}$ is the complement of the graph $\bigcup_{i=1}^{p}\bigcup_{j=1}^{q} G_{ij}$, where for each $i = 1,2,\dots,p$, $j = 1,2,\dots,q$, G_{ij} is a complete graph with vertex set $\{u_i, v_j, w_{ij}\}$; see [10, Thm. 4.5] for details. Then we may use a similar logic as in (i) above and rewrite the matrix as follows [19, Cor. 4.1]:

$$A(\varGamma_{\mathrm{L}(D_{2n})}) = \begin{pmatrix} D & B \\ B^t & O \end{pmatrix}_{(\sigma(pq)-1)\times(\sigma(pq)-1)},$$

where

$$D = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}_{(pq)\times(pq)}, \quad O = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{(p+q)\times(p+q)},$$

and

$$B = \begin{pmatrix} W_1 & \cdots & T_1 \\ \vdots & \ddots & \vdots \\ W_q & \cdots & T_q \end{pmatrix}_{(\sigma(pq)-1)\times(p+q)}$$

with

$$W_1 = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 1 \end{pmatrix}_{p\times q}, \quad W_2 = \begin{pmatrix} 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 \end{pmatrix}_{p\times q}, \quad \dots,$$

$$W_q = \begin{pmatrix} 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}_{p\times q}, \quad \text{and} \quad T_i = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}_{p\times p}.$$

Now for the function $g(n)$ described in (15) and $m = \sigma(pq) - 1$, we have

$$\sigma(n)^2 - g(n) = \sum_{i=1}^{m} \lambda_i^2. \tag{16}$$

Then the spectrum of $A(\mathit{\Gamma}_{\mathrm{L}(D_{2n})})$ for $n = pq$ with $p \geqslant 5$ and $q \geqslant 7$ is

Type	Eigenvalues	Multiplicity
λ_1	0	1
λ_2	-1	$\frac{p-1}{2}(2q-2)$
λ_3^+	$\frac{1}{2}(-1+\sqrt{\sigma(3p)-3})$	$q-1$
λ_3^-	$\frac{1}{2}(-1-\sqrt{\sigma(3p)-3})$	$q-1$
λ_4^+	$\frac{1}{2}(-1+\sqrt{\sigma(3q)-3})$	$p-1$
λ_4^-	$\frac{1}{2}(-1-\sqrt{\sigma(3q)-3})$	$p-1$
λ_5^+	$\frac{pq-1}{2}+\sqrt{b}$	1
λ_5^-	$\frac{pq-1}{2}-\sqrt{b}$	1

Here b can be obtained using (16), which is given by

$$b = \frac{1}{2}\left(\sigma(n)^2 - g(n) - \sum_{i=1}^{m-2} \lambda_i^2\right) - \frac{1}{4}(pq-1)^2, \tag{17}$$

and the spectrum of $A(\mathit{\Gamma}_{\mathrm{L}(D_{2n})})$ for the smallest $p = 3$ and $n = 3q$ with $q \geqslant 5$ is

Type	Eigenvalues	Multiplicity
λ_1	0	1
λ_2	-1	$2q-2$
λ_3^+	$\frac{1}{2}(-1+\sqrt{13})$	$q-1$
λ_3^-	$\frac{1}{2}(-1-\sqrt{13})$	$q-1$
λ_4^+	$\frac{1}{2}(-1+\sqrt{\sigma(3q)-3})$	$p-1$
λ_4^-	$\frac{1}{2}(-1-\sqrt{\sigma(3q)-3})$	$p-1$
λ_5^+	$\frac{3q-1}{2}+\sqrt{(\frac{3q-1}{2})(\frac{7q-3}{2})+(\frac{q+5}{2})}$	1
λ_5^-	$\frac{3q-1}{2}-\sqrt{(\frac{3q-1}{2})(\frac{7q-3}{2})+(\frac{q+5}{2})}$	1

Consequently, point (ii) holds because for $p = 3$ and $q \geqslant 5$, we obtain

$$\begin{aligned}\mathcal{E}(\mathit{\Gamma}_{\mathrm{L}(D_{2n})}) = 2q - 2 &+ \frac{q-1}{2}\bigl(|-1+\sqrt{13}| + |-1-\sqrt{13}|\bigr) \\ &+ \left|-1+\sqrt{\sigma(3q)-3}\right| + \left|-1-\sqrt{\sigma(3q)-3}\right| \\ &+ \left|\frac{3q-1}{2} + \sqrt{\left(\frac{3q-1}{2}\right)\left(\frac{7q-3}{2}\right) + \left(\frac{q+5}{2}\right)}\right| \\ &+ \left|\frac{3q-1}{2} - \sqrt{\left(\frac{3q-1}{2}\right)\left(\frac{7q-3}{2}\right) + \left(\frac{q+5}{2}\right)}\right|.\end{aligned}$$

(iii) If $p = 5$ and $q \geqslant 7$, then we may consider exactly the same argument, which is used in (ii) above, but now

$$b = \left(\frac{5q-1}{2}\right)\left(\frac{9q+3}{2}\right) + \left(\frac{13-q}{2}\right).$$

This implies

$$\begin{aligned}\mathcal{E}(\Gamma_{\mathrm{L}(D_{2n})}) &= 2q - 2 + \frac{q-1}{2}\bigl(\bigl|-1+\sqrt{21}\bigr| + \bigl|-1-\sqrt{21}\bigr|\bigr) \\ &\quad + 7\bigl(\bigl|-1+\sqrt{\sigma(3q)-3}\bigr| + \bigl|-1-\sqrt{\sigma(3q)-3}\bigr|\bigr) \\ &\quad + \left|\frac{5q-1}{2} + \sqrt{\left(\frac{5q-1}{2}\right)\left(\frac{9q+3}{2}\right) + \left(\frac{13-q}{2}\right)}\right| \\ &\quad + \left|\frac{5q-1}{2} - \sqrt{\left(\frac{5q-1}{2}\right)\left(\frac{9q+3}{2}\right) + \left(\frac{13-q}{2}\right)}\right|.\end{aligned}$$

Then we get (iii).

(iv) The same method applies for $p \geqslant 7$ and $q > 7$ and b in (17). That means

$$\begin{aligned}\mathcal{E}(\Gamma_{\mathrm{L}(D_{2n})}) &= 2q - 2 + \frac{q-1}{2}\bigl(\bigl|-1+\sqrt{\sigma(3p)-3}\bigr| + \bigl|-1-\sqrt{\sigma(3p)-3}\bigr|\bigr) \\ &\quad + \frac{p-1}{2}\bigl(\bigl|-1+\sqrt{\sigma(3q)-3}\bigr| + \bigl|-1-\sqrt{\sigma(3q)-3}\bigr|\bigr) \\ &\quad + \left|\frac{pq-1}{2} + \sqrt{b}\right| + \left|\frac{pq-1}{2} - \sqrt{b}\right|.\end{aligned}$$

Hence the result is proved completely. □

Remark 6. In order to compute $\mathcal{E}(\Gamma_{\mathrm{L}(D_{2n})})$ for $n = 2^\alpha$ with $\alpha \geqslant 3$ and $n = p^\beta$ with odd prime and $\beta \geqslant 3$, we found many computational difficulties. Note that the results can be derived also from [28, 29], but explicit combinatorial formulas as per Theorems 1 and 2 are not currently available.

From [25, 26] we know that the extraspecial p-group $E(p^3)$ of order p^3 and exponent p (with p odd prime) is a p-group whose center $Z(E(p^3)) = [E(p^3), E(p^3)] = \Phi(E(p^3))$ has order p and is equal to the commutator subgroup $[E(p^3), E(p^3)]$ and to the Frattini subgroup $\Phi(E(p^3))$ as well. Moreover, the central quotient $E(p^3)/Z(E(p^3))$ turns out to be a p-elementary abelian group. Among these groups, notable examples include Heisenberg p-groups H_p over the field with p-elements (p odd). In particular, we may consider the Heisenberg 3-group

$$H_3 = \bigl\langle a, b, c \bigm| a^3 = b^3 = c^3 = 1,\ [a, c] = [b, c] = 1,\ [a, b] = c \bigr\rangle,$$

which is a metabelian nonmodular nilpotent group by Lemma 1. Later on, we will apply the following result to determine whether a p-group is modular or not.

Corollary 2. *Let* $\mathcal{E}(\Gamma_{\mathrm{L}(E(p^3))})$ *be the energy of* $\Gamma_{\mathrm{L}(E(p^3))}$*. Then*

$$\mathcal{E}(\Gamma_{\mathrm{L}(E(p^3))}) = 2p^2.$$

Proof. It is well known that $E(p^3)$ has

$$\frac{p^3-1}{p-1} = p^2+p+1$$

cyclic subgroups, and it is easy to check that p^2+p of them are nonnormal of order p and only one is normal cyclic and isomorphic to C_p. In addition, it is very clear what is the sublattice of the normal subgroups in $E(p^3)$; we have just the trivial subgroup, $E(p^3)$, and $p+1$ normal subgroups, which are isomorphic to C_{p^2}. Hence $|\mathrm{L}(E(p^3))| = p^2+2p+4$. Note that among the p^2+p nonnormal subgroups, each p of them, together with the normal cyclic subgroup, forms a subgroup of order p^2, which means that these p subgroups are mutually commuting, but do not commute with the remaining p^2, which are indeed nonnormal. Hence $\Gamma_{\mathrm{L}(E(p^3))}$ is regular with $|V(\Gamma_{\mathrm{L}(E(p^3))})| = p^2+p$, and we may apply Theorem 1(ii), getting

$$A(\Gamma_{\mathrm{L}(E(p^3))}) = \begin{pmatrix} O & J & \cdots & J \\ J & O & \cdots & J \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \cdots & O \end{pmatrix}$$

of size $(p^2+p)\times(p^2+p)$, where O is the $p\times p$ zero matrix, and J is the $p\times p$ all-ones matrix. Therefore the spectrum of $A(\Gamma_{\mathrm{L}(E(p^3))})$ is 0 with multiplicity p^2-1, p^2 with multiplicity 1, and $-p$ with multiplicity p. Hence

$$\mathcal{E}(\Gamma_{\mathrm{L}(E(p^3))}) = \bigl(p^2-1\bigr)\cdot 0 + p^2 + p|-p| = 2p^2. \qquad \square$$

Example 1. One can check that $\operatorname{spec}(A(\Gamma_{\mathrm{L}(D_6)})) = \{-1,-1,2\}$. This implies that $\mathcal{E}(\Gamma_{\mathrm{L}(D_6)}) = 4$ and $\operatorname{sd}(D_6) = 5/6$. Using Proposition 1, it is easy to check that

$$\frac{7}{9} < \operatorname{sd}(D_6) < \frac{8}{9}.$$

If we consider D_8, then $\operatorname{spec}(A(\Gamma_{\mathrm{L}(D_8)})) = \{-1,0,1,2\}$, $\mathcal{E}(\Gamma_{\mathrm{L}(D_8)}) = 4$, hence

$$\frac{92}{100} \leqslant \operatorname{sd}(D_8) < \frac{96}{100}.$$

For $p=3$, by applying Corollary 2 we get $\mathcal{E}(\Gamma_{\mathrm{L}(E(p^3))}) = 2(3)^2 = 18$, but

$$\operatorname{spec}(A(\Gamma_{\mathrm{L}(E(p^3))})) = \{0,0,0,0,0,0,0,0,9,-3,-3,-3\},$$

hence $\mathcal{E}(\Gamma_{\mathrm{L}(E(p^3))}) = 18$. Furthermore, this implies

$$\frac{199}{361} < \operatorname{sd}\bigl(E(p^3)\bigr) = \frac{253}{361} < \frac{343}{361}.$$

Note that Proposition 1 puts strong restrictions on the upper and lower bounds for the subgroup commutativity degree, but we can actually say more:

Corollary 3. *There are no nonquasihamiltonian groups G with*

$$\mathrm{sd}(G) \in \left[1 - \frac{\mathcal{E}(\Gamma_{\mathrm{L}(G)})}{|\mathrm{L}(G)|^2}, 1\right].$$

Remark 7. The complement of $\Gamma_{\mathrm{L}(G)}$ may be the empty graph of size n, namely the graph with n vertices and zero edges. For instance, the complement of $\Gamma_{\mathrm{L}(D_6)}$ is isomorphic to the permutability graph of nonnormal subgroups of group D_6, $\Gamma_N(D_6)$, which is disconnected and has size 3; see [20, Exa. 2.9] and [2, 9, 10].

4 Energetic bounds for sd(G)

We begin this section with some facts, which can be found in [27, Prop. 2.4]. If N is a normal subgroup of a group G, then the following inequality holds:

$$\begin{aligned}|\mathrm{L}(G)|^2\,\mathrm{sd}(G) \geqslant \left(|\mathrm{L}(N)| + \left|\mathrm{L}\left(\frac{G}{N}\right)\right| - 1\right)^2 + \left(\mathrm{sd}(N) - 1\right)|\mathrm{L}(N)|^2 \\ + \left(\mathrm{sd}\left(\frac{G}{N}\right) - 1\right)\left|\mathrm{L}\left(\frac{G}{N}\right)\right|^2. \qquad (18)\end{aligned}$$

Recall that a metabelian group is a group whose commutator subgroup is abelian. Equivalently, a group G is metabelian if and only if there is an abelian normal subgroup N such that the quotient group G/N is abelian. Thus, if G is metabelian, then $\mathrm{sd}(N) = \mathrm{sd}(G/N) = 1$, and (18) reduces to

$$\mathrm{sd}(G) \geqslant \left(\frac{|\mathrm{L}(N)| + |\mathrm{L}(\frac{G}{N})| - 1}{|\mathrm{L}(G)|}\right)^2.$$

This means that if G is a metabelian quasihamiltonian group, then the bound above is trivial, and so the bound becomes really significant only if G is a metabelian non-quasihamiltonian group and N is a proper nontrivial normal subgroup. For instance, this happens in the following situation.

Remark 8. If $G = D_6$ and $N \simeq C_3$, then we have

$$\left(\frac{|\mathrm{L}(N)| + |\mathrm{L}(\frac{D_6}{N})| - 1}{|\mathrm{L}(D_6)|}\right)^2 = \frac{1}{2} < \frac{7}{9} = 1 - \frac{[\mathcal{E}(\Gamma_{\mathrm{L}(D_6)})]^2}{2|\mathrm{L}(D_6)|^2}.$$

This means that

$$1 - \frac{[\mathcal{E}(\Gamma_{\mathrm{L}(D_6)})]^2}{2|\mathrm{L}(D_6)|^2}$$

is a better lower bound for $\mathrm{sd}(D_6)$. Moreover, if $G = D_8$ and $N \simeq C_4$, we have

$$\left(\frac{|\mathrm{L}(N)| + |\mathrm{L}(\frac{D_8}{N})| - 1}{|\mathrm{L}(D_8)|}\right)^2 = \frac{1}{50} < \frac{92}{100} = 1 - \frac{[\mathcal{E}(\Gamma_{\mathrm{L}(D_8)})]^2}{2|\mathrm{L}(D_8)|^2}.$$

We can generalize the previous observations via the following result:

Corollary 4. *Assume that G is a metabelian nonquasihamiltonian group and $\mathcal{E}(\Gamma_{\mathrm{L}(G)})$ is the energy of the corresponding $\Gamma_{\mathrm{L}(G)}$. Then*

$$\left(\frac{|\mathrm{L}(N)| + |\mathrm{L}(G/N)| - 1}{|\mathrm{L}(G)|}\right)^2 \leqslant 1 - \frac{[\mathcal{E}(\Gamma_{\mathrm{L}(G)})]^2}{2|\mathrm{L}(G)|^2}.$$

Next, assume that the normal subgroup N in (18) is of prime index. Then $|\mathrm{L}(G/N)|=2$ and $\mathrm{sd}(G/N) = 1$. This implies

$$\mathrm{sd}(G) \geqslant \frac{1}{|\mathrm{L}(G)|^2}\big(\mathrm{sd}(N)\big|\mathrm{L}(N)\big|^2 + 2\big|\mathrm{L}(N)\big| + 1\big). \tag{19}$$

Moreover, if G is solvable and $\{1\} = G_0 \subset G_1 \subset \cdots \subset G_k = G$ is a series of subgroups of G such that all factor groups G_i/G_{i-1} are cyclic of prime order (see [25] concerning the main properties of solvable groups), then using (19), we have

$$\mathrm{sd}(G_i)\big|\mathrm{L}(G_i)\big|^2 \geqslant \mathrm{sd}(G_{i-1})\big|\mathrm{L}(G_{i-1})\big|^2 + 2\big|\mathrm{L}(G_{i-1})\big| + 1.$$

Therefore, as shown in [27, Cor. 2.7], we have in this situation that

$$\mathrm{sd}(G) \geqslant \frac{1}{|\mathrm{L}(G)|^2}\left(2\sum_{i=1}^{k}\big|\mathrm{L}(G_{i-1})\big| + k + 1\right). \tag{20}$$

In addition, if G is a p-group of order p^k, which has a cyclic maximal subgroup, then one can further assume that all G_{i-1} are cyclic, and one obtains that a p-group of order p^k, which has a cyclic maximal subgroup, should necessarily satisfy the lower bound

$$\mathrm{sd}(G) \geqslant \left(\frac{k+1}{|\mathrm{L}(G)|}\right)^2. \tag{21}$$

Details of the argument can be found in [27, Cor. 2.8]. However, the lower bounds of $\mathrm{sd}(G)$ in (19), (20), and (21) are all less than $1 - [\mathcal{E}(\Gamma_{\mathrm{L}(G)})]^2/(2|\mathrm{L}(G)|^2)$ with respect to the same assumption for a nonquasihamiltonian group G.

Therefore we may prove the following result, which is a special case for the lower bound of $\mathrm{sd}(G)$ given in Proposition 1. This bound is given in terms of the energy of $\Gamma_{\mathrm{L}(G)}$ and of the determinant of $A(\Gamma_{\mathrm{L}(G)})$ whenever all the eigenvalues are nonzero. For example, this is what we note in the case of dihedral groups.

Theorem 3. *Let G be a nonquasihamiltonian group, and $\mathcal{E}(\Gamma_{\mathrm{L}(G)})$ denote the energy of $\Gamma_{\mathrm{L}(G)}$. Then*

$$1 - \frac{1}{|\mathrm{L}(G)|^2}\big(\mathcal{E}(\Gamma_{\mathrm{L}(G)})^2 - m(m-1)\,\big|\det\big(A(\Gamma_{\mathrm{L}(G)})\big)\big|^{2/m}\big) \leqslant \mathrm{sd}(G),$$

where $m = |V(\Gamma_{\mathrm{L}(G)})|$, and $\det(A(\Gamma_{\mathrm{L}(G)}))$ is the determinant of $A(\Gamma_{\mathrm{L}(G)})$.

Proof. Assume that $\operatorname{spec}(A(\Gamma_{\mathrm{L}(G)})) = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$ with $m = |V(\Gamma_{\mathrm{L}(G)}))|$ and $\det(A(\Gamma_{\mathrm{L}(G)})) \neq 0$. Of course, if $\det(A(\Gamma_{\mathrm{L}(G)})) = 0$, then this bound is the same as with the lower bound for $\operatorname{sd}(G)$ in Proposition 1. Now from (14) we have

$$1 - \operatorname{sd}(G) = \frac{1}{|\mathrm{L}(G)|^2}\biggl(\mathcal{E}(\Gamma_{\mathrm{L}(G)})^2 - \sum_{j\neq i} |\lambda_j||\lambda_i|\biggr).$$

Since the geometric mean of positive numbers is not greater than their arithmetic mean (see [14, p. 79]), we have

$$\begin{aligned}\frac{1}{m(m-1)}\sum_{j\neq i}|\lambda_j||\lambda_i| &\geqslant \prod_{j\neq i}\bigl(|\lambda_j\|\lambda_i|\bigr)^{1/(m(m-1))} \\ &= \prod_{j=1}^{m}\bigl(|\lambda_j\bigr)^{2/m} = \bigl|\det\bigl(A(\Gamma_{\mathrm{L}(G)})\bigr)\bigr|^{2/m}.\end{aligned}$$

This implies

$$\sum_{j\neq i}|\lambda_j||\lambda_i| \geqslant m(m-1)\bigl|\det\bigl(A(\Gamma_{\mathrm{L}(G)})\bigr)\bigr|^{2/m}$$

from which the result follows. □

The upper bound for $\operatorname{sd}(G)$, which is shown below, may achieve the equality for some groups. This indicates that it is a better upper bound for $\operatorname{sd}(G)$ than the bound stated in Proposition 1.

Theorem 4. *Let G be a nonquasihamiltonian group, and let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m$ denote the ordered eigenvalues of the spectrum of $A(\Gamma_{\mathrm{L}(G)})$. Then*

$$\operatorname{sd}(G) \leqslant 1 - \biggl(\frac{(\mathcal{E}(\Gamma_{\mathrm{L}(G)}) - \lambda_1)^2}{(m-1)|\mathrm{L}(G)|^2} + \frac{\lambda_1^2}{|\mathrm{L}(G)|^2}\biggr).$$

Proof. Assume that G is a nonquasihamiltonian group and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m$ are given. Then from [14, Thm. 5.3] we have

$$\mathcal{E}(\Gamma_{\mathrm{L}(G)}) \leqslant \lambda_1^2 + \sqrt{(m-1)\bigl(\bigl|E(\Gamma_{\mathrm{L}(G)})\bigr| - \lambda_1^2\bigr)}.$$

Therefore, using [19, Lemma 2.5], we obtain

$$\frac{(\mathcal{E}(\Gamma_{\mathrm{L}(G)}) - \lambda_1)^2}{m-1} - \lambda_1^2 \leqslant 2\bigl|E\bigl(\Gamma_{\mathrm{L}(G)}\bigr)\bigr| = \bigl|\mathrm{L}(G)\bigr|^2\bigl(1 - \operatorname{sd}(G)\bigr).$$

Hence the result follows. □

There was a naive conjecture [13], stating that, among all graphs of order m, the complete graph K_m has the maximal energy. But this conjecture was soon shown to be false by Godsil in the 1980s. He presented graphs, called hyperenergetic, whose energy exceeds $\mathcal{E}(K_n)$; for details, see [14, Chap. 8]. For example, $\Gamma_{\mathrm{L}(D_6)}$ is hyperenergetic, and we have the following result for the class of dihedral groups.

Corollary 5. *Let $\mathcal{E}(\Gamma_{\mathrm{L}(D_{2n})})$ be the energy $\Gamma_{\mathrm{L}(D_{2n})}$. Then $\Gamma_{\mathrm{L}(D_{2n})}$ is hyperenergetic if and only if n is odd prime.*

Proof. For any $n \geqslant 3$, $\Gamma_{\mathrm{L}(D_{2n})}$ is complete if and only if n is odd prime; see [19, Cor. 2.3]. Hence the result follows from Theorem 1. □

The previous corollary does not apply when D_{2n} is a 2-group for n power of 2. However, it can help in the following situation:

Corollary 6. *Let $\mathcal{E}(\Gamma_{\mathrm{L}(D_{2^\alpha})})$ for some integer $\alpha \geqslant 3$ be the energy of $\Gamma_{\mathrm{L}(D_{2^\alpha})}$. Then $\Gamma_{\mathrm{L}(D_{2^\alpha})}$ is never hyperenergetic.*

Proof. This follows from Corollary 5 and the structural properties of D_{2n}. □

5 Two new criteria of nilpotence

In order to end with the final results of the present paragraph, we must recall from [26] that a section of a group G is a group of the form H/N, where H is a subgroup of G, and N is a normal subgroup of H. This notion plays a fundamental role in lattice theory, as illustrated in several parts of the monograph of Schmidt [26]. If S is a subgroup of G for which we have the corresponding nonpermutability graph of subgroups $\Gamma_{\mathrm{L}(S)}$ and $\Gamma_{\mathrm{L}(G)}$, then we may consider the optimal quantities

$$\left(\frac{\mathcal{E}(\Gamma_{\mathrm{L}(S)})^2 - 2\sum_{i<j}^m |\lambda_i||\lambda_j|}{|\mathrm{L}(S)|^2}\right)^* = \max\left\{\frac{\mathcal{E}(\Gamma_{\mathrm{L}(S)})^2 - 2\sum_{i<j}^m |\lambda_i||\lambda_j|}{|\mathrm{L}(S)|^2} \;\middle|\; S \text{ is a section of } G\right\}.$$

These turn out to provide a new criterion for detecting nilpotence in arbitrary groups, and this is probably the first criterion where the notion of energy is used in connection with the notion of subgroup commutativity degree.

Theorem 5. *Let G be a p-group with a section S such that $\Gamma_{\mathrm{L}(S)}$ is a subgraph of $\Gamma_{\mathrm{L}(G)}$, and let*

$$\left(\frac{\mathcal{E}(\Gamma_{\mathrm{L}(S)})^2 - 2\sum_{i<j}^m |\lambda_i||\lambda_j|}{|\mathrm{L}(S)|^2}\right)^* < \frac{54}{361}.$$

Then G is a modular nilpotent group.

Proof. Clearly, G is nilpotent as it is a p-group. Assume that G is nonmodular. Then using Lemma 1, we have that G contains a section S isomorphic to D_8 if $p = 2$ or to $E(p^3)$ for $p > 2$. Consider the subgraph $\Gamma_{\mathrm{L}(S)}$ of $\Gamma_{\mathrm{L}(G)}$ corresponding to $S \simeq D_8$ or $S \simeq E(p^3)$. From Corollary 2 we have

$$\frac{1}{|\mathrm{L}(E(p^3))|^2}\left(\mathcal{E}(\Gamma_{\mathrm{L}(E(p^3))})^2 - 2\sum_{i<j}^m |\lambda_i||\lambda_j|\right) = \frac{4p^4}{(p^2+2p+4)^2} - \frac{(3p^4-p^3)}{(p^2+2p+4)^2} = \frac{p^4-p^3}{(p^2+2p+4)^2}.$$

Now, for all odd prime $p \geqslant 5$, we have

$$\frac{1}{|\mathrm{L}(E(p^3))|^2}\left(\mathcal{E}(\Gamma_{\mathrm{L}(E(p^3))})^2 - 2\sum_{i<j}^{m}|\lambda_i||\lambda_j|\right) > \frac{54}{361},$$

and if $p = 3$,

$$\frac{1}{|\mathrm{L}(E(p^3))|^2}\left(\mathcal{E}(\Gamma_{\mathrm{L}(E(p^3))})^2 - 2\sum_{i<j}^{m}|\lambda_i||\lambda_j|\right) = \frac{54}{361}.$$

Furthermore, from Example 1 we have that

$$\frac{1}{|\mathrm{L}(D_8)|^2}\left(\mathcal{E}(\Gamma_{\mathrm{L}(D_8)})^2 - 2\sum_{i<j}^{m}|\lambda_i||\lambda_j|\right) = \frac{4}{25} > \frac{54}{361}$$

implies that

$$\frac{1}{|\mathrm{L}(S)|^2}\left(\mathcal{E}(\Gamma_{\mathrm{L}(S)})^2 - 2\sum_{i<j}^{m}|\lambda_i||\lambda_j|\right) \geqslant \frac{54}{361},$$

but it contradicts our assumptions. Hence the result follows. □

By taking the poset $\mathrm{L}_1(G)$ of cyclic subgroups of G, instead of $\mathrm{L}(G)$, we may sharpen the previous optimal quantity, considering

$$\left(\frac{\mathcal{E}(\Gamma_{\mathrm{L}_1(S)})^2 - 2\sum_{i<j}^{m}|\lambda_i||\lambda_j|}{|\mathrm{L}_1(S)|^2}\right)^* = \max\left\{\frac{\mathcal{E}(\Gamma_{\mathrm{L}_1(S)})^2 - 2\sum_{i<j}^{m}|\lambda_i||\lambda_j|}{|\mathrm{L}_1(S)|^2} \;\middle|\; S \text{ is a section of } G\right\}.$$

Theorem 6. *Let G be a nonquasihamiltonian group containing a section S of G such that $\Gamma_{\mathrm{L}_1(S)}$ subgraph of $\Gamma_{\mathrm{L}(G)}$, and let*

$$\left(\frac{\mathcal{E}(\Gamma_{\mathrm{L}_1(S)})^2 - 2\sum_{i<j}^{m}|\lambda_i||\lambda_j|}{|\mathrm{L}_1(S)|^2}\right)^* < \frac{6}{25}.$$

Then G is nilpotent.

Proof. Assume that G is nonquasihamiltonian and $\Gamma_{\mathrm{L}_1(S)}$ is subgraph of $\Gamma_{\mathrm{L}(G)}$. We will show by induction on $|G|$ that if G is nonnilpotent, then

$$\frac{1}{|L_1(S)|^2}\left(\mathcal{E}(\Gamma_{\mathrm{L}_1(S)})^2 - 2\sum_{i<j}^{m}|\lambda_i||\lambda_j|\right) \geqslant \frac{6}{25}.$$

This means that there is a section S of G such that the above inequality is satisfied. For $|G| = 6$, we just have to consider $G \simeq D_6$ and $\Gamma_{\mathrm{L}_1(D_6)} = \Gamma_{\mathrm{L}(D_6)}$, so this leads to the conclusion $S = G$. Assume that this is true for all nonnilpotent groups with order $< |G|$. We consider now the two cases below.

Case 1: G has a proper nonnilpotent subgroup H. Then, by the inductive hypothesis, H has a section S with

$$\frac{1}{|L_1(S)|^2}\left(\mathcal{E}(\varGamma_{\mathrm{L}_1(S)})^2 - 2\sum_{i<j}^{m}|\lambda_i||\lambda_j|\right) \geqslant \frac{6}{25}.$$

Thus, we are done since S is likewise a section of G.

Case 2: All proper subgroups of G are nilpotent. Then G is a minimal nonnilpotent group, and from [25, pp. 247–258] we have that G is a solvable group of order $p^m q^n$ (where p and q are different primes) with a unique Sylow p-subgroup P and a cyclic Sylow q-subgroup Q such that G is a semidirect product of P by Q. In addition, we have that if $Q = \langle y\rangle$, then $y^q \in Z(G)$, $Z(G) = \varPhi(G) = \varPhi(P) \times \langle y^q\rangle$, $G' = P$, $P' = (G')' = \varPhi(P)$, and $|P/P'| = p^r$, where r is the order of p modulo q. If P is abelian, then P is an elementary abelian p-group of order p^r, and P is a minimal normal subgroup of G; and if P is nonabelian, then $Z(P) = P' = \varPhi(P)$ and $|P/Z(P)| = p^r$. Here we are able to take $S = G/Z(G)$ as it is also a nonquasihamiltonian and a minimal nonnilpotent group of order $p^r q$, which can be written as a semidirect product of an elementary abelian p-group P_1 of order p^r by a cyclic group Q_1 of order q. Note that $\mathrm{L}_1(S)$ contains the trivial subgroup 1 of the subgroups of order p in P_1 and of the subgroups of type $\langle yx\rangle \in Q_1 = \langle x\rangle$ with $y \in P_1$ and $x \in S$; see [17]. This means $L_1(S) = L_1(P_1) \cup \{Q_1^x \mid x \in S\}$ and

$$\left|L_1(S)\right| = \frac{p^r - 1}{p - 1} + 1 + p^r = \frac{p^{r+1} + p - 2}{p - 1}.$$

Furthermore, it is easy to check that the normal subgroups of S are all the subgroups in P_1 together with S and any subgroup of type $\langle y_i x\rangle \in Q_1$ does not commute with $\langle y_j x\rangle \in Q_1$ if $i \neq j$ for all $i, j = 1, 2, \ldots, p^r$. Hence $\varGamma_{\mathrm{L}_1(S)}$ is a complete graph with $m = |V(\varGamma_{\mathrm{L}_1(S)})| = p^r$, and its spectrum is $p^r - 1$ with multiplicity 1 and -1 with multiplicity $p^r - 1$. Therefore by applying Lemma 2, for all $r \geqslant 1$ and prime $p \geqslant 3$, we obtain

$$\begin{aligned}\frac{1}{|L_1(S)|^2}&\left(\mathcal{E}(\varGamma_{\mathrm{L}_1(S)})^2 - 2\sum_{i<j}^{m}|\lambda_i||\lambda_j|\right)\\ &= \frac{(2p^r - 2)^2 - (3p^r - 7p^r + 4)}{(\frac{p^{r+1}+p-2}{p-1})^2} = \frac{(p^{2r} - p^r)(p-1)^2}{(p^{r+1} + p - 2)^2} \geqslant \frac{6}{25},\end{aligned}$$

contradicting the assumption. Hence the result follows. □

6 On the discrete nonlinear analysis on $\varGamma_{\mathrm{L}(G)}$

We want to end our paper with some possible lines of future research, arising from the considerations and the results, which we have seen until here. We briefly adapt a few notions of Chung et al. [5–7], as recently described for the case of noncommuting graphs of finite groups in [23]. The underlying idea is that more general methods could be developed when geometric measure theory and nonlinear analysis are involved in the graph to be studied.

For the nonpermutability graph of subgroups $\Gamma_{\mathrm{L}(G)}$ of G in (2), we may consider a subset $\Omega \subseteq V(\Gamma_{\mathrm{L}(G)})$ and its boundary

$$\partial\Omega = \bigl\{XY \bigm| X \in \Omega \text{ and } Y \in V(\Gamma_{\mathrm{L}(G)}) - \Omega\bigr\}.$$

Since the graph is unweighted (and unoriented), we may associate a *unitary edge weight* $1 = \sigma_{XY} > 0$ for each edge $XY \in E(\Gamma_{\mathrm{L}(G)})$, and so, for any $S \subseteq E(\Gamma_{\mathrm{L}(G)})$, we may look at

$$\sigma(S) = \sum_{XY \in S} \sigma_{XY} \quad \text{with } \sigma_{XY} = 1,$$

extending $\sigma_{XY} = 0$ to those X and Y, which are not adjacent. This gives a symmetric function from $V(\Gamma_{\mathrm{L}(G)}) \times V(\Gamma_{\mathrm{L}(G)})$ to $(0, +\infty)$ with discrete nonzero values. In fact, (6) can be interpreted as the positive discrete measure

$$\mu_X = \deg(X) = \bigl|\{YX \mid Y \sim X\}\bigr|,$$

which counts the number of neighbors of the vertex X, and after all, it is not so different from $\sigma(S)$, which is localized at S. There is consequently a global positive discrete measure

$$\mu : \Omega \subseteq V(\Gamma_{\mathrm{L}(G)}) \longmapsto \mu(\Omega) = \sum_{X \in \Omega} \mu_X \in]0, +\infty[.$$

In particular,

$$\mu(\Omega) = \sum_{X \in \Omega} \deg(X).$$

Following [5, 7, 23], it is possible to define the *gradient operator* of $\Gamma_{\mathrm{L}(G)}$:

$$\nabla : f \in \mathbb{R}^{V(\Gamma_{\mathrm{L}(G)}) \times V(\Gamma_{\mathrm{L}(G)})} \longmapsto \nabla f = \nabla_{XY} f = f(Y) - f(X) \in \mathbb{R},$$

where $\mathbb{R}^{V(\Gamma_{\mathrm{L}(G)}) \times V(\Gamma_{\mathrm{L}(G)})}$ is the set of all functions from $V(\Gamma_{\mathrm{L}(G)}) \times V(\Gamma_{\mathrm{L}(G)})$ to $\mathbb{R}$, and the symbol ∇_{XY} emphasizes that there is a dependence on $X, Y \in V(\Gamma_{\mathrm{L}(G)})$ in the definition of ∇. Consequently,

$$\Delta : f \in \mathbb{R}^{V(\Gamma_{\mathrm{L}(G)})} \longmapsto \Delta f(X) = \frac{1}{\mu_X} \sum_{Y : Y \sim X} \nabla_{XY} f \in \mathbb{R}$$

is the *Laplace operator* of $\Gamma_{\mathrm{L}(G)}$. A variation of the Green's formula shows that

$$\sum_{X \in \Omega} \Delta f(X) \mu_X = \sum_{\substack{X \in \Omega \\ Y \in V(\Gamma_{\mathrm{L}(G)}) - \Omega}} \nabla_{XY} f = \sum_{Z \in \partial\Omega} \nabla_Z f,$$

and for all $f, g \in \mathbb{R}^{V(\mathit{\Gamma}_{\mathrm{L}(G)})}$, we find that

$$\sum_{X \in V(\mathit{\Gamma}_{\mathrm{L}(G)})} \Delta f(X) g(X) \mu_X = -\frac{1}{2} \sum_{X,Y \in V(\mathit{\Gamma}_{\mathrm{L}(G)})} (\nabla_{XY} f)(\nabla_{XY} g)$$
$$= -\sum_{Z \in E(\mathit{\Gamma}_{\mathrm{L}(G)})} (\nabla_Z f)(\nabla_Z g).$$

The above property of the Laplace operators is typical from the nonstandard analysis and of the geometric measure theory on compact manifolds, but Chung et al. [5–7] showed that similar notions are meaningful for graphs.

The *graph distance* $\rho_\xi(X)$ between $X \in V(\mathit{\Gamma}_{\mathrm{L}(G)})$ and the fixed vertex ξ in $V(\mathit{\Gamma}_{\mathrm{L}(G)})$ is the number of edges in a shortest path connecting them, and so

$$\rho : (\xi, X) \in V(\mathit{\Gamma}_{\mathrm{L}(G)}) \times V(\mathit{\Gamma}_{\mathrm{L}(G)}) \longmapsto \rho(\xi, X) = \rho_\xi(X) \in \mathbb{N},$$

where

$$\rho_\xi : X \in V(\mathit{\Gamma}_{\mathrm{L}(G)}) \longmapsto \rho_\xi(X) \in \mathbb{N},$$

is the *geodesic distance*. Moreover, ρ_ξ allows us to consider balls, and so we have

$$B_\xi(r) = \bigl\{X \in V(\mathit{\Gamma}_{\mathrm{L}(G)}) \bigm| \rho_\xi(X) < r\bigr\}$$

as in every metric space for a given $r > 0$. Since $\mathit{\Gamma}_{\mathrm{L}(G)}$ is unweighted, we have

$$\mu_X^\xi = \sum_{\substack{Y:\, Y \sim X \\ \rho_\xi(Y) < \rho_\xi(X)}} \sigma_{XY} \quad \text{with } \sigma_{XY} = 1,$$

which allows us to introduce what is known in geometric analysis and optimization theory [5, 7] as the *relative isoperimetric dimension* of $\mathit{\Gamma}_{\mathrm{L}(G)}$, namely

$$\nu_r = \inf\left\{ \frac{\mu_X}{\mu_X^\xi} \;\middle|\; \xi \in V(\mathit{\Gamma}_{\mathrm{L}(G)}),\ X \in B_\xi(r) \right\}.$$

The $P(\delta, \iota, R_0)$ *property* (or *Chung–Grigoryan–Yau property*) works for any finite weighted (or unweighted) graph, and in our context, it may be formulated as:

(i) $|\nabla_{XY} \rho_\xi| \leqslant 1$ for any $\xi, X, Y \in V(\mathit{\Gamma}_{\mathrm{L}(G)})$;

(ii) there exist a function $q_\xi(X)$ and the constants $\iota \geqslant 1$ and $\delta, R_0 > 0$ such that:
- $q_\xi(X) \geqslant 0$ for all $X \in V(\mathit{\Gamma}_{\mathrm{L}(G)})$, and $q_\xi(X) = 0$ if and only if $X = \xi$;
- $|\nabla_{XY} q_\xi| \leqslant \rho_\xi(X) + \iota$ for all $\xi \in V(\mathit{\Gamma}_{\mathrm{L}(G)})$ and $X, Y \in B_\xi(R_0)$;
- $\Delta q_\xi(X) \geqslant \delta$ for all $\xi \in V$ and $X \in B_\xi(R_0)$;

(iii) $n = \delta \;\nu_{R_0+1} \geqslant 1$.

The presence of a discrete isoperimetric inequality can be deduced from the Chung–Grigoryan–Yau property. In recent years, several works in graph theory have focused on this topic.

Open problem. We know from [7, Thm. 6.3] that a weighted (or unweighted) graph $(\mathit{\Gamma}, \sigma)$ with $P(\delta, \iota, R_0)$ satisfies the isoperimetric inequality

$$\sigma(\partial \mathit{\Omega}) \geqslant c\, \mu(\mathit{\Omega})^{1-1/n}, \tag{22}$$

where

$$c = \frac{\omega' \omega^{1/(n-1)}}{4^{n+3} \nu_{R_0+1} \iota \mathrm{e}^{2n}}, \qquad \omega = \inf\bigl\{\mu_X \bigm| X \in V(\mathit{\Gamma}_{\mathrm{L}(G)})\bigr\},$$

and

$$\omega' = \inf\bigl\{\sigma_{XY} \bigm| X \sim Y,\ X, Y \in V(\mathit{\Gamma}_{\mathrm{L}(G)})\bigr\}.$$

Note that (12) shows that $|E(\mathit{\Gamma}_{\mathrm{L}(G)})|$ is proportional to $\mathrm{sd}(G)$ and this is a fundamental relation, which allows us to find bounds for $\mathrm{sd}(G)$ of energetic nature. Specifically for $\mathit{\Gamma}_{\mathrm{L}(G)}$, we have the isoperimetric inequality (22), but we do not know how this is related to the bounds of spectral nature, which we found in Theorems 1–4. In fact, the same relation (12) might allow us to find isoperimetric inequalities for $\mathrm{sd}(G)$ via the Chung–Grigoryan–Yau property. We have not explored this area yet since the techniques of the spectral analysis on graphs are quite peculiar, but we think it is worth spending more time on it.

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