Nonlinear Analysis: Modelling and Control, Online First, 1–19
https://doi.org/10.15388/namc.2026.31.44762

Observer-based distributed optimization for linear multiagent systems with external disturbances*

Xuening Xu[a], **Zhiyong Yu**[a], **Haijun Jiang**[a,b]

[a]College of Mathematics and System Science,
Xinjiang University,
Urumqi 830017, China
yzygsts@163.com

[b]School of Mathematical Sciences,
Xinjiang Normal University,
Urumqi 830017, China

Received: April 29, 2025 / **Revised:** October 1, 2025 / **Published online:** January 7, 2026

Abstract. This paper addresses the distributed optimization problem in linear multiagent systems (MASs) under external disturbances. Firstly, an observation system is designed by utilizing the output values of agents, which can eliminate external disturbances of system. Secondly, an event-triggered control algorithm is proposed through the gradient information of local cost functions, and its convergence is rigorously established using the Lyapunov stability and looped functional theory. This novel event-triggered protocol incorporates dwell time within the threshold function, effectively eliminating Zeno behavior. By leveraging the looped functional technique, more relaxed conditions are derived for solving the distributed optimization problem. Finally, the validity and feasibility of the proposed protocol are substantiated through numerical simulation.

Keywords: external disturbances, multiagent system, distributed optimization, looped functional method.

1 Introduction

The distributed optimization of MASs has aroused considerable attention in the recent decade due to its broad applications, including but not limited to smart microgrids [26], economic dispatch [24], and so on. The primary goal of solving the distributed optimization problems is to develop appropriate algorithms that ensure the global objective function is minimized. The distributed control methods, leveraging the entirety of available

*This work was supported in part by the National Natural Science Foundation of China (grant Nos. 62363033 and 62163035), in part by the Natural Science Foundation of Xinjiang Uygur Autonomous Region (grant No. 2023D01C162), in part by Tianshan Talent Training Program, (grant Nos. 2022TSYCLJ0004 and 2023TSYCCX0102), and in part by the Excellent Doctor Innovation Program of Xinjiang University (grant No. XJU2024BS042).

computational power, are more practical compared to conventional centralized control techniques.

Research on distributed optimization algorithms can be broadly categorized into discrete and continuous forms. Discrete distributed optimization algorithms have been presented in [7, 10, 17]. However, in practical applications such as autonomous underwater vehicles, UAVs, and manipulators, the dynamic behaviors of these systems are continuous. Consequently, lots of continuous distributed optimization algorithms have been proposed in [5, 16, 21]. It is noteworthy that these algorithms primarily address single-integrator dynamical systems. Given that many practical dynamical models cannot be accurately described by integral systems, general linear dynamical systems have gained significant attention in distributed cooperative control.

Recently, distributed optimization in linear MASs has yielded promising results. For example, the gradient descent method has been employed to solve unconstrained optimization problems in [3, 4]. Additionally, the Lagrange multiplier method has been utilized to address the distributed optimization problems for linear systems subject to equality constraints in [19, 28]. However, when designing distributed optimization algorithms for MASs, practical challenges such as external disturbances, communication delays, packet loss, and other unforeseen faults may arise. Consequently, various studies [11, 12, 15, 25] have examined distributed optimization problems under various real-world scenarios.

It is important to note that the above research primarily focuses on continuous communication among MASs for information transmission, which can result in unnecessary communication overhead. To address this issue, the event-triggered mechanism [8, 9, 27] has proven to be an effective approach for reducing communication costs, and it has garnered significant attention from researchers. To solve the distributed optimization problem, very recently, some distributed event-triggered optimization algorithms have been developed. In [18], a novel event-triggered mechanism was proposed to address the consensus problem in MASs, and it introduced a dwell time to effectively prevent Zeno behavior. The distributed algorithm was designed for resource allocation problems via event-triggered communication [13]. Specifically, the event-triggered zero-gradient-sum distributed optimization algorithms were introduced to tackle various convex distributed optimization problems in [22, 30].

Motivated by the above analysis, the goal of this paper is to design the distributed optimization algorithm for large-scale linear MASs with external disturbances via the event-triggered mechanism. The following difficulties and challenges may be encountered: 1. Develop a estimator to eliminate external disturbances. Meanwhile, a reasonable control protocol is established based on this observation value. 2. Design a suitable event-triggered control protocol such that (a) the states of agents realize the consensus under the external disturbances; (b) after MASs achieve the consensus, the state of MASs is able to achieve the optimal solution. Our main contributions can be summarized in three key aspects.

(i) Unlike previous works [14, 29], we introduce a distributed observer based on the system's state outputs, enabling us to design an effective distributed control protocol using these observations. This approach is particularly valuable when the system's state is not readily available. Therefore, our proposed distributed control protocol can be viewed as a generalization of existing results.

(ii) In [23, 31], the authors delved into continuous distributed optimization problems for single- or double-integral dynamical MASs. In contrast, this paper extends the scope by considering the distributed optimization problem for general linear MASs. An event-triggered control mechanism is developed to optimize communication costs, and the dwell time T is introduced into the event trigger condition to effectively exclude Zeno behavior.

(iii) A looped functional method is introduced to analyze the stability of linear MASs, effectively overcoming the limitations of Lyapunov–Krasovskii functional. This approach eliminates the need for derivatives of Lyapunov functions to be strictly negative, indicating that stability can be achieved under less conservative conditions by employing the looped functional method.

The subsequent sections of this study are delineated as follows. Section 2 provides the fundamental preliminaries. Our primary results are presented in Section 3. The feasibility is verified by applying an example in Section 4. Finally, we give the conclusions and outline potential future directions.

Notations. Let $\mathbb{R}^n$ and $\mathbb{N}$ denote the sets of n-dimensional vectors and natural numbers, respectively. The notation $\mathbf{1}_n$ (or $\mathbf{0}_n$) represents $n \times 1$ column vector with all elements set to 1 (or 0), and I_n signifies n-dimensional identity matrix. For any matrix $\mathcal{A}$, $\mathcal{A}^{\mathrm{T}}$ represents its transpose, and $\otimes$ is the Kronecker product. If square matrix $\mathcal{A} > 0$, then $\mathcal{A}$ is a positive definite matrix, otherwise, $\mathcal{A}$ is a seminegative definite matrix. $\mathcal{A}^{-1}$ denotes its inverse, and $\mathrm{sym}(\mathcal{A}) = \mathcal{A} + \mathcal{A}^{\mathrm{T}}$. $*$ indicates a symmetric term in a symmetric matrix. The smallest and largest eigenvalues of matrix $\mathcal{A}$ are represented by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively. For vectors $x_1, x_2, \dots, x_n$, $\mathrm{col}(x_1, x_2, \dots, x_n) = [{x_1}^{\mathrm{T}}, {x_2}^{\mathrm{T}}, \dots, {x_n}^{\mathrm{T}}]^{\mathrm{T}}$. Let $\|\cdot\|$ denotes the 2-norm of a matrix, and the gradient of a function f is indicated by ∇f.

2 Preliminaries

Consider a network graph $G = (V, \aleph, A)$ with N agents, where $V = \{1, 2, \dots, N\}$ and $\aleph \subseteq V \times V$ represent the nonempty node set and edge set, respectively. The weighted adjacency matrix is denoted by $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ with $a_{ii} = 0$, $a_{ij} > 0$ if and only if $(j, i) \in \aleph$, otherwise, $a_{ij} = 0$. A path between nodes i and j is a vertex sequence of different edges that connect distinct nodes from i to j. The neighbor set of node i is defined as $N_i = \{j \in V \mid (j, i) \in \aleph\}$. The graph is undirected and connected if there exists a path between any two nodes. Furthermore, the Laplacian matrix is denoted as $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$, where $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ii} = -a_{ij}$ for $i \neq j$.

Consider a linear MAS governed by N agents with the following dynamics:

$$\begin{aligned} \dot{x}_i(t) &= \mathscr{A} x_i(t) + \mathscr{B} u_i(t) + \mathscr{D} d_i(t), \\ y_i(t) &= \mathscr{C} x_i(t), \quad i = 1, 2, \dots, N, \end{aligned} \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ and $y_i(t) \in \mathbb{R}^r$ denote the state and output of the ith agent, $u_i(t) \in \mathbb{R}^m$ represents the control input. $\mathscr{A} \in \mathbb{R}^{n\times n}$, $\mathscr{B} \in \mathbb{R}^{n\times m}$, $\mathscr{C} \in \mathbb{R}^{r\times n}$, and $\mathscr{D} \in \mathbb{R}^{n\times s}$ are constant matrices in which $(\mathscr{A}, \mathscr{B})$ is stabilizable, $(\mathscr{D}, \mathscr{S})$ and $(\mathscr{A}, \mathscr{C})$ are observable. Suppose that the disturbance $d_i(t) \in \mathbb{R}^s$ is generated by an external system ($\dot{d}_i(t) = \mathscr{S} d_i(t)$, $\mathscr{S} \in \mathbb{R}^{s\times s}$) and is matched, then there exists a matrix $\mathscr{F} \in \mathbb{R}^{m\times s}$ satisfying $\mathscr{D} = \mathscr{B}\mathscr{F}$.

Furthermore, we present a distributed optimization problem as follows:

$$\min \mathcal{F}(x) = \sum_{i=1}^{N} f_i(x_i) \quad \text{s.t. } (\mathcal{L} \otimes I_n)x = 0, \tag{2}$$

where $x = \operatorname{col}(x_1, x_2, \ldots, x_N)$, $f_i(x_i)$ and $\mathcal{F}(x)$ are local cost function of agent i and global objective function, respectively.

Remark 1. In [31] and [23], the authors considered the distributed optimization problem of the simple first-order integrator system, highlighting its limitations in practical applications. From the perspective of the system model, this paper addresses the distributed optimization problem of linear systems subject to external disturbances. Furthermore, in contrast to traditional pure optimization problems, this study explores optimization issues in MASs using a distributed strategy, significantly improving robustness and applicability.

The aim of this study is to devise an appropriate controller for linear MAS (1), ensuring that all agents' states converge to the optimal solution of the distributed optimization problem in (2). To achieve this goal, specific assumptions are required concerning the management of linear MAS (1) and their communication topology.

Assumption 1. The topological graph G is undirected and connected.

Assumption 2. For each agent $i \in \mathcal{V}$, the local cost function $f_i(\cdot)$ is differentiable and ω_i-strongly convex with $\omega_i > 0$. Furthermore, the gradient $\nabla f_i(\cdot)$ satisfies m_i-Lipschitz with constant $m_i > 0$, i.e., $\|\nabla f_i(x) - \nabla f_i(y)\| \leqslant m_i \|x - y\|$ for all $x, y \in \mathbb{R}^n$.

Assumption 3. The eigenvalues of the matrix $\mathscr{S}$ have negative real parts.

Remark 2. Assumptions 2 and 3 are commonly utilized in studies such as [14] and [31]. Assumption 2 guarantees the existence and uniqueness of the optimal solution to the distributed optimization problem (2). Assumption 3 is often applied to disturbance rejection and output regulation. While practical applications involve both matched and unmatched external disturbances, this study focuses exclusively on matched disturbances. Future work will explore more general disturbance scenarios such as plant-model mismatch, transient disturbances, and so on.

Lemma 1. (*See* [6]). *If the matrix pair* $(\mathscr{S}, \mathscr{D})$ *is observable, then* $(\bar{\mathscr{A}}, J)$ *is observable, where* $\bar{\mathscr{A}} = \left(\begin{smallmatrix} \mathscr{A} - \mathscr{G}\mathscr{C} & \mathscr{D} \\ 0 & \mathscr{S} \end{smallmatrix}\right)$ *and* $J = [I \ \mathbf{0}]$. *Further, there exists a matrix* $\mathscr{P} > 0$ *with appropriate dimension such that*

$$P\bar{\mathscr{A}} + \bar{\mathscr{A}}^{\mathrm{T}} P - 2J^{\mathrm{T}} J < 0.$$

Lemma 2. (*See* [2]). *For a matrix* $R = \begin{pmatrix} R_1 & R_2 \\ R_2^{\mathrm{T}} & R_3 \end{pmatrix} < 0$, *the following conditions are equivalent*:

(i) $R_3 < 0,\ R_1 - R_2 R_3^{-1} R_2^{\mathrm{T}} < 0;$
(ii) $R_1 < 0,\ R_3 - R_2^{\mathrm{T}} R_1^{-1} R_2 < 0.$

Lemma 3. (*See* [1].) *Consider the differential equation as follows*:

$$\dot{z} = h(z, t), \tag{3}$$

where $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *is a nonlinear function, and* $z = 0$ *is an equilibrium point of Eq.* (3). *If there exists a continuous radially unbounded function* $V : \mathbb{R}^n \to \mathbb{R}$, *which satisfies* $V(z) = 0 \Leftrightarrow z = 0$ *and* $\dot{V}(z) < 0$ *in* $\mathbb{R} \setminus \{0\}$, *then* $z = 0$ *is asymptotically stable.*

3 Main results

In this section, we establish an event-triggered algorithm with a sequence of triggered instants given by $0 = t_0^i < t_1^i < t_2^i < \cdots < t_k^i < \cdots$, where $\lim_{k\to\infty} t_k^i = +\infty$. The controller of agent i will be activated only at prescribed instants t_k^i, $k \in \mathbb{N}$, which are determined by predefined conditions. To proceed, an observation system is reconstructed for the linear MAS (1) based on the event-triggered instant

$$\begin{aligned}
\dot{\hat{x}}_i(t) &= \mathscr{A}\hat{x}_i(t) + \mathscr{B}u_i(t) + \mathscr{D}\hat{d}_i(t) + \mathscr{G}\big(y_i(t) - \hat{y}_i(t)\big) \\
&\quad + b\Gamma_1\mathscr{G}\sum_{j=1}^N a_{ij}\big(y_i\big(t_k^i\big) - y_j\big(t_k^i\big) - \hat{y}_i\big(t_k^i\big) + \hat{y}_j\big(t_k^i\big)\big), \\
\dot{\hat{d}}_i(t) &= \mathscr{S}\hat{d}_i(t) + b\Gamma_d\mathscr{G}\sum_{j=1}^N a_{ij}\big(y_i\big(t_k^i\big) - y_j\big(t_k^i\big) - \hat{y}_i\big(t_k^i\big) + \hat{y}_j\big(t_k^i\big)\big), \\
\dot{\hat{y}}_i(t) &= \mathscr{C}\hat{x}_i(t),
\end{aligned} \tag{4}$$

where $\hat{x}_i(t) \in \mathbb{R}^n$, $\hat{y}_i(t) \in \mathbb{R}^r$, and $\hat{d}_i(t) \in \mathbb{R}^s$ represent the observations of $x_i(t)$, $y_i(t)$, and $d_i(t)$, respectively. $y_i(t_k^i)$, $\hat{y}_i(t_k^i)$ represent the output state, and output observation state at the kth trigger instant of agent i, respectively. Here b is a positive real constant, and $\mathscr{G} \in \mathbb{R}^{n\times r}$ is an undetermined feedback matrix. The control protocol $u_i(t)$ is designed as follows:

$$\begin{aligned}
u_i(t) &= -\alpha\mathscr{B}^{\mathrm{T}}\nabla f_i\big(\hat{x}_i(t)\big) - \mathscr{K}\hat{x}_i(t) - \mathscr{F}\hat{d}_i(t) \\
&\quad - \Pi\sum_{j=1}^N a_{ij}\big[\hat{y}_j\big(t_k^i\big) - \hat{y}_i\big(t_k^i\big)\big] \quad \forall t \in \big[t_k^i, t_{k+1}^i\big),
\end{aligned} \tag{5}$$

where $\alpha > 0$ is constant. Π is a feedback matrix, and $\mathscr{K}$ is a constant gain matrix with suitable dimension such that $\mathscr{A} - \mathscr{B}\mathscr{K}$ is Hurwitz stable. Let $e_i^1(t) = x_i(t) - \hat{x}_i(t)$, $e_i^2(t) = d_i(t) - \hat{d}_i(t)$, $e_i(t) = \mathrm{col}(e_i^1(t), e_i^2(t))$, $i = 1, 2, \ldots, N$. In combination with (1), (4), and (5), one gets

$$\begin{aligned}
\dot{x}_i(t) &= (\mathscr{A} - \mathscr{B}\mathscr{K})x_i(t) + \mathscr{B}[K, F]e_i(t) + \mathscr{B}\Pi\mathscr{C}\sum_{j=1}^{N} l_{ij}\hat{x}_j\big(t_k^i\big) \\
&\quad - \alpha\mathscr{B}\mathscr{B}^{\mathrm{T}}\nabla f_i\big(\hat{x}_i(t)\big), \\
\dot{\hat{x}}_i(t) &= (\mathscr{A} - \mathscr{B}\mathscr{K})\hat{x}_i(t) + \mathscr{B}\Pi\mathscr{C}\sum_{j=1}^{N} l_{ij}\hat{x}_j(t_k^i) + \mathscr{G}\mathscr{C}Je_i(t) \\
&\quad - \alpha\mathscr{B}\mathscr{B}^{\mathrm{T}}\nabla f_i\big(\hat{x}_i(t)\big) + b\Gamma_1\mathscr{G}\mathscr{C}J\sum_{j=1}^{N} l_{ij}e_j\big(t_k^i\big), \\
\dot{e}_i(t) &= \bar{A}e_i(t) + b\Gamma\mathscr{G}\mathscr{C}J\sum_{j=1}^{N} l_{ij}e_j\big(t_k^i\big),
\end{aligned}$$

where $\Gamma = \mathrm{col}(\Gamma_1, \Gamma_d) = P^{-1}J^{\mathrm{T}}$. Furthermore, an orthogonal matrix transformation and an error difference variable transformation are carried out, respectively. We construct error variables as follows:

$$\begin{aligned}
\tilde{x}_i(t) &= x_i(t) - \frac{1}{N}\sum_{j=1}^{N} x_j(t), \qquad \tilde{\hat{x}}_i(t) = \hat{x}_i(t) - \frac{1}{N}\sum_{j=1}^{N} \hat{x}_j(t), \\
\tilde{\eta}_i(t) &= \eta_i(t) - \frac{1}{N}\sum_{j=1}^{N} \eta_j(t), \qquad \nabla\tilde{f}_i\big(\tilde{\hat{x}}_i(t)\big) = \nabla f_i\big(\hat{x}_i(t)\big) - \frac{1}{N}\sum_{j=1}^{N} \nabla f_j\big(\hat{x}_j(t)\big).
\end{aligned} \tag{6}$$

It is noted that the compact forms of (6) are

$$\begin{gathered}
\tilde{x}(t) = (\mathscr{H} \otimes I_n)x(t), \qquad \tilde{\hat{x}}(t) = (\mathscr{H} \otimes I_n)\hat{x}(t), \qquad \tilde{E}(t) = (\mathscr{H} \otimes I_{n+s})E(t), \\
\nabla\tilde{f}\big(\tilde{\hat{x}}(t)\big) = (\mathscr{H} \otimes I_n)\nabla f\big(\hat{x}(t)\big),
\end{gathered}$$

where $\mathscr{H} = I_N - (1/N)\mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}$ with $\mathcal{L}\mathscr{H} = \mathscr{H}\mathcal{L} = \mathcal{L}$, and $\mathscr{H}\mathbf{1}_N = 0$. Combining these compact forms yields

$$\begin{aligned}
\dot{\bar{\hat{x}}}(t) &= \big(I_N \otimes (\mathscr{A} - \mathscr{B}\mathscr{K})\big)\bar{\hat{x}}(t) + (\Xi \otimes \mathscr{B}\Pi\mathscr{C})\bar{\hat{x}}(t_k) + (I_N \otimes \mathscr{G}\mathscr{C}J)\bar{e}(t) \\
&\quad - \alpha\big(I_N \otimes \mathscr{B}\mathscr{B}^{\mathrm{T}}\big)\nabla\bar{f}\big(\bar{\hat{x}}(t)\big) + b(\Xi \otimes \Gamma_1\mathscr{G}\mathscr{C}J)\bar{e}(t_k), \\
\dot{\bar{e}}(t) &= (I_N \otimes \bar{\mathscr{A}})\bar{e}(t) + b(\Xi \otimes \Gamma\mathscr{G}\mathscr{C}J)\bar{e}(t_k),
\end{aligned} \tag{7}$$

where $\bar{\hat{x}}(t_k) = \mathrm{col}(\bar{\hat{x}}_1(t_k^1), \bar{\hat{x}}_2(t_k^2), \dots, \bar{\hat{x}}_N(t_k^N))$ and $\bar{e}(t_k) = \mathrm{col}(\bar{e}_1(t_k^1), \bar{e}_2(t_k^2), \dots, \bar{e}_N(t_k^N))$. Define the vector $z(t) = \mathrm{col}(\bar{\hat{x}}(t), \bar{e}(t)) \in \mathbb{R}^{N(2n+s)}$, then system (7) can be represented as following closed-loop system:

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}z(t_k) - \alpha B_f\nabla\bar{f}\big(Hz(t)\big), \tag{8}$$

where

$$\mathcal{A} = \begin{pmatrix} I_N \otimes (\mathscr{A} - \mathscr{B}\mathscr{K}) & I_N \otimes \mathscr{G}\mathscr{C}J \\ 0 & I_N \otimes \bar{\mathscr{A}} \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \varXi \otimes \mathscr{B}\varPi\mathscr{C} & b\varXi \otimes \varGamma_1\mathscr{G}\mathscr{C}J \\ 0 & b\varXi \otimes \varGamma\mathscr{G}\mathscr{C}J \end{pmatrix},$$

$$B_f = \begin{pmatrix} I_N \otimes \mathscr{B}\mathscr{B}^{\mathrm{T}} \\ 0 \end{pmatrix}, \quad \text{and} \quad H = [I_{Nn}\ 0].$$

Suppose that $\zeta_i(t) = z_i(t_k^i) - z_i(t)$ is observation error. Then the following generic threshold function is introduced to determine the sampling instants:

$$t_{k+1}^i = \min\bigl\{t \geqslant t_k^i + T \bigm| \zeta_i^{\mathrm{T}}(t)Q_\zeta\zeta_i(t) - y_{ai}^{\mathrm{T}}(t)Q_\epsilon^{-1}y_{ai}(t) \geqslant 0\bigr\}, \tag{9}$$

where T is a dwell time, $y_{ai}(t) = \mathrm{col}(\bar{\tilde{x}}_i(t), \bar{e}_{yi}(t))$ with $\bar{e}_{yi}(t) = y_i(t) - \hat{y}_i(t)$ denotes the output error, Q_ζ and Q_ϵ are positive definite matrices.

Remark 3. In [20], the MAS was also taken into consideration, but it does not reduce the communication cost between agents. As observed from the term $\varPi\sum_{j=1}^N a_{ij}[\hat{y}_j(t_k^i) - \hat{y}_i(t_k^i)]$ in protocol (5), the communication frequency has been reduced due to the controller only updating when the event occurs. Hence, this research can be regarded as an improvement of previous works. Furthermore, it can be observed from the event-trigger condition (9) that each agent must wait for time units before making the threshold judgment. As a result, the event-triggered mechanism designed in this paper inherently prevents Zeno behavior. It is worth noting that due to the introduction of the dwell time T, the stability analysis of the system must be conducted in two intervals $[t_k^i,\, t_k^i + T)$ and $[t_k^i + T,\, t_{k+1}^i)$.

To ensure a strict decrease in the overall variance of the Lyapunov function during the dwell time, a looped functional technique is taken into consideration. From a Lyapunov perspective the basic idea consists in considering a Lyapunov candidate function $\mathscr{V}(x)$ that satisfies the following conditions:

(i) $\mathscr{V}(x(t_k + T)) < \mathscr{V}(x(t_k))$,
(ii) $\dot{\mathscr{V}}(x(t)) < 0$ for all $t \in [t_k + T, t_{k+1})$.

Remark 4. Noteworthy, the application of the looped functional approach has proven successful in the stability analysis of linear MASs, as documented in [20]. Additionally, condition (i) does not necessitate strict monotonicity of that $\mathscr{V}(x(t))$ within the interval $[t_k,\, t_k + T)$. In other words, it is not required that $\dot{\mathscr{V}}(x(t)) < 0$ hold throughout this interval. To meet condition (i), the control signal must remain constant during the interval $[t_k,\, t_k + T)$.

Lemma 4. (*See* [18].) *For* $0 < \mu_1 < \mu_2$ *and* $1 \leqslant p$, $V(x(t)) : \mathbb{R}^n \to \mathbb{R}^+$ *is a differentiable function and satisfies*

$$\mu_1\bigl|x(t)\bigr|^p \leqslant V\bigl(x(t)\bigr) \leqslant \mu_2\bigl|x(t)\bigr|^p \quad \forall x(t) \in \mathbb{R}^n.$$

Then the following two statements are equivalent:

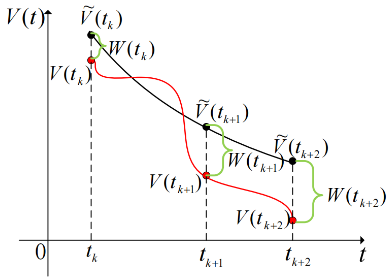

Figure 1. The illustration of Lemma 4.

(i) *For all* $k \in \mathbb{N}$*, the Lyapunov function exhibits a strictly decreasing trend with each increment, i.e.,*

$$\Delta V = V(t_{k+1}) - V(t_k) < 0.$$

(ii) *For all* $t \in [t_k, t_{k+1})$*, there exists a differentiable function* $W(t)$ *that fulfills the condition*

$$\Delta W = W(t_{k+1}) - W(t_k) \geqslant 0,$$

and it satisfies

$$\dot{\tilde{\mathscr{V}}}(t) = \dot{W}(t) + \dot{V}(t) < 0.$$

Lemma 4 is clearly illustrated in Fig. 1. As shown in the figure, the established Lyapunov function does not require strict monotonicity, it only needs to satisfy specific conditions at the sampling time. Compared to the traditional Lyapunov–Krasovskii function, this approach effectively relaxes the sufficient conditions for system stability, making it more suited to practical applications.

Theorem 1. *Under Assumptions* 1–3*, if there exist positive definite matrices* Q_ζ*,* Q_ϵ*,* Q*, and* R*, matrices* Q_1*,* Q_2*,* X_1*,* X_2*,* Y_1*,* X_{1a}*, and* X_{2a} *with appropriate dimensions to satisfy the following inequalities*

$$\varSigma_1 + T(\varSigma_2 + \varSigma_3) < 0, \tag{10}$$

$$\begin{bmatrix} \varSigma_1 - T\varSigma_3 & TN \\ * & -TR \end{bmatrix} < 0, \qquad \begin{bmatrix} \varSigma_a & \varSigma_b^{\mathrm{T}} \\ * & -Q_\epsilon \end{bmatrix} < 0, \tag{11}$$

where

$$\begin{aligned}
\varSigma_1 &= \mathrm{sym}\bigl\{\iota_1^{\mathrm{T}} Q \iota_1 - N\iota_{12} - \iota_{12}^{\mathrm{T}} Q_2 \iota_2 \\
&\quad + \bigl(\iota_1^{\mathrm{T}} X_1^{\mathrm{T}} + \iota_3^{\mathrm{T}} X_2^{\mathrm{T}}\bigr)\bigl[\mathcal{A}\iota_1 + \mathcal{B}\iota_2 - \iota_3 - \alpha B_f H^{\mathrm{T}} H \iota_4\bigr]\bigr\} - \iota_{12}^{\mathrm{T}} Q_1 \iota_{12}, \\
\varSigma_2 &= \mathrm{sym}\bigl\{\iota_3^{\mathrm{T}} Q_1 \iota_{12} + \iota_3^{\mathrm{T}} Q_2 \iota_2\bigr\} + \iota_3^{\mathrm{T}} Y_1 \iota_3, \qquad \varSigma_3 = \iota_2^{\mathrm{T}} Y \iota_2, \\
\varSigma_a &= \mathrm{sym}\bigl\{\iota_1^{\mathrm{T}} Q \iota_2 + \bigl(\iota_1^{\mathrm{T}} X_{1a}^{\mathrm{T}} + \iota_2^{\mathrm{T}} X_{2a}^{\mathrm{T}}\bigr)\bigl[(\mathcal{A} + \mathcal{B})\iota_1 - \iota_2 + \mathcal{B}\iota_3 - \alpha B_f H^{\mathrm{T}} \iota_4\bigr] \\
&\quad + \hat{m}\iota_1^{\mathrm{T}} H^{\mathrm{T}} \iota_4 - \iota_4^{\mathrm{T}} \iota_4\bigr\} - \iota_3^{\mathrm{T}} Q_\zeta \iota_3, \\
\varSigma_b &= C_a \iota_1,
\end{aligned}$$

$\iota_1 = [I_{N(2n+s)}\ 0\ 0\ 0]$, $\iota_2 = [0\ I_{N(2n+s)}\ 0\ 0]$, $\iota_3 = [0\ 0\ I_{N(2n+s)}]$, $\iota_4 = [0\ 0\ 0\ I_{Nn}]$, *and* $\iota_{12} = \iota_1 - \iota_2$, *then the linear MAS* (1) *with controller* (5) *can converge to the optimal solution of problem* (2)*.*

Proof. Firstly, the stability of system (8) is discussed in the dwell time interval $[t_k, t_k+T)$. We choose the Lyapunov function $V(z(t)) = z^{\mathrm{T}}(t)Qz(t)$ with $Q > 0$ and define $\mathscr{X}_k(\tau(t)) = z(t_k + \tau(t)) = z(t)$ with $\tau(t) \in [0, T)$. Then

$$\dot{\mathscr{X}}_k\big(\tau(t)\big) = \mathcal{A}\mathscr{X}_k\big(\tau(t)\big) + \mathcal{B}\mathscr{X}_k(0) - \alpha B_f \nabla \bar{f}\big(H\mathscr{X}_k\big(\tau(t)\big)\big). \tag{12}$$

Furthermore, we have

$$V\big(z(t)\big) = V\big(\mathscr{X}_k\big(\tau(t)\big)\big) = \mathscr{X}_k^{\mathrm{T}}\big(\tau(t)\big)Q\mathscr{X}_k\big(\tau(t)\big), \quad \tau(t) \in [0, T).$$

Inspired by [18], we design the following looped functional:

$$\begin{aligned} &W\big(\tau(t), \mathscr{X}_k\big) \\ &\quad = \big(T - \tau(t)\big)\big(\mathscr{X}_k\big(\tau(t)\big) - \mathscr{X}_k(0)\big)^{\mathrm{T}}\big[Q_1\big(\mathscr{X}_k\big(\tau(t)\big) - \mathscr{X}_k(0)\big) + 2Q_2\mathscr{X}_k(0)\big] \\ &\qquad + \big(T - \tau(t)\big)\tau(t)\mathscr{X}_k^{\mathrm{T}}(0)Y\mathscr{X}_k(0) - \big(T - \tau(t)\big)\int_0^{\tau(t)} \dot{\mathscr{X}}_k^{\mathrm{T}}(s)Y_1\dot{\mathscr{X}}_k(s)\,\mathrm{d}s. \end{aligned}$$

It can be founded that $W(0, \mathscr{X}_k) = W(T, \mathscr{X}_k) = 0$. Now, we define $\tilde{V}(\tau(t), \mathscr{X}_k) = V(\mathscr{X}_k(\tau(t))) + W(\tau(t), \mathscr{X}_k)$, and it is easy to conclude that

$$\begin{aligned} &\dot{\tilde{V}}\big(\tau(t), \mathscr{X}_k\big) \\ &\quad = \big(T - \tau(t)\big)\dot{\mathscr{X}}_k^{\mathrm{T}}\big(\tau(t)\big)\big[Y_1\dot{\mathscr{X}}_k\big(\tau(t)\big) + 2Q_1\big(\mathscr{X}_k\big(\tau(t)\big) - \mathscr{X}_k(0)\big) + 2Q_2\mathscr{X}_k(0)\big] \\ &\qquad + 2\mathscr{X}_k^{\mathrm{T}}\big(\tau(t)\big)Q\dot{\mathscr{X}}_k\big(\tau(t)\big) - \int_0^{\tau(t)} \dot{\mathscr{X}}_k^{\mathrm{T}}(s)Y_1\dot{\mathscr{X}}_k(s)\,\mathrm{d}s \\ &\qquad - \big(\mathscr{X}_k\big(\tau(t)\big) - \mathscr{X}_k(0)\big)^{\mathrm{T}}\big[Q_1\big(\mathscr{X}_k\big(\tau(t)\big) - \mathscr{X}_k(0)\big) + 2Q_2\mathscr{X}_k(0)\big] \\ &\qquad + \big(T - 2\tau(t)\big)\mathscr{X}_k^{\mathrm{T}}(0)Y\mathscr{X}_k(0). \end{aligned} \tag{13}$$

To show that $\dot{\tilde{V}}(\tau(t), \mathscr{X}_k) < 0$, the augmented vector $\varsigma_k(\tau(t)) = \mathrm{col}(\mathscr{X}_k(\tau(t)), \mathscr{X}_k(0), \dot{\mathscr{X}}_k(\tau(t)), \nabla\bar{f}(\tau(t)))$ is constructed in which $\nabla\bar{f}(\tau(t)) = \nabla\bar{f}(H\mathscr{X}_k(\tau(t)))$ for any matrix with appropriate dimensions X_1 and X_2. The following zero equation can be generated through the coupling relation between the components of $\varsigma_k(\tau(t))$ and Eq. (12):

$$2\big(\mathscr{X}_k^{\mathrm{T}}\big(\tau(t)\big)X_1^{\mathrm{T}} + \dot{\mathscr{X}}_k^{\mathrm{T}}\big(\tau(t)\big)X_2^{\mathrm{T}}\big)X_0\varsigma_k\big(\tau(t)\big) = 0, \tag{14}$$

where $X_0 = [\mathcal{A}\ \mathcal{B}\ -I_{N(2n+s)}\ -\alpha B_f]$. Subsequently, incorporating Eq. (14) into (13), we obtain

$$\begin{aligned}
&\dot{\tilde{V}}(\tau(t), \mathscr{X}_k) \\
&\quad = 2\mathscr{X}_k^{\mathrm{T}}\big(\tau(t)\big)Q\dot{\mathscr{X}}_k\big(\tau(t)\big) + \big(T-\tau(t)\big)\dot{\mathscr{X}}_k^{\mathrm{T}}\big(\tau(t)\big) \\
&\qquad \times \big[Y_1\dot{\mathscr{X}}_k(\tau(t)\big) + 2Q_1\big(\mathscr{X}_k\big(\tau(t)\big) - \mathscr{X}_k(0)\big) + 2Q_2\mathscr{X}_k(0)\big] \\
&\qquad - \big(\mathscr{X}_k\big(\tau(t)\big) - \mathscr{X}_k(0)\big)^{\mathrm{T}}\big[Q_1\big(\mathscr{X}_k\big(\tau(t)\big) - \mathscr{X}_k(0)\big) + 2Q_2\mathscr{X}_k(0)\big] \\
&\qquad + \big(T-2\tau(t)\big)\mathscr{X}_k^{\mathrm{T}}(0)Y\mathscr{X}_k(0) + \dot{\mathscr{X}}_k^{\mathrm{T}}(\tau(t)X_2^{\mathrm{T}})X_0\varsigma_k\big(\tau(t)\big) \\
&\qquad - \int\limits_0^{\tau(t)} \dot{\mathscr{X}}_k^{\mathrm{T}}(s)Y_1\dot{\mathscr{X}}_k(s)\,\mathrm{d}s + 2(\mathscr{X}_k^{\mathrm{T}}(\tau(t)X_1^{\mathrm{T}}.
\end{aligned}$$

Notably, it is found in [20] that

$$\begin{aligned}
\int\limits_0^{\tau(t)} \dot{\mathscr{X}}_k^{\mathrm{T}}(s)Y_1\dot{\mathscr{X}}_k(s)\,\mathrm{d}s \geqslant{}& 2\varsigma_k^{\mathrm{T}}\big(\tau(t)\big)N\big(\mathscr{X}_k\big(\tau(t)\big) - \mathscr{X}_k(0)\big) \\
&- \tau(t)\varsigma_k^{\mathrm{T}}\big(\tau(t)\big)NR^{-1}N^{\mathrm{T}}\varsigma_k\big(\tau(t)\big).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\dot{\tilde{V}}\big(\tau(t), \mathscr{X}_k\big) \\
&\quad \leqslant 2\mathscr{X}_k^{\mathrm{T}}\big(\tau(t)\big)Q\dot{\mathscr{X}}_k\big(\tau(t)\big) + \big(T-\tau(t)\big)\dot{\mathscr{X}}_k^{\mathrm{T}}(\tau(t)\big) \\
&\qquad \times \big[Y_1\dot{\mathscr{X}}_k(\tau(t)\big) + 2Q_1\big(\mathscr{X}_k\big(\tau(t)\big) - \mathscr{X}_k(0)\big) + 2Q_2\mathscr{X}_k(0)\big] \\
&\qquad - \big(\mathscr{X}_k\big(\tau(t)\big) - \mathscr{X}_k(0)\big)^{\mathrm{T}}\big[Q_1\big(\mathscr{X}_k\big(\tau(t)\big) - \mathscr{X}_k(0)\big) + 2Q_2\mathscr{X}_k(0)\big] \\
&\qquad - 2\varsigma_k^{\mathrm{T}}\big(\tau(t)\big)N\big(\mathscr{X}_k\big(\tau(t)\big) - \mathscr{X}_k(0)\big) \\
&\qquad + 2(\mathscr{X}_k^{\mathrm{T}}\big(\tau(t)X_1^{\mathrm{T}}\big) + \dot{\mathscr{X}}_k^{\mathrm{T}}\big(\tau(t)X_2^{\mathrm{T}}\big)X_0\varsigma_k\big(\tau(t)\big) \\
&\qquad + \big(T-2\tau(t)\big)\mathscr{X}_k^{\mathrm{T}}(0)Y\mathscr{X}_k(0) + \tau(t)\varsigma_k^{\mathrm{T}}\big(\tau(t)\big)NR^{-1}N^{\mathrm{T}}\varsigma_k\big(\tau(t)\big).
\end{aligned} \tag{15}$$

Carrying out some algebraic operations, (15) can be rewritten as

$$\begin{aligned}
\dot{\tilde{V}}\big(\tau(t), \mathscr{X}_k\big) \leqslant{}& \varsigma_k^{\mathrm{T}}\big(\tau(t)\big)\big[\varSigma_1 + \big(T-\tau(t)\big)\varSigma_2 \\
&+ \big(T-2\tau(t)\big)\varSigma_3 + \tau(t)NR^{-1}N^{\mathrm{T}}\big]\varsigma_k\big(\tau(t)\big),
\end{aligned} \tag{16}$$

where the definition of $\varSigma_1$, $\varSigma_2$, and $\varSigma_3$ has been given in Theorem 1.

Notice that the right side of (16) is affine with respect to $\tau(t)$. Let us substitute $\tau(t) = 0$ and $\tau(t) = T$ into (16), this leads to the following conditions:

$$\begin{aligned}
&\varSigma_1 + T(\varSigma_2 + \varSigma_3) < 0, \\
&\varSigma_1 - T\varSigma_3 + TNR^{-1}N^{\mathrm{T}} < 0.
\end{aligned} \tag{17}$$

Applying Lemma 1 to (17) yields conditions (11), ensuring that $\dot{\tilde{V}}(\tau(t), \mathscr{X}_k) < 0$ for all $\tau(t) \in [0, T]$ and

$$\int_0^\tau \dot{\tilde{V}}(\tau(t), \mathscr{X}_k)\, \mathrm{d}\tau(t) = V\big(\mathscr{X}_k(T)\big) - V\big(\mathscr{X}_k(0)\big) + W(T, \mathscr{X}_k) - W(0, \mathscr{X}_k) < 0.$$

Since $W(0, \mathscr{X}_k) = W(T, \mathscr{X}_k) = 0$, one may conclude that

$$V\big(\mathscr{X}_k(T)\big) - V\big(\mathscr{X}_k(0)\big) = V\big(z(t_k + T)\big) - V\big(z(t_k)\big) < 0.$$

Secondly, we consider the stability condition for $t \in [t_k + T,\, t_{k+1})$. Based on $\zeta(t) = z(t_k) - z(t)$, system (8) is reformulated as follows:

$$\dot{z}(t) = (\mathcal{A} + \mathcal{B})z(t) + \mathcal{B}\zeta(t) - \alpha B_f \nabla \bar{f}\big(Hz(t)\big).$$

From Assumption 2

$$\big\|\nabla \bar{f}\big(\bar{\hat{x}}(t)\big)\big\|^2 \leqslant \hat{m}\bar{\hat{x}}^{\mathrm{T}}(t)\nabla \bar{f}\big(\bar{\hat{x}}(t)\big),$$

where $\hat{m} = \max\{m_1, m_2, \dots, m_N\}$. For any $\delta > 0$, the above inequality is also equivalent to

$$0 \leqslant \delta\hat{m}\bar{\hat{x}}^{\mathrm{T}}(t)\nabla \bar{f}\big(\bar{\hat{x}}(t)\big) - \delta\nabla \bar{f}^{\mathrm{T}}\big(\bar{\hat{x}}(t)\big)\nabla \bar{f}\big(\bar{\hat{x}}(t)\big).$$

Therefore,

$$2\hat{m}z^{\mathrm{T}}(t)H^{\mathrm{T}}\nabla \bar{f}\big(Hz(t)\big) - 2\nabla \bar{f}^{\mathrm{T}}\big(Hz(t)\big)\nabla \bar{f}\big(Hz(t)\big) < 0.$$

Supposing that

$$\begin{aligned}\mathscr{W}(t) = \dot{V}(z) - g\big(\zeta(t), y_a(t)\big) + 2\hat{m}z^{\mathrm{T}}(t)H^{\mathrm{T}}\nabla \bar{f}\big(Hz(t)\big) \\ - 2\nabla \bar{f}^{\mathrm{T}}\big(Hz(t)\big)\nabla \bar{f}\big(Hz(t)\big),\end{aligned}$$

if we ensure $\mathscr{W}(t) < 0$, then

$$\begin{aligned}\dot{V}(z) &< g\big(\zeta(t), y_a(t)\big) - 2\hat{m}z^{\mathrm{T}}(t)H^{\mathrm{T}}\nabla \bar{f}\big(Hz(t)\big) \\ &\quad + 2\nabla \bar{f}^{\mathrm{T}}\big(Hz(t)\big)\nabla \bar{f}\big(Hz(t)\big) \\ &< 0\end{aligned}$$

for any $t \in [t_k + T,\, t_{k+1})$ by the event-triggered condition (9).

Constructing a vector $\varsigma(t) = \mathrm{col}(z(t), \dot{z}(t), \zeta(t), \nabla \bar{f}(t))$, where $\nabla \bar{f}(t) = \nabla \bar{f}(Hz(t))$, so we can rephrase $\mathscr{W}(t)$ as

$$\begin{aligned}\mathscr{W}(t) = \varsigma^{\mathrm{T}}(t)\big\{\mathrm{sym}\big\{\iota_1^{\mathrm{T}}Q\iota_2 + \hat{m}\iota_1^{\mathrm{T}}H^{\mathrm{T}}\iota_4 - \iota_4^{\mathrm{T}}\iota_4\big\} \\ - \iota_3^{\mathrm{T}}Q_\zeta\iota_3 + \iota_1^{\mathrm{T}}C_a^{\mathrm{T}}Q_\epsilon^{-1}C_a\iota_1\big\}\varsigma(t).\end{aligned}$$

Similar to (14), the following zero equation can be generated for any matrices X_{1a} and X_{2a} with appropriate dimensions:

$$2\big(z^{\mathrm{T}}(t)X_{1a}^{\mathrm{T}} + \dot{z}^{\mathrm{T}}(t)X_{2a}^{\mathrm{T}}\big)X_{0a}\varsigma(t) = 0,$$

where $X_{0a} = [\mathcal{A} + \mathcal{B}\ \ -I_{N(2n+s)}\ \ \mathcal{B}\ \ -\alpha B_f]$. The zero equation can be added to $\mathscr{W}(t)$, we can deduce that

$$\begin{aligned}\mathscr{W}(t) &= \varsigma^{\mathrm{T}}(t)\big\{\mathrm{sym}\big\{\iota_1^{\mathrm{T}}Q\iota_2 + \hat{m}\iota_1^{\mathrm{T}}H^{\mathrm{T}}\iota_4 - \iota_4^{\mathrm{T}}\iota_4 \\ &\quad + \big(\iota_1^{\mathrm{T}}X_{1a}^{\mathrm{T}} + \iota_2^{\mathrm{T}}X_{2a}^{\mathrm{T}}\big)X_{0a}\big\} - \iota_3^{\mathrm{T}}Q_\zeta\iota_3 + \iota_1^{\mathrm{T}}C_a^{\mathrm{T}}Q_\epsilon^{-1}C_a\iota_1\big\}\varsigma(t) \\ &= \varsigma^{\mathrm{T}}(t)\big(\Sigma_a + \Sigma_b^{\mathrm{T}}Q_\epsilon^{-1}\Sigma_b\big)\varsigma(t).\end{aligned}$$

According to condition (10), we can get that $\mathscr{W}(t) < 0$. Further, through the above analysis, $\dot{V}(z) < 0$ for all $t \in [t_k + T,\, t_{k+1})$.

Next, we will demonstrate that the average consensus value is the optimal solution of the global objective function $F(x)$. Let $x^* = (1/N)\sum_{j=1}^N x_j(t)$ be an optimal solution to the optimization problem (2). Since all agents achieve the same state, thus one obtains that $e_i(t) = 0$ and $\sum_{j=1}^N l_{ij}\hat{x}_i(t - \tau(t)) = 0$. Furthermore, we have

$$\begin{aligned}\dot{x}^*(t) &= \frac{1}{N}\sum_{j=1}^N \dot{x}_j(t) \\ &= \frac{1}{N}\sum_{j=1}^N (\mathscr{A} - \mathscr{B}\mathscr{K})x_j(t) + \frac{1}{N}\sum_{j=1}^N \mathscr{B}\big(-\alpha\mathscr{B}^{\mathrm{T}}\nabla f_j\big(x_j(t)\big)\big) \\ &= (\mathscr{A} - \mathscr{B}\mathscr{K})x^*(t) + \frac{1}{N}\sum_{i=1}^N \mathscr{B}\big(-\alpha\mathscr{B}^{\mathrm{T}}\nabla f_i\big(x_i(t)\big)\big).\end{aligned} \tag{18}$$

Based on Eq. (18), it is readily seen that

$$\begin{aligned}&\frac{\mathrm{d}(\sum_{i=1}^N x_i(t) - Nx^*(t))}{\mathrm{d}t} \\ &\quad = \big(1_N^{\mathrm{T}} \otimes I_n\big)\dot{x}(t) - N\dot{x}^*(t) \\ &\quad = \big(1_N^{\mathrm{T}} \otimes I_n\big)\big[\big(I_N \otimes (\mathscr{A} - \mathscr{B}\mathscr{K})\big)x(t) + \big(I_N \otimes \mathscr{B}[\mathscr{K}, \mathscr{F}]\big)e(t) \\ &\qquad + (\mathcal{L} \otimes \mathscr{B}\Pi\mathscr{C})\hat{x}\big(t - \tau(t)\big) - \alpha\big(I_N \otimes \mathscr{B}\mathscr{B}^{\mathrm{T}}\big)\nabla f\big(x(t)\big)\big] \\ &\qquad - N(\mathscr{A} - \mathscr{B}\mathscr{K})x^*(t) + \alpha\big(1_N^{\mathrm{T}} \otimes \mathscr{B}\mathscr{B}^{\mathrm{T}}\big)\nabla f\big(x(t)\big) \\ &\quad = (\mathscr{A} - \mathscr{B}\mathscr{K})\left(\sum_{i=1}^N x_i(t) - Nx^*(t)\right).\end{aligned}$$

Because $\mathscr{A} - \mathscr{B}\mathscr{K}$ is Hurwitz stable, then $(\mathscr{A} - \mathscr{B}\mathscr{K})^{\mathrm{T}}\Psi + \Psi(\mathscr{A} - \mathscr{B}\mathscr{K}) + 2I_n < 0$, where matrix $\Psi > 0$. Let us select Lyapunov function

$$\bar{V}(t) = \left(\sum_{i=1}^N x_i(t) - Nx^*(t)\right)^{\mathrm{T}}\Psi\left(\sum_{i=1}^N x_i(t) - Nx^*(t)\right).$$

We compute the derivative of $\bar{V}$ as follows:

$$\dot{\bar{V}}(t) = \left(\sum_{i=1}^{N} x_i(t) - Nx^*(t)\right)^{\mathrm{T}} \mathscr{S} \left(\sum_{i=1}^{N} x_i(t) - Nx^*(t)\right)$$
$$< -2\left\| \sum_{i=1}^{N} x_i(t) - Nx^*(t)\right\|^2 < 0,$$

where

$$\mathscr{S} = \left((\mathscr{A} - \mathscr{B}\mathscr{K})^{\mathrm{T}}\Psi + \Psi(\mathscr{A} - \mathscr{B}\mathscr{K})\right).$$

Then one has

$$\lim_{t\to\infty} x_i(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(t) = x^*. \qquad \square$$

Remark 5. Note that the construction of the looped functional $W(\tau(t), \mathscr{X}_k)$ does not necessarily require positive definiteness, which represents an improvement over the classical Lyapunov–Krasovskii functional approach. In Theorem 1, it is rigorously proven that the MAS (1) asymptotically converges to the optimal value of the optimization problem (2) under the event-triggered mechanism (9) with the dwell time T. Furthermore, compared to the traditional event-triggered control strategy, which requires the exclusion of Zeno behavior, this paper automatically eliminates Zeno behavior by introducing a dwell time T into the event-triggered mechanism.

4 Simulation example

Consider the linear MAS (1) consisting of four agents under the network topology shown in Fig. 2. The selection of all matrices is provided as follows:

$$\mathscr{A} = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}, \qquad \mathscr{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathscr{C} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix},$$
$$\mathscr{F} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathscr{S} = \begin{bmatrix} -2 & 1 \\ -1 & -1.5 \end{bmatrix}.$$

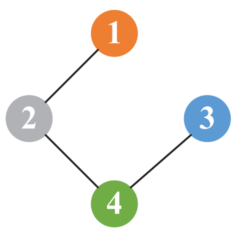

Figure 2. The topology graph.

Based on the matrix provided above, we can calculate $\mathscr{D} = \mathscr{B}\mathscr{F} = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$. By solving LMIs in (10)–(11) and considering the above parameters, the corresponding solutions are

$$\Pi = 1, \qquad \mathscr{K} = [8.241 \;\; 2.831], \qquad \mathscr{G} = [3 \;\; 12]^{\mathrm{T}},$$

$$P = \begin{bmatrix} 0.0203 & 0 & 0 & 0 \\ 0 & 0.0203 & 0 & 0 \\ 0 & 0 & 15.0183 & 0.0963 \\ 0 & 0 & 0.0963 & 19.0083 \end{bmatrix}.$$

The local cost functions are given as follows:

$$f_1(x_{11}, x_{12}) = 0.1x_{11}^2 + 0.1x_{12}^2 + 2x_{11}x_{12}^2 + 0.1x_{11} + 0.2x_{12} + 0.1,$$
$$f_2(x_{21}, x_{22}) = 0.1x_{21}^2 + 0.1x_{22}^2 + 2x_{21}x_{22}^2 + 0.2x_{21} + 0.3x_{22} + 0.2,$$
$$f_3(x_{31}, x_{32}) = 0.1x_{31}^2 + 0.1x_{32}^2 + 2x_{31}x_{32}^2 + 0.3x_{31} + 0.4x_{32} + 0.3,$$
$$f_4(x_{41}, x_{42}) = 0.1x_{41}^2 + 0.1x_{42}^2 + 2x_{41}x_{42}^2 + 0.4x_{41} + 0.5x_{42} + 0.4.$$

Based on Theorem 1, the distributed optimization problem (2) is addressed by appropriately selecting parameters $\alpha = 18$, $b = 10$ via event-triggered control algorithm (5). The theoretical analysis suggests that the state trajectories of MASs will converge towards the average value of the initial states $x_1(0) = [5 \;\; 8]^{\mathrm{T}}$, $x_2(0) = [-3 \;\; -5]^{\mathrm{T}}$, $x_3(0) = [3 \;\; 5]^{\mathrm{T}}$, $x_4(0) = [-5 \;\; -8]^{\mathrm{T}}$, the state trajectories of $\bar{x}_i(t)$ and $\hat{\bar{x}}_i(t)$ under the event-triggered algorithm (9) with $T = 0.2$ are illustrated in Figs. 3–4. Under the threshold parameter $Q_\zeta = [5 \;\; 0; 0 \;\; 5]$ and $Q_\epsilon = 1$, the control input $u_i(t)$ and the event-triggered instants graph can be obtained as shown in Fig. 5. Further, the trajectory graph of the objective function is obtained, as shown in Fig. 6.

From Figs. 3–4 it can be observed that the state error of each agent gradually converges to zero, demonstrating that under the proposed control protocol (5), each component of the system successfully converges to its corresponding average state. Figure 5 shows the graph of control input $u_i(t)$ and the event-triggered instants. Additionally, as seen in Fig. 6, the states of all agents stabilize over time with the optimal values of the global objective functions reaching 0.815. To further investigate the effect of increasing the dwell time T in the event-triggered mechanism (9), we analyze the variation in triggered instants by adjusting the dwell time. Without loss of generality, keeping other parameters and initial conditions constant, simulations were conducted for $T = 0.15$ and $T = 0.35$ with the corresponding results presented in Figs. 7–14.

From the figures below it can be observed that as the dwell time T decreases from 0.2 to 0.15, the number of triggered instants increases, indicating a rise in the communication frequency between agents in the MASs. In contrast, when the dwell time is increased to 0.35, the number of event triggers decreases significantly, while the system remains stable. A detailed comparison of the number of event triggers is presented in Fig. 15, where it is clearly observed that as the dwell time increases, the communication frequency between agents decreases. Therefore, appropriately increasing the dwell time T can help reduce communication frequency. However, it is important to note that the dwell time cannot be increased indefinitely, as a larger dwell time may prevent the trigger protocol from meeting its threshold conditions, thereby compromising its control effectiveness.

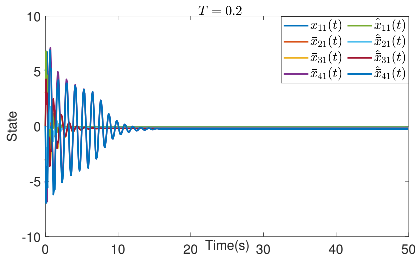

Figure 3. The trajectory of $\bar{x}_{i1}(t)$ and $\hat{\bar{x}}_{i1}(t)$.

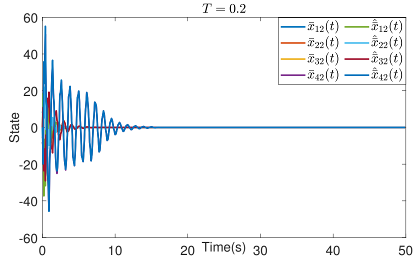

Figure 4. The trajectories of $\bar{x}_{i2}(t)$ and $\hat{\bar{x}}_{i2}(t)$.

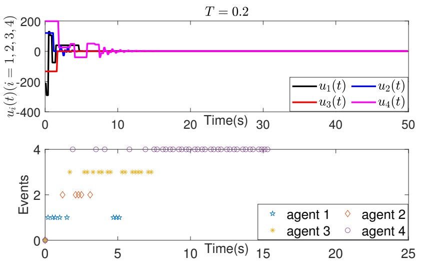

Figure 5. The control input $u_i(t)$ and triggered instants.

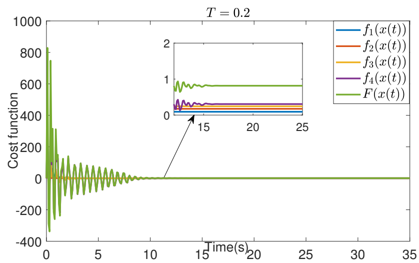

Figure 6. The diagrams of $f_i(x_i(t))$ and $F(x(t))$.

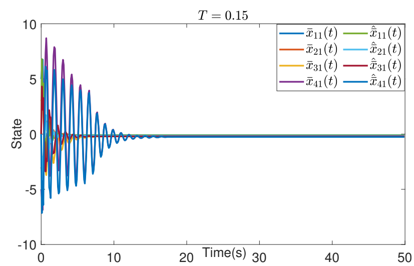

Figure 7. The trajectory of $\bar{x}_{i1}(t)$ and $\hat{\bar{x}}_{i1}(t)$.

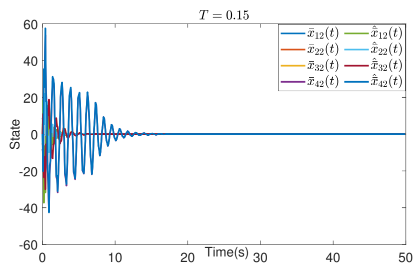

Figure 8. The trajectories of $\bar{x}_{i2}(t)$ and $\hat{\bar{x}}_{i2}(t)$.

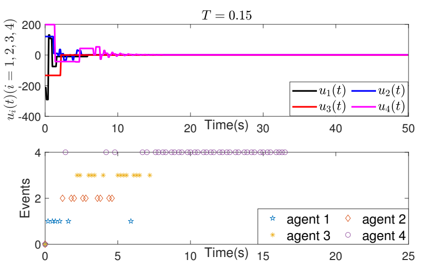

Figure 9. The control input $u_i(t)$ and triggered instants.

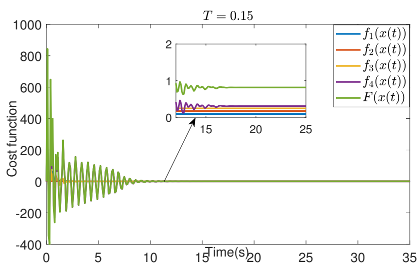

Figure 10. The diagrams of $f_i(x_i(t))$ and $F(x(t))$.

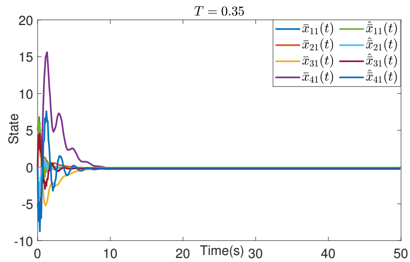

Figure 11. The trajectory of $\bar{x}_{i1}(t)$ and $\hat{\bar{x}}_{i1}(t)$.

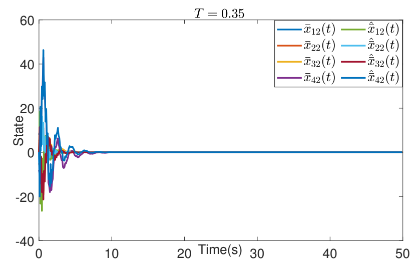

Figure 12. The trajectories of $\bar{x}_{i2}(t)$ and $\hat{\bar{x}}_{i2}(t)$.

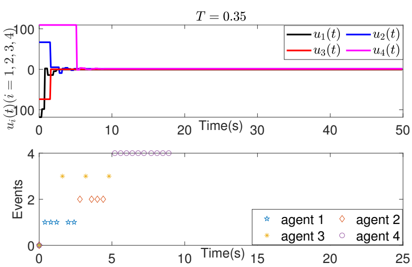

Figure 13. The control input $u_i(t)$ and triggered instants.

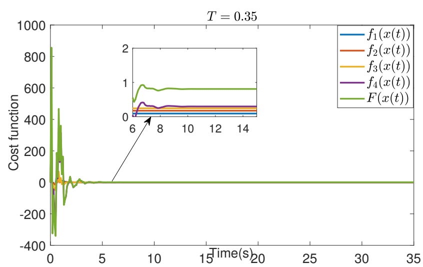

Figure 14. The diagrams of $f_i(x_i(t))$ and $F(x(t))$.

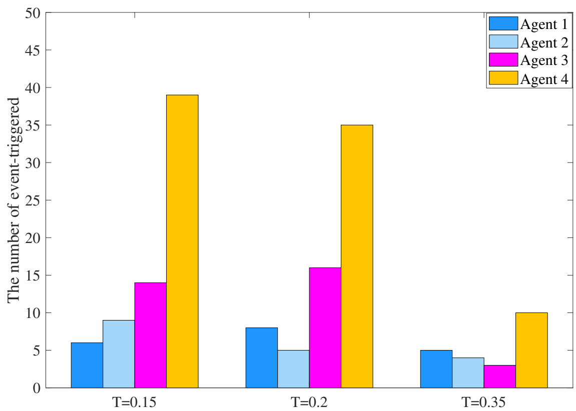

Figure 15. The comparison graph of the number of triggered-instants.

5 Conclusion

This paper has discussed the distributed optimization problem of linear MASs with external disturbances over undirected networks. A new observation system was established to avoid the use of state information. Moreover, by utilizing the observation information, a distributed event-triggered control protocol was designed, enabling all agents to achieve state consensus and reach the optimal value of the objective function.

In this paper, we only considered the asymptotic-time distributed optimization problem for global objective functions. Further work will focus on the prescribed-time distributed optimization problem.

Conflicts of interest. The authors declare no conflicts of interest.

References

1. D. Boley, B. Datta, *Numerical Methods for Linear Control Systems*, Birkhäuser, Boston, MA, 2004.
2. S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.
3. J. Cai, C. Yi, An adaptive gradient-descent-based neural networks for the on-line solution of linear time variant equations and its applications, *Inf. Sci.*, **622**:34–45, 2023, https://doi.org/10.1016/j.ins.2022.11.157.
4. G.P. Chander, S. Das, Hesitant t-spherical fuzzy linear regression model based decision making approach using gradient descent method, *Eng. Appl. Artif. Intell.*, **122**:106074, 2023, https://doi.org/10.1016/j.engappai.2023.106074.
5. G. Chen, P. Yi, Y. Hong, J. Chen, Distributed optimization with projection-free dynamics: A Frank–Volfe perspective, *IEEE Trans. Cybern.*, **51**(1):599–610, 2024, https://doi.org/10.1109/TCYB.2023.3284822.
6. Z. Ding, Consensus disturbance rejection with disturbance observers, *IEEE Trans. Ind. Electron.*, **62**(9):5829–5837, 2015, https://doi.org/10.1109/TIE.2015.2442218.
7. M. Doostmohammadian, W. Jiang, M. Liaquat, A. Aghasi, H. Zarrabi, Discretized distributed optimization over dynamic digraphs, *IEEE Trans. Autom. Sci. Eng.*, **22**:2758–2767, 2023, https://doi.org/10.1109/TASE.2024.3383894.
8. X. Ge, Q.-L. Han, L. Ding, Y.-L. Wang, X.-M. Zhang, Dynamic event-triggered distributed coordination control and its applications: A survey of trends and techniques, *IEEE Trans. Syst. Man Cybern.*, **50**(9):3112–3125, 2020, https://doi.org/10.1109/TSMC.2020.3010825.
9. Z. Hu, R. Su, K. Zhang, Z. Xu, R. Ma, Resilient event-triggered model predictive control for adaptive cruise control under sensor attacks, *IEEE/CAA J. Autom. Sin.*, **10**(3):807–809, 2023, https://doi.org/10.1109/JAS.2023.123111.
10. Y. Huang, Z. Meng, J. Sun, Distributed constrained optimization for second-order multiagent systems via event-based communication, *IEEE Trans. Syst. Man Cybern.*, **54**(9):5317–5326, 2024, https://doi.org/10.1109/TSMC.2024.3405453.
11. J. Kang, G. Guo, G. Yang, Distributed optimization of uncertain multiagent systems with disturbances and actuator faults via exosystem observer-based output regulation, *IEEE Trans. Circuits Syst. I Regul. Pap.*, **70**(2):897–909, 2023, https://doi.org/10.1109/TCSI.2022.3221097.

12. J. Li, X. Chang, Y. Xue, J. Su, Z. Li, W. Guan, H. Sun, Privacy-preserving neurodynamic distributed energy management for integrated energy system considering packet losses, *IEEE Trans. Ind. Inf.*, **21**(2):1882–1891, 2025, `https://doi.org/10.1109/TII.2024.3495760`.
13. R. Li, G.-H. Yang, Distributed event-triggered algorithm designs for resource allocation problems via a universal scalar function-based analysis, *IEEE Trans. Cybern.*, **54**(4):2224–2234, 2024, `https://doi.org/10.1109/TCYB.2022.3219449`.
14. S. Li, X. Nian, Z. Deng, Distributed optimization of general linear multi-agent systems with external disturbance, *J. Franklin Inst.*, **358**(11):5951–5970, 2021, `https://doi.org/10.1016/j.jfranklin.2021.05.024`.
15. Y. Li, B. Huang, J. Dai, D.W. Gao, Q. Sun, H. Zhang, Distributed resilient initialization-free Jacobi descent algorithm for constrained optimization against DoS attacks, *IEEE Trans. Autom. Sci. Eng.*, **21**(3):3332–3343, 2024, `https://doi.org/10.1109/TASE.2023.3278865`.
16. H. Liu, W. Yu, G. Wen, W.X. Zheng, Distributed algorithm over time-varying unbalanced graphs for optimization problem subject to multiple local constraints, *IEEE Trans. Control Network Syst.*, **12**(1):387–402, 2025, `https://doi.org/10.1109/TCNS.2024.3469048`.
17. M. Luan, G. Wen, H. Liu, T. Huang, G. Chen, W. Yu, Distributed discrete-time convex optimization with closed convex set constraints: Linearly convergent algorithm design, *IEEE Trans. Cybern.*, **54**(4):2271–2283, 2024, `https://doi.org/10.1109/TCYB.2023.3270185`.
18. L.G. Moreira, J.M. Gomes da Silva Jr., S. Tarbouriech, A. Seuret, Observer-based event-triggered control for systems with slope-restricted nonlinearities, *Int. J. Robust Nonlinear Control*, **30**:7409–7428, 2020, `https://doi.org/10.1002/rnc.5185`.
19. K. Sakurama, M. Miura, Distributed constraint optimization on networked multi-agent systems, *Applied Math. Comput.*, **292**:272–281, 2017, `https://doi.org/10.1016/j.amc.2016.07.008`.
20. L. Shanmugam, Y. Joo, Further stability and stabilization condition for sampled-data control systems via the looped functional method, *IEEE Trans. Circuits Syst. II Express Briefs*, **68**(1):301–305, 2021, `https://doi.org/10.1109/TCSII.2020.2994591`.
21. X. Shi, J. Liu, C. Sun, Distributed multi-timescale algorithm for nonconvex optimization problem: Acontrol perspective, *Neural Netw.*, **186**:107257, 2025, `https://doi.org/10.1016/j.neunet.2025.107257`.
22. X. Shi, Y. Wan, G. Wen, X. Yu, Event-triggered zero-gradient-sum distributed constrained optimization over jointly connected balanced digraphs, *IEEE Trans. Network Sci. Eng.*, **12**(4):3389–3399, 2025, `https://doi.org/10.1109/TNSE.2025.3559905`.
23. C. Sun, M. Ye, G. Hu, Distributed time-varying quadratic optimization for multiple agents under undirected graphs, *IEEE Trans. Autom. Control*, **62**(7):687–3694, 2017, `https://doi.org/10.1109/TAC.2017.2673240`.
24. L. Sun, D. Ding, H. Dong, X. Yi, Distributed economic dispatch of microgrids based on ADMM algorithms with encryption-decryption rules, *IEEE Trans. Autom. Sci. Eng.*, **22**:8427–8438, 2025, `https://doi.org/10.1109/TASE.2024.3485922`.

25. M. Wei, W. Yu, H. Liu, Q. Xu, Distributed weakly convex optimization under random time-delay interference, *IEEE Trans. Network Sci. Eng.*, **11**(1):212–224, 2024, https://doi.org/10.1109/TNSE.2023.3294414.

26. L. Xu, Q. Guo, H. Zeng, Y. Yang, K. Feng, H. Sun, Communication topology optimization for time-delay cyber–physical microgrids under distributed control, *IEEE Trans. Smart Grid*, **15**(5):5087–5101, 2024, https://doi.org/10.1109/TSG.2024.3384229.

27. H. Ye, Y. Song, Z. Zhang, C. Wen, Global dynamic event-triggered control for nonlinear systems with sensor and actuator faults: A matrix-pencil-based approach, *IEEE Trans. Autom. Control*, **69**(3):200–2014, 2024, https://doi.org/10.1109/TAC.2023.3313634.

28. Y. Yu, X. Li, L. Li, L. Xie, Distributed online optimization for heterogeneous linear multi-agent systems with coupled constraints, *Automatica*, **159**:111407, 2024, https://doi.org/10.1016/j.automatica.2023.111407.

29. H. Zhang, Y. Li, Z. Wang, Y. Ding, H. Yan, Distributed optimal control of nonlinear system based on policy gradient with external disturbance, *IEEE Trans. Network Sci. Eng.*, **11**(1):872–885, 2024, https://doi.org/10.1109/TNSE.2023.3309816.

30. Z. Zhao, G. Chen, Event-triggered scheme for zero-gradient-sum optimization under directed networks with time delay, *Int. J. Syst. Sci.*, **52**(1):47–56, 2021, https://doi.org/10.1080/00207721.2020.1819467.

31. Y. Zhu, W. Yu, G. Wen, W. Ren, Continuous-time coordination algorithm for distributed convex optimization over weight-unbalanced directed networks, *IEEE Trans. Circuits Syst. II Express Briefs*, **66**(7):1202–1206, 2019, https://doi.org/10.1109/TCSII.2018.2878250.