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Stability and stabilization for positive switched delay systems under mode-dependent interval dwell time*

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Abstract. This paper explores the issues of exponential stability and positive stabilization of positive switched systems with time-varying delays. Using the Lyapunov–Razumikhin approach, we derive exponential stability criteria for positive switched delay systems (PSDSs) across various switching behaviors, including nondecreasing and decreasing switching patterns. Following this, we delve into the design of state feedback control to achieve positive stabilization of these systems, leveraging the established stability conditions. The paper culminates with numerical examples that confirm the validity of our theoretical findings.

Keywords: positive switched systems, exponential stability, positive stabilization, time-varying delay, mode-dependent interval dwell time.

1 Introduction

With the development of modern control theory and technology, switched systems [4, 11, 13, 24, 33], which are a particular type of hybrid dynamic systems, have attracted extensive attention from both academia and industry. These systems consist of multiple subsystems and a switching rule that governs the switching between these subsystems, and have a widespread presence in numerous practical applications, including power systems, networked control systems, and mechanical systems. Positive switched systems [20, 23, 25, 30] represent an important branch within the theory of switched systems, characterized by the nonnegativity of all state variables. This nonnegativity reflects the physical properties of certain real systems, such as population dynamics, biochemical processes, and economic management systems.

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Stability analysis is a critical aspect of the theory of dynamic systems, primarily concerned with investigating the stability characteristics of system states over time. This analysis involves not only assessing the stability of a system after a control strategy has been implemented but also designing an appropriate controller based on stability analysis. Consequently, the study of system stability is of considerable importance. Notably, even when all subsystems are stable, an ill-chosen switching rule can destabilize the overall switched system. Conversely, with an appropriate switching rule, even if some or all subsystems are unstable [16, 17, 19], the switched system can maintain stability. Therefore, designing an appropriate switching rule to stabilize the switched system is a common control strategy. Additionally, state feedback controller [7, 9, 15, 27] offers another effective approach for achieving system stability. By introducing a state feedback loop, these controllers adjust the system state in real time, aiming to achieve the desired stable state. Careful design of a state feedback controller can yield substantial enhancements in both the dynamic performance and stability of the system. In contrast to research on the stabilization of normal switched dynamical systems [3, 5, 12, 21], this paper requires that the states of the switched systems remain positive at all times, an approach referred to as positive stabilization.

Time delay from controller, actuator, and measurement elements is a widespread phenomenon in engineering, frequently resulting in oscillation and performance degradation during the dynamic process. The investigation of time delay in control systems has garnered significant attention [1, 2, 8, 10, 14, 26, 34]. Among these, Chen and Zheng [2] addressed the time delay by employing the Lyapunov–Razumikhin technique, thereby deriving stability conditions. Zong et al. [34] addressed the issue of finite-time H_1 control for a class of discrete-time switched time-delay systems by constructing a Lyapunov-like functional. Zhang et al. [26] studied the fixed-time and finite-time stability of switched delay systems using the Lyapunov–Krasovskii technique. Chen et al. [1] developed a time-dependent Lyapunov function to investigate the stability and L_2-gain performance of linear time-delay systems. It is noteworthy that only the Lyapunov–Razumikhin method is capable of handling rapid time-varying delay. Nonetheless, literature on the stability analysis of positive switched systems with time-varying delay utilizing the Lyapunov–Razumikhin method remains scarce.

Switching behaviors can sometimes cause an increase in the Lyapunov function, which can adversely affect the stability of the system. Conversely, within a switching sequence, these behaviors can also play a stabilizing role. Using a discretized Lyapunov–Krasovskii functional, the stability of positive switched delay systems was investigated in [28], where all subsystems were unstable, implying that every switching action contributes to stability. Zhou employed the Lyapunov–Krasovskii functional to establish an exponential stability criterion for positive switched delay systems, as cited in [32], which includes a switching behavior that involves nondecreasing switching behavior and decreasing switching behavior. Considering these two types of switching behaviors, this paper addresses the exponential stability of positive switched delay systems and positive stabilization.

Building upon the aforementioned background, this paper centers on the study of time-varying positive switched delay system (PSDS), examining in depth the issues of exponential stability and positive stabilization control. It is crucial to emphasize the distinctions between the current work and previous research efforts:

(i) This paper employs the Lyapunov–Razumikhin technique to address time delay, rather than the Lyapunov–Krasovskii method, as in [28] and [32]. The method employed in this paper is capable of studying the stability of systems with rapid time delay. In addition, the Lyapunov function constructed in this paper is discretized, which can reduce the conservatism of the obtained results.

(ii) Unlike the literature, which concentrates on fixed delay system, the findings presented in this paper are applicable to time-varying delay.

(iii) In contrast to the common approach of utilizing a single pair of maximum and minimum dwell times for all subsystems, this study is based on mode-dependent interval dwell-time method, where each subsystem is assigned a pair of maximum and minimum dwell times.

(iv) The stability results obtained in this paper can address scenarios where stable subsystems are present, as well as those where all subsystems are unstable. These criteria are verified through numerical examples. Based on these findings, a design for a state feedback controller is proposed.

The structure of the remainder of this paper is as follows. Section 2 provides the preliminaries. Section 3 presents the sufficient conditions for exponential stability of PSDS under mode-dependent interval dwell time. Based on the stability results, the positive stabilization problem of the system is further investigated in Section 4. Section 5 presents numerical examples to verify the effectiveness of the conclusions in this paper. Section 6 summarizes the entire paper.

Notations. Let $\mathbb{N}$ denote the set of natural numbers. A nonnegative vector x is defined as a vector in which all elements are nonnegative. Similarly, a nonnegative matrix A is defined as a matrix in which all elements are nonnegative. $\mathbb{R}^n$ ($\mathbb{R}^n_+$) and $\mathbb{R}^{n\times n}$ ($\mathbb{R}^{n\times n}_+$) represent n-dimensional (nonnegative) real vector space and $(n\times n)$-dimensional (nonnegative) real matrix space, respectively. $\mathbb{M}_n$ denotes the set of Metzler matrices, whose nondiagonal elements are nonnegative. I represents the identity matrix of the appropriate dimension. A vector $x \succ 0$ ($x \succcurlyeq 0$) means that x is a positive (nonnegative) vector, conversely, $x \prec 0$ ($x \preccurlyeq 0$) means that x is a negative (nonpositive) vector. The function $x(t)$ equals its right-hand limit at t, given by $x(t) = x(t^+)$. $\|\cdot\|$ is the Euclidean norm. $A^\top$ is the transpose of matrix A. $\mathbb{L} = \{0, 1, \dots, L\}$ and $\mathbb{L}^- = \{0, 1, \dots, L-1\}$.

2 Preliminaries

Consider the system defined as follows:

$$\begin{aligned} &\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}x\big(t-\omega(t)\big) + D_{\sigma(t)}u(t), \quad t \geqslant 0, \\ &x(\delta) = \rho(\delta), \quad \delta \in [-\hat{\omega}, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^n$ is the control input. The time-varying delay $\omega(t)$ is constrained such that $0 \leqslant \omega(t) \leqslant \hat{\omega}$, where $\hat{\omega}$ is a known positive constant. $\rho(\delta)$:

$[-\hat{\omega}, 0] \to \mathbb{R}^n$ is an initial function. The piecewise function $\sigma(t) : [0, +\infty) \to \mathcal{P} = \{1, 2, \ldots, p\}$ is the switching signal. The time sequence t_k for $k \in \mathbb{N}$ forms a sequence characterized by $0 < t_1 < \cdots < t_k < \cdots$ and $\lim_{k\to+\infty} t_k = +\infty$. τ_k represents the dwell time on the $\sigma(t_k)$th subsystem. For each $i \in \mathcal{P}$, A_i, B_i, and D_i are known constant system matrices with appropriate dimensions.

Definition 1. (See [31].) System (1) with the switching signal $\sigma(t)$ is said to be positive if, given the initial state $\rho(t) \succcurlyeq 0$ for $t \in [-\hat{\omega}, 0]$ and the input vector $u(t) \succcurlyeq 0$, the state trajectory satisfies $x(t) \succcurlyeq 0$ for all $t \geqslant 0$.

Definition 2. (See [22].) System (1) with the switching signal $\sigma(t)$ is said to be exponentially stable if there exist constants $p_0 > 0$, $q_0 > 0$ such that, for any initial function $\rho(t) \succcurlyeq 0$, the solution of system (1) satisfies $\|x(t)\| \leqslant p_0 \mathrm{e}^{-q_0 t} \|\rho\|_{\hat{\omega}}$, $t \geqslant 0$, where $\|\rho\|_{\hat{\omega}} = \sup_{-\hat{\omega} \leqslant t \leqslant 0} \|\rho(t)\|$.

Definition 3. (See [29].) A switching signal $\sigma(t)$ is called a mode-dependent interval dwell-time (MDIDT) switching signal if for each mode $i \in \mathcal{P}$, there exist constants τ_{i1} and τ_{i2} such that $\tau_{i1} = \inf_{k\in\mathbb{N}}\{\tau_k\colon \sigma(t_k) = i,\ i \in \mathcal{P}\}$ and $\tau_{i2} = \sup_{k\in\mathbb{N}}\{\tau_k\colon \sigma(t_k) = i,\ i \in \mathcal{P}\}$, where $\tau_k = t_{k+1} - t_k$.

Definition 4. (See [29].) The region D is called the stability region for admissible upper bound and lower bound of dwell time, which ensures the exponential stability of the switched system (1).

Lemma 1. (*See* [18].) *System* (1) *is positive if* $A_i \in \mathbb{M}_n$, $B_i \in \mathbb{R}_+^{n\times n}$, *and* $D_i \in \mathbb{R}_+^{n\times n}$ *for all* $i \in \mathcal{P}$.

Lemma 2. (*See* [6].) *A matrix* $A_i \in \mathbb{M}_n$ *for all* $i \in \mathcal{P}$ *if and only if there exists a scalar* $\hbar$ *for which the inequality* $A_i + \hbar I \geqslant 0$ *is satisfied.*

The interval $[t_k, t_{k+1})$ is partitioned into L equal subintervals, each denoted by $C_{k,q} = [t_k + q\ell_k,\ t_k + (q+1)\ell_k)$ for $q \in \mathbb{L}^-$, where $\ell_k = (t_{k+1} - t_k)/L$ is the length of each subinterval. It follows that $\cup_{q=0}^{L-1} C_{k,q} = [t_k, t_{k+1})$ and $C_{k,m} \cap C_{k,n} = \emptyset$ for $m \neq n$. For each subinterval $C_{k,q}$, let $\sigma(t_k) = i$, and we select positive constant vectors $\xi_{i,q} \in \mathbb{R}_+^n$, $\xi_{i,q+1} \in \mathbb{R}_+^n$, $q \in \mathbb{L}^-$. By using the convex combination technique, a positive time-varying function $\xi_i(t)$ can be described as

$$\xi_i(t) = f(t)\xi_{i,q+1} + \tilde{f}(t)\xi_{i,q}, \quad t \in C_{k,q}, \tag{2}$$

where $f(t) = (t - t_k - q\ell_k)/\ell_k$ and $\tilde{f}(t) = (t_k + (q+1)\ell_k - t)/\ell_k$. Note that $\tilde{f}(t) = 1 - f(t)$. Similarly, we define

$$\varepsilon_i(t) = f(t)\varepsilon_{i,\,q+1} + \tilde{f}(t)\varepsilon_{i,q}, \quad t \in C_{k,q}.$$

Then, taking the derivation of $\xi(t)$, we obtain

$$\dot{\xi}_i(t) = \frac{\xi_{i,q+1} - \xi_{i,q}}{\ell_k} = L\left(\frac{\xi_{i,q+1} - \xi_{i,q}}{t_{k+1} - t_k}\right), \quad t \in C_{k,q}. \tag{3}$$

Since $\tau_{i1} \leqslant t_{k+1} - t_k \leqslant \tau_{i2}$, by using the convex combination technique again, there exists a function $g(t) \in [0,1]$ such that

$$\begin{aligned}\dot{\xi}_i(t) &= L\left(\frac{\xi_{i,q+1} - \xi_{i,q}}{t_{k+1} - t_k}\right)\\ &= g(t)\frac{L}{\tau_{i2}}(\xi_{i,q+1} - \xi_{i,q}) + \tilde{g}(t)\frac{L}{\tau_{i1}}(\xi_{i,q+1} - \xi_{i,q}),\end{aligned} \tag{4}$$

where $\tilde{g}(t) = 1 - g(t)$.

3 Exponential stability analysis

In this section, we investigate the exponential stability of system (1) with $u(t) = 0$. The system (1) is simplified and rewritten as

$$\begin{aligned}&\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}x\big(t - \omega(t)\big), \quad t \geqslant 0,\\ &x(\delta) = \rho(\delta), \quad \delta \in [-\hat{\omega}, 0],\end{aligned} \tag{5}$$

where $A_i \in \mathbb{M}_n$ and $B_i \in \mathbb{R}_+^{n\times n}$ with the other symbols defined as in system (1). By Lemma 1, system (5) is a PSDS. We proceed to utilize the Lyapunov–Razumikhin method to establish the exponential stability criteria for this PSDS.

First, we consider the case of nondecreasing switching behavior.

Theorem 1. *Given an integer $L > 0$, suppose there exist vectors $\xi_{i,q} \succ 0$, $\xi_{i,q+1} \succ 0$ for $q \in \mathbb{L}^-$ and $i \in \mathcal{P}$, a vector $\xi^* \succ 0$, and constants $\beta_i > 0$, $\theta \geqslant 1$, $\alpha > 0$ such that*

$$\begin{aligned}&\frac{L}{\tau_{i1}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q}^\top A_i + (\beta_i + \alpha)\xi_{i,q}^\top \prec 0,\\ &\frac{L}{\tau_{i2}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q}^\top A_i + (\beta_i + \alpha)\xi_{i,q}^\top \prec 0,\\ &\frac{L}{\tau_{i1}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q+1}^\top A_i + (\beta_i + \alpha)\xi_{i,q+1}^\top \prec 0,\\ &\frac{L}{\tau_{i2}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q+1}^\top A_i + (\beta_i + \alpha)\xi_{i,q+1}^\top \prec 0,\end{aligned} \tag{6}$$

$$\xi_{i,q}^\top B_i - y_i \mathrm{e}^{-\alpha\hat{\omega}}\xi^* \prec 0, \quad q \in \mathbb{L}, \tag{7}$$

$$\xi^* \preccurlyeq \xi_{i,q}, \quad q \in \mathbb{L}, \tag{8}$$

$$\xi_{j,0} \preccurlyeq \theta\xi_{i,L}, \quad i \neq j,\ j \in \mathcal{P}. \tag{9}$$

Then system (5) *is exponentially stable under the MDIDT switching signal with $\tau_1 > \ln(\theta)/\alpha$, where $\tau_1 = \min_{i\in\mathcal{P}}\{\tau_{i1}\}$.*

Proof. For $t \in [t_k, t_{k+1})$, when $\sigma(t) = i$, we construct a copositive time-varying Lyapunov function as follows:

$$V_i\big(t, x(t)\big) = \mathrm{e}^{\alpha t}\xi_i^\top(t)x(t), \tag{10}$$

where α is a positive constant. From (6) there exists $\mu \geqslant 1$ such that

$$\begin{aligned}
&\frac{L}{\tau_{i1}}\left(\xi_{i,q+1}^{\top}-\xi_{i,q}^{\top}\right)+\xi_{i,q}^{\top}A_i+(\beta_i\mu+\alpha)\xi_{i,q}^{\top}\prec 0\\
&\frac{L}{\tau_{i2}}\left(\xi_{i,q+1}^{\top}-\xi_{i,q}^{\top}\right)+\xi_{i,q}^{\top}A_i+(\beta_i\mu+\alpha)\xi_{i,q}^{\top}\prec 0\\
&\frac{L}{\tau_{i1}}\left(\xi_{i,q+1}^{\top}-\xi_{i,q}^{\top}\right)+\xi_{i,q+1}^{\top}A_i+(\beta_i\mu+\alpha)\xi_{i,q+1}^{\top}\prec 0\\
&\frac{L}{\tau_{i2}}\left(\xi_{i,q+1}^{\top}-\xi_{i,q}^{\top}\right)+\xi_{i,q+1}^{\top}A_i+(\beta_i\mu+\alpha)\xi_{i,q+1}^{\top}\prec 0.
\end{aligned} \tag{11}$$

There exist two positive constants $\overline{\xi}$ and $\underline{\xi}$ such that $\overline{\xi}\mathbf{1} \succcurlyeq \xi_{i,q}$ and $\underline{\xi}\mathbf{1} \preccurlyeq \xi_{i,q}$ for all $i \in \mathcal{P}$ and $q \in \mathbb{L}$ with $\mathbf{1} = (1,1,\dots,1)^{\top}$ being a vector of the same dimension as $\xi_{i,q}$. Next, we aim to prove that

$$V\big(t,x(t)\big) < \mu\overline{\xi}\|\rho\|_{\hat{\omega}}\theta^k, \quad t \in [t_k,t_{k+1}),\ k \in \mathbb{N}. \tag{12}$$

The proof is divided into two steps based on mathematical derivations.

(i) For the time interval $t \in [0,t_1)$, we verify that (12) holds.

At $t=0$, it is evident that $V(t,x(t)) < \mu\overline{\xi}\|\rho\|_{\hat{\omega}}$. Next, we will demonstrate that

$$V\big(t,x(t)\big) < \mu\overline{\xi}\|\rho\|_{\hat{\omega}}, \quad t \in (0,t_1).$$

Supposed that there exists a $t \in (0,t_1)$ such that $V(t,x(t)) \geqslant \mu\overline{\xi}\|\rho\|_{\hat{\omega}}$. Set

$$t^* = \inf\big\{t \in (0,t_1)\colon V\big(t,x(t)\big) \geqslant \mu\overline{\xi}\|\rho\|_{\hat{\omega}}\big\}$$

and

$$t^{\circ} = \sup\big\{t \in [0,t^*)\colon V\big(t,x(t)\big) \leqslant \overline{\xi}\|\rho\|_{\hat{\omega}}\big\}.$$

Given that $V(t,x(t))$ is continuous on $t \in [0,t_1)$, we obtain $V(t^*,x(t^*)) = \mu\overline{\xi}\|\rho\|_{\hat{\omega}}$ and $V(t^{\circ},x(t^{\circ})) = \overline{\xi}\|\rho\|_{\hat{\omega}}$. Consequently, for any $t \in [t^{\circ},t^*]$, it yields that

$$V\big(t+\delta,\ x(t+\delta)\big) \leqslant \mu V\big(t,x(t)\big), \quad \delta \in [-\hat{\omega},0]. \tag{13}$$

Taking the derivative of $V(t,x(t))$ with respect to t, $t \in [t^{\circ},t^*]$, for any positive constant β_i, one can derive from (8) and (13)

$$\begin{aligned}
\dot{V}\big(t,x(t)\big) &= \mathrm{e}^{\alpha t}\big[\dot{\xi}_i^{\top}(t)x(t)+\xi_i^{\top}(t)\big(A_ix(t)+B_ix\big(t-\omega(t)\big)\big)+\alpha\xi_i^{\top}(t)x(t)\big]\\
&\leqslant \mathrm{e}^{\alpha t}\big[\dot{\xi}_i^{\top}(t)+\xi_i^{\top}(t)A_i+\alpha\xi_i^{\top}(t)\big]x(t)+\mathrm{e}^{\alpha t}\xi_i^{\top}(t)B_ix\big(t-\omega(t)\big)\\
&\quad+\beta_i\big[\mu V\big(t,x(t)\big)-V\big(t-\omega(t),\ x\big(t-\omega(t)\big)\big)\big]\\
&\leqslant \mathrm{e}^{\alpha t}\big[\dot{\xi}_i^{\top}(t)+\xi_i^{\top}(t)A_i+\alpha\xi_i^{\top}(t)\big]x(t)+\mathrm{e}^{\alpha t}\xi_i^{\top}(t)B_ix\big(t-\omega(t)\big)\\
&\quad+\beta_i\mathrm{e}^{\alpha t}\big[\mu\xi_i(t)x(t)-\mathrm{e}^{-\alpha\hat{\omega}}\xi_j\big(t-\omega(t)\big)x\big(t-\omega(t)\big)\big]\\
&\leqslant \mathrm{e}^{\alpha t}\big[\dot{\xi}_i^{\top}(t)+\xi_i^{\top}(t)A_i+(\alpha+\beta_i\mu)\xi_i^{\top}(t)\big]x(t)\\
&\quad+\mathrm{e}^{\alpha t}\big[\xi_i^{\top}(t)B_i-\beta_i\mathrm{e}^{-\alpha\hat{\omega}}\xi^*\big]x\big(t-\omega(t)\big).
\end{aligned} \tag{14}$$

By (2)–(4) and (14), we can infer that

$$\begin{aligned}\dot V\bigl(t,x(t)\bigr) \leqslant \mathrm{e}^{\alpha t}\biggl[& f(t)g(t)\biggl(\frac{L}{\tau_{i2}}\bigl(\xi_{i,q+1}^{\top}-\xi_{i,q}^{\top}\bigr)+\xi_{i,q+1}^{\top}A_i+(\alpha+\beta_i\mu)\xi_{i,q+1}^{\top}\biggr)\\
&+f(t)\tilde g(t)\biggl(\frac{L}{\tau_{i1}}\bigl(\xi_{i,q+1}^{\top}-\xi_{i,q}^{\top}\bigr)+\xi_{i,q+1}^{\top}A_i+(\alpha+\beta_i\mu)\xi_{i,q+1}^{\top}\biggr)\\
&+\tilde f(t)g(t)\biggl(\frac{L}{\tau_{i2}}\bigl(\xi_{i,q+1}^{\top}-\xi_{i,q}^{\top}\bigr)+\xi_{i,q}^{\top}A_i+(\alpha+\beta_i\mu)\xi_{i,q}^{\top}\biggr)\\
&+\tilde f(t)\tilde g(t)\biggl(\frac{L}{\tau_{i1}}\bigl(\xi_{i,q+1}^{\top}-\xi_{i,q}^{\top}\bigr)+\xi_{i,q}^{\top}A_i+(\alpha+\beta_i\mu)\xi_{i,q}^{\top}\biggr)\biggr]x(t)\\
&+\mathrm{e}^{\alpha t}\bigl[f(t)\bigl(\xi_{i,q+1}^{\top}B_i-\beta_i\mathrm{e}^{-\alpha\hat\omega}\xi^{*}\bigr)\\
&+\tilde f(t)\bigl(\xi_{i,q}^{\top}B_i-\beta_i\mathrm{e}^{-\alpha\hat\omega}\xi^{*}\bigr)\bigr]x\bigl(t-\omega(t)\bigr).\end{aligned}$$

From (7) and (11) it follows that

$$\dot V\bigl(t,x(t)\bigr)<0,\quad t\in[t^{\circ},t^{*}],$$

which means that $V(t^*,x(t^*)) < V(t^{\circ},x(t^{\circ}))$. This leads to a contradiction, implying that (12) holds for $t\in[0,t_1)$.

(ii) Within the time interval $t\in[t_k,t_{k+1})$, we illustrate that (12) is true.

Suppose (12) is true for $[t_{k-1},t_k)$, where $k=1,2,\dots,r$. Then we proceed to demonstrate that (12) remains valid on $[t_r,t_{r+1})$. According to (9), we have

$$V\bigl(t_r,x(t_r)\bigr)\leqslant\theta V\bigl(t_r^{-},\,x(t_r^{-})\bigr)<\mu\bar\xi\|\rho\|_{\hat\omega}\theta^{r}.$$

Assume that there exists $t\in(t_r,t_{r+1})$ such that $V(t,x(t))\geqslant\mu\bar\xi\|\rho\|_{\hat\omega}\theta^{r}$, and let

$$t^{*}=\inf\bigl\{t\in(t_r,t_{r+1})\colon\ V\bigl(t,x(t)\bigr)\geqslant\mu\bar\xi\|\rho\|_{\hat\omega}\theta^{r}\bigr\}$$

and

$$t^{\circ}=\sup\bigl\{t\in[t_r,t^{*})\colon\ V\bigl(t,x(t)\bigr)\leqslant\bar\xi\|\rho\|_{\hat\omega}\theta^{r}\bigr\}.$$

Note that if $\{t\in[t_r,t^*)\colon\ V(t,x(t))\leqslant\bar\xi\|\rho\|_{\hat\omega}\theta^{r}\}=\emptyset$, we set $t^{\circ}=t_r$. Therefore, it is evident that

$$V(t+\delta,\,x(t+\delta))\leqslant\mu V\bigl(t,x(t)\bigr),\quad t\in[t^{\circ},t^{*}],\ \delta\in[-\hat\omega,0).$$

Similar to the case $t\in[0,t_1)$, by combining (7), (8), and (11), we obtain $\dot V(t,x(t))<0$ for $t\in[t^{\circ},t^{*}]\subset[t_r,t_{r+1})$, which implies that $V(t^*,x(t^*))<V(t^{\circ},x(t^{\circ}))$. This leads to a contradiction, suggesting that (12) holds for $t\in[t_r,t_{r+1})$.

Consequently, $V(t,x(t))<\mu\bar\xi\|\rho\|_{\hat\omega}\theta^{k}$ holds on $t\in[t_k,t_{k+1})$, $k\in\mathbb{N}$, by mathematical induction.

Finally, we prove that system (5) is exponentially stable. The definition of $V(t,x(t))$ implies that
$$V\big(t,x(t)\big) \geqslant \underline{\xi}\mathrm{e}^{\alpha t}\big\|x(t)\big\|.$$
From (12) it follows that
$$\begin{aligned}\underline{\xi}\mathrm{e}^{\alpha t}\big\|x(t)\big\| &\leqslant \mu\bar{\xi}\theta^k\|\rho\|_{\hat{\omega}} = \mu\bar{\xi}\mathrm{e}^{k\ln\theta}\|\rho\|_{\hat{\omega}}\\ &\leqslant \mu\bar{\xi}\mathrm{e}^{t\ln(\theta)/\tau_1}\|\rho\|_{\hat{\omega}}.\end{aligned}$$
We then derive that
$$\big\|x(t)\big\| \leqslant p_0\mathrm{e}^{(\ln(\theta)/\tau_1-\alpha)t}\|\rho\|_{\hat{\omega}},$$
where $p_0 = \mu\bar{\xi}/\underline{\xi}$. It is straightforward to see that system (5) is exponentially stable. This completes the proof. □

Remark 1. L is a given positive integer, and as L increases, the conclusions become less conservative, whereas the computational effort required for the analysis increases. The comparison in conservativeness for different values of L can be observed in Example 1.

Remark 2. Theorem 1 addresses the case of nondecreasing switching behavior, which means that the Lyapunov function remains constant or increases following a switch.

When $\tau_{i2} \equiv \tau_2$ and $\tau_{i1} \equiv \tau_1$, the MDIDT switching signal is simplified to the interval dwell-time (IDT) switching signal. Subsequently, we will study the stability criterion for system (5) under IDT switching signal.

Corollary 1. *Given an integer $L > 0$, suppose there exist vectors $\xi_{i,q} \succ 0$, $\xi_{i,q+1} \succ 0$ for $q \in \mathbb{L}^-$ and $i \in \mathcal{P}$, a vector $\xi^* \succ 0$, and constants $\beta_i > 0$, $\theta \geqslant 1$, $\alpha > 0$ such that*
$$\begin{aligned}&\frac{L}{\tau_1}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q}^\top A_i + (\beta_i+\alpha)\xi_{i,q}^\top \prec 0,\\ &\frac{L}{\tau_2}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q}^\top A_i + (\beta_i+\alpha)\xi_{i,q}^\top \prec 0,\\ &\frac{L}{\tau_1}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q+1}^\top A_i + (\beta_i+\alpha)\xi_{i,q+1}^\top \prec 0,\\ &\frac{L}{\tau_2}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q+1}^\top A_i + (\beta_i+\alpha)\xi_{i,q+1}^\top \prec 0,\end{aligned} \tag{15}$$
$$\xi_{i,q}^\top B_i - \beta_i\xi^* \prec 0, \quad q \in \mathbb{L}, \tag{16}$$
$$\xi^* \preccurlyeq \xi_{i,q}, \quad q \in \mathbb{L}, \tag{17}$$
$$\xi_{j,0} \preccurlyeq \theta\xi_{i,L}, \quad i \neq j,\ j \in \mathcal{P}. \tag{18}$$
Then system (5) *is exponentially stable under the MDIDT switching signal with $\tau_1 > \ln(\theta)/\alpha$.*

Next, we consider the case of decreasing switching behavior.

Theorem 2. *Given an integer $L > 0$, suppose there exist vectors $\xi_{i,q} \succ 0$, $\xi_{i,q+1} \succ 0$ for $q \in \mathbb{L}^-$ and $i \in \mathcal{P}$, a vector $\xi^* \succ 0$, and constants $\beta_i > 0$, $0 < \theta < 1$, $\alpha > 0$ such that*

$$\begin{aligned} &\frac{L}{\tau_{i1}}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q}^\top A_i + (\beta_i - \alpha)\xi_{i,q}^\top \prec 0,\\ &\frac{L}{\tau_{i2}}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q}^\top A_i + (\beta_i - \alpha)\xi_{i,q}^\top \prec 0,\\ &\frac{L}{\tau_{i1}}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q+1}^\top A_i + (\beta_i - \alpha)\xi_{i,q+1}^\top \prec 0,\\ &\frac{L}{\tau_{i2}}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q+1}^\top A_i + (\beta_i - \alpha)\xi_{i,q+1}^\top \prec 0, \end{aligned} \tag{19}$$

$$\xi_{i,q}^\top B_i - \beta_i \xi^* \prec 0, \quad q \in \mathbb{L}, \tag{20}$$

$$\xi^* \preccurlyeq \xi_{i,q}, \quad q \in \mathbb{L}, \tag{21}$$

$$\xi_{j,0} \preccurlyeq \theta \xi_{i,L}, \quad i \neq j,\ j \in \mathcal{P}.$$

Then system (5) *is exponentially stable under the MDIDT switching signal with $\tau_2 > -\ln(\theta)/\alpha$, where $\tau_2 = \max_{i\in\mathcal{P}}\{\tau_{i2}\}$.*

Proof. For $t \in [t_k, t_{k+1})$, as $\sigma(t) = i$, we construct a copositive time-varying Lyapunov function as follows:

$$V_i\big(t, x(t)\big) = \mathrm{e}^{-\alpha t}\xi_i^\top(t)x(t), \tag{22}$$

where α is a positive constant. From (19) and (20) there exists $0 < \mu < 1$ such that

$$\begin{aligned} &\frac{L}{\tau_{i1}}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q}^\top A_i + \left(\beta_i\frac{1}{\mu} - \alpha\right)\xi_{i,q}^\top \prec 0,\\ &\frac{L}{\tau_{i2}}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q}^\top A_i + \left(\beta_i\frac{1}{\mu} - \alpha\right)\xi_{i,q}^\top \prec 0,\\ &\frac{L}{\tau_{i1}}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q+1}^\top A_i + \left(\beta_i\frac{1}{\mu} - \alpha\right)\xi_{i,q+1}^\top \prec 0,\\ &\frac{L}{\tau_{i2}}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q+1}^\top A_i + \left(\beta_i\frac{1}{\mu} - \alpha\right)\xi_{i,q+1}^\top \prec 0. \end{aligned} \tag{23}$$

There exist two positive constants $\overline{\xi}$ and $\underline{\xi}$ such that $\overline{\xi}\mathbf{1} \succcurlyeq \xi_{i,q}$ and $\underline{\xi}\mathbf{1} \preccurlyeq \xi_{i,q}$ for all $i \in \mathcal{P}$ and $q \in \mathbb{L}$ with $\mathbf{1} = (1, 1, \dots, 1)^\top$ being a vector of the same dimension as $\xi_{i,q}$. Next, we aim to prove that

$$V\big(t, x(t)\big) < \frac{1}{\mu}\overline{\xi}\|\rho\|_{\hat{\omega}}\theta^k, \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{N}. \tag{24}$$

The proof is divided into two steps based on mathematical derivations.

(i) For the time interval $t \in [0, t_1)$, we verify that (24) is true.

When $t = 0$, $V(t, x(t)) < \overline{\xi}\|\rho\|_{\hat{\omega}}/\mu$ holds. Next, we provide a proof that

$$V\big(t, x(t)\big) < \frac{1}{\mu}\overline{\xi}\|\rho\|_{\hat{\omega}}, \quad t \in (0, t_1).$$

It is assumed that there exists a $t \in (0, t_1)$ such that $V(t, x(t)) \geqslant \overline{\xi}\|\rho\|_{\hat{\omega}}/\mu$, and we set

$$t^* = \inf\left\{t \in (0, t_1)\colon\ V(t, x(t)) \geqslant \frac{1}{\mu}\overline{\xi}\|\rho\|_{\hat{\omega}}\right\}$$

and

$$t^\circ = \sup\bigl\{t \in [0, t^*)\colon\ V(t, x(t)) \leqslant \overline{\xi}\|\rho\|_{\hat{\omega}}\bigr\}.$$

As the function $V(t, x(t))$ is continuous on $[0, t_1)$, we have $V(t^*, x(t^*)) = \overline{\xi}\|\rho\|_{\hat{\omega}}/\mu$ and $V(t^\circ, x(t^\circ)) = \overline{\xi}\|\rho\|_{\hat{\omega}}$. Hence, for any $t \in [t^\circ, t^*]$, we arrive at

$$V\bigl(t + \delta,\, x(t + \delta)\bigr) \leqslant \frac{1}{\mu}V\bigl(t, x(t)\bigr), \quad \delta \in [-\hat{\omega}, 0]. \tag{25}$$

Taking the derivative of $V(t, x(t))$, $t \in [t^\circ, t^*]$, for any $\beta_i > 0$, we can derive the following from (21) and (25):

$$\begin{aligned}
\dot{V}\bigl(t, x(t)\bigr) &= \mathrm{e}^{-\alpha t}\bigl[\dot{\xi}_i^\top(t)x(t) + \xi_i^\top(t)\bigl(A_i x(t) + B_i x\bigl(t - \omega(t)\bigr)\bigr) - \alpha\xi_i^\top(t)x(t)\bigr] \\
&\leqslant \mathrm{e}^{-\alpha t}\bigl[\dot{\xi}_i^\top(t) + \xi_i^\top(t)A_i - \alpha\xi_i^\top(t)\bigr]x(t) + \mathrm{e}^{-\alpha t}\xi_i^\top(t)B_i x\bigl(t - \omega(t)\bigr) \\
&\quad + \beta_i\left[\frac{1}{\mu}V\bigl(t, x(t)\bigr) - V\bigl(t - \omega(t),\, x\bigl(t - \omega(t)\bigr)\bigr)\right] \\
&\leqslant \mathrm{e}^{-\alpha t}\bigl[\dot{\xi}_i^\top(t) + \xi_i^\top(t)A_i - \alpha\xi_i^\top(t)\bigr]x(t) + \mathrm{e}^{-\alpha t}\xi_i^\top(t)B_i x\bigl(t - \omega(t)\bigr) \\
&\quad + \beta_i\mathrm{e}^{-\alpha t}\left[\frac{1}{\mu}\xi_i(t)x(t) - \xi_j\bigl(t - \omega(t)\bigr)x\bigl(t - \omega(t)\bigr)\right] \\
&\leqslant \mathrm{e}^{-\alpha t}\left[\dot{\xi}_i^\top(t) + \xi_i^\top(t)A_i + \left(\beta_i\frac{1}{\mu} - \alpha\right)\xi_i^\top(t)\right]x(t) \\
&\quad + \mathrm{e}^{-\alpha t}\bigl[\xi_i^\top(t)B_i - \beta_i\xi^*\bigr]x\bigl(t - \omega(t)\bigr).
\end{aligned} \tag{26}$$

In light of (2)–(4) and (26), the following is established:

$$\begin{aligned}
\dot{V}\bigl(t, x(t)\bigr) \leqslant{}& \mathrm{e}^{-\alpha t}\left[f(t)g(t)\left(\frac{L}{\tau_{i2}}\bigl(\xi_{i,q+1}^\top - \xi_{i,q}^\top\bigr) + \xi_{i,q+1}^\top A_i + \left(\beta_i\frac{1}{\mu} - \alpha\right)\xi_{i,q+1}^\top\right)\right. \\
&+ f(t)\tilde{g}(t)\left(\frac{L}{\tau_{i1}}\bigl(\xi_{i,q+1}^\top - \xi_{i,q}^\top\bigr) + \xi_{i,q+1}^\top A_i + \left(\beta_i\frac{1}{\mu} - \alpha)\xi_{i,q+1}^\top\right)\right. \\
&+ \tilde{f}(t)g(t)\left(\frac{L}{\tau_{i2}}\bigl(\xi_{i,q+1}^\top - \xi_{i,q}^\top\bigr) + \xi_{i,q}^\top A_i + \left(\beta_i\frac{1}{\mu} - \alpha)\xi_{i,q}^\top\right)\right. \\
&\left.+ \tilde{f}(t)\tilde{g}(t)\left(\frac{L}{\tau_{i1}}\bigl(\xi_{i,q+1}^\top - \xi_{i,q}^\top\bigr) + \xi_{i,q}^\top A_i + \left(\beta_i\frac{1}{\mu} - \alpha\right)\xi_{i,q}^\top\right)\right]x(t) \\
&+ \mathrm{e}^{-\alpha t}\bigl[f(t)\bigl(\xi_{i,q+1}^\top B_i - \beta_i\xi^*\bigr) \\
&+ \tilde{f}(t)\bigl(\xi_{i,q}^\top B_i - \beta_i\xi^*\bigr)\bigr]x\bigl(t - \omega(t)\bigr).
\end{aligned}$$

By combining (20) and (23), we obtain that

$$\dot{V}(t, x(t) < 0, \quad t \in [t^\circ, t^*],$$

which means that $V(t^*, x(t^*)) < V(t^\circ, x(t^\circ))$. This leads to a contradiction, implying that (12) holds for $t \in [0, t_1)$. A contradiction arises, implying that (24) holds for $t \in [0, t_1)$.

(ii) Within the time interval $t \in [t_k, t_{k+1})$, we illustrate that (24) is true.

Assuming that (25) holds on $[t_{k-1}, t_k)$, $k = 1, 2, \ldots, r$, we next show that (24) also holds on $[t_r, t_{r+1})$. From (21) we have

$$V\bigl(t_r, x(t_r)\bigr) \leqslant \theta V\bigl(t_r^-, x(t_r^-)\bigr) < \frac{1}{\mu}\overline{\xi}\|\rho\|_{\hat{\omega}}\theta^r.$$

Assume that there exists $t \in (t_r, t_{r+1})$ such that $V(t, x(t)) \geqslant \overline{\xi}\|\rho\|_{\hat{\omega}}\theta^r/\mu$, and let

$$t^* = \inf\biggl\{t \in (t_r, t_{r+1})\colon V\bigl(t, x(t)\bigr) \geqslant \frac{1}{\mu}\overline{\xi}\|\rho\|_{\hat{\omega}}\theta^r\biggr\}$$

and

$$t^\circ = \sup\bigl\{t \in [t_r, t^*)\colon V\bigl(t, x(t)\bigr) \leqslant \overline{\xi}\|\rho\|_{\hat{\omega}}\theta^r\bigr\}.$$

Note that if $\{t \in [t_r, t^*)\colon V(t, x(t)) \leqslant \overline{\xi}\|\rho\|_{\hat{\omega}}\theta^r\} = \emptyset$, we set $t^\circ = t_r$. Hence, it yields that

$$V\bigl(t + \delta, x(t + \delta)\bigr) \leqslant \frac{1}{\mu}V\bigl(t, x(t)\bigr), \quad t \in [t^\circ, t^*],\ \delta \in [-\hat{\omega}, 0).$$

As in the scenario with $t \in [0, t_1)$, based on (20), (21), and (23), we can conclude that $\dot{V}(t, x(t)) < 0$, $t \in [t^\circ, t^*] \subset [t_r, t_{r+1})$, which means that $V(t^*, x(t^*)) < V(t^\circ, x(t^\circ))$. This is a contradiction, which implies that (24) holds for $t \in [t_r, t_{r+1})$.

Consequently, $V(t, x(t)) < \overline{\xi}\|\rho\|_{\hat{\omega}}\theta^k/\mu$ holds on $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, by mathematical induction.

Finally, we prove that system (5) is exponentially stable. The definition of $V(t, x(t))$ implies that

$$V\bigl(t, x(t)\bigr) \geqslant \underline{\xi}\mathrm{e}^{-\alpha t}\bigl\|x(t)\bigr\|.$$

Then the following can be inferred from (24):

$$\underline{\xi}\mathrm{e}^{-\alpha t}\bigl\|x(t)\bigr\| \leqslant \frac{1}{\mu}\overline{\xi}\theta^k\|\rho\|_{\hat{\omega}} = \frac{1}{\mu}\overline{\xi}\mathrm{e}^{k\ln\theta}\|\rho\|_{\hat{\omega}} \leqslant \frac{1}{\mu}\overline{\xi}\mathrm{e}^{t\ln(\theta)/\tau_2}\|\rho\|_{\hat{\omega}}.$$

We then derive that

$$\bigl\|x(t)\bigr\| \leqslant p_0\mathrm{e}^{(\ln(\theta)/\tau_2 + \alpha)t}\|\rho\|_{\hat{\omega}},$$

where $p_0 = (\overline{\xi}/\underline{\xi})/\mu$. It is straightforward to see that system (5) is exponentially stable. This completes the proof. □

Remark 3. Theorem 2 addresses the case of decreasing switching behavior, where switching leads to a reduction in the Lyapunov function, thereby exerting a positive influence on the stability of the system.

Remark 4. The conclusion demonstrated in Theorem 2 is independent of the time-varying delay $\omega(t)$. Hence, it can be applied to scenarios where $\omega(t)$ is unknown.

We will also study the stability condition for system (5) under IDT switching signal.

Corollary 2. *Given an integer $L > 0$, suppose there exist vectors $\xi_{i,q} \succ 0$, $\xi_{i,q+1} \succ 0$ for $q \in \mathbb{L}^-$ and $i \in \mathcal{P}$, a vector $\xi^* \succ 0$, and constants $\beta_i > 0$, $0 < \theta < 1$, $\alpha > 0$ such that*

$$\begin{aligned} &\frac{L}{\tau_1}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q}^\top A_i + (\beta_i - \alpha)\xi_{i,q}^\top \prec 0, \\ &\frac{L}{\tau_2}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q}^\top A_i + (\beta_i - \alpha)\xi_{i,q}^\top \prec 0, \\ &\frac{L}{\tau_1}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q+1}^\top A_i + (\beta_i - \alpha)\xi_{i,q+1}^\top \prec 0, \\ &\frac{L}{\tau_2}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q+1}^\top A_i + (\beta_i - \alpha)\xi_{i,q+1}^\top \prec 0, \end{aligned} \tag{27}$$

$$\xi_{i,q}^\top B_i - \beta_i \xi^* \prec 0, \quad q \in \mathbb{L}, \tag{28}$$

$$\xi^* \preccurlyeq \xi_{i,q}, \quad q \in \mathbb{L}, \tag{29}$$

$$\xi_{j,0} \preccurlyeq \theta \xi_{i,L}, \quad i \neq j,\ j \in \mathcal{P}. \tag{30}$$

Then system (5) *is exponentially stable under the IDT switching signal with $\tau_2 > -\ln(\theta)/\alpha$.*

Remark 5. In contrast to the majority of research [17–19, 28, 32], we do not specify a maximum dwell time τ_2 and a minimum dwell time τ_1. Considering that inequalities (15)–(18) ((27)–(30)) result in mutual constraints between τ_2 and τ_1, the stability region is the most intuitive method for characterizing the relationship between the maximum and minimum dwell times, as well as the admissible region for dwell time.

4 Positive stabilization

Assuming that system (1) is unstable without a control input, we aim to stabilize it using state feedback. The proposed state feedback controller is given by

$$u(t) = K_{\sigma(t)}(t)x(t),$$

where $K_i(t) \in \mathbb{R}^{n\times n}$ denotes the time-varying gain matrix for $i \in \mathcal{P}$. This controller transforms (1) into the closed-loop system

$$\begin{aligned} &\dot{x}(t) = \left(A_{\sigma(t)} + D_{\sigma(t)}K_{\sigma(t)}(t)\right)x(t) + B_{\sigma(t)}x\big(t - \omega(t)\big), \quad t \geqslant 0, \\ &x(\delta) = \rho(\delta), \quad \delta \in [-\hat{\omega}, 0], \end{aligned} \tag{31}$$

with matrices $A_i \in \mathbb{M}_n$, $B_i \in \mathbb{R}_+^{n\times n}$, and $D_i \in \mathbb{R}_+^{n\times n}$, where the rest of the symbol definitions are consistent with those in system (1).

Remark 6. Different from the stabilization control design for general systems, positive stabilization necessitates ensuring that the closed-loop system (31) remains positive for any $t \geqslant 0$.

First, we consider the issue of positive stabilization in the nondecreasing switching behavior.

Theorem 3. *Given an integer $L > 0$, suppose there exist vectors $\xi_{i,q} \succ 0$, $\xi_{i,q+1} \succ 0$, $\vartheta_i \succ 0$, $\varepsilon_{i,q}$, $\varepsilon_{i,q+1}$, $q \in \mathbb{L}^-$, $i \in \mathcal{P}$, a vector $\xi^* \succ 0$, and constants $\beta_i > 0$, $\theta \geqslant 1$, $\alpha > 0$, and $\hbar$ such that*

$$\xi_{i,q}^\top D_i \vartheta_i A_i + D_i \vartheta_i \varepsilon_{i,q}^\top + \hbar I \geqslant 0, \quad q \in \mathbb{L}, \tag{32}$$

$$\begin{aligned} &\frac{L}{\tau_{i1}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q}^\top A_i + (\beta_i + \alpha)\xi_{i,q}^\top + \varepsilon_{i,q}^\top \prec 0, \\ &\frac{L}{\tau_{i2}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q}^\top A_i + (\beta_i + \alpha)\xi_{i,q}^\top + \varepsilon_{i,q}^\top \prec 0, \\ &\frac{L}{\tau_{i1}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q+1}^\top A_i + (\beta_i + \alpha)\xi_{i,q+1}^\top + \varepsilon_{i,q+1}^\top \prec 0, \\ &\frac{L}{\tau_{i2}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q+1}^\top A_i + (\beta_i + \alpha)\xi_{i,q+1}^\top + \varepsilon_{i,q+1}^\top \prec 0, \end{aligned} \tag{33}$$

$$\begin{gathered} \xi_{i,q}^\top B_i - \beta_i \xi^* \prec 0, \quad q \in \mathbb{L}, \\ \xi^* \preccurlyeq \xi_{i,q}, \quad q \in \mathbb{L}, \\ \xi_{j,0} \preccurlyeq \theta \xi_{i,L}, \quad i \neq j,\ j \in \mathcal{P}. \end{gathered} \tag{34}$$

Then system (31) *is positive and exponentially stable under the MDIDT switching signal with $\tau_1 > \ln(\theta)/\alpha$, where $\tau_1 = \min_{i\in\mathcal{P}}\{\tau_{i1}\}$. Moreover, the controller can be designed as follows*:

$$u(t) = K_i(t)x(t) = \frac{\vartheta_i \varepsilon_i^\top(t)}{\xi_i^\top(t) D_i \vartheta_i} x(t), \quad i \in \mathcal{P}. \tag{35}$$

Proof. First, we verify that system (31) is positive. Based on (32), one has

$$\begin{aligned} &\xi_i^\top(t) D_i \vartheta_i A_i + D_i \vartheta_i \varepsilon_i^\top(t) + \hbar I \\ &\quad = f(t)\big(\xi_{i,q}^\top D_i \vartheta_i A_i + D_i \vartheta_i \varepsilon_{i,q}^\top + \hbar I\big) \\ &\qquad + \tilde{f}(t)\big(\xi_{i,q+1}^\top D_i \vartheta_i A_i + D_i \vartheta_i \varepsilon_{i,q+1}^\top + \hbar I\big) \\ &\quad \geqslant 0. \end{aligned} \tag{36}$$

From (36) we obtain

$$A_i + \frac{D_i \vartheta_i \varepsilon_i^\top(t)}{\xi_i^\top(t) D_i \vartheta_i} + \frac{\hbar}{\xi_i^\top(t) D_i \vartheta_i} I \geqslant 0.$$

By Lemma 2, $A_i + D_i \vartheta_i \varepsilon_i^\top(t)/(\xi_i^\top(t) D_i \vartheta_i)$, $i \in \mathcal{P}$, is a Metzler matrix, from which it can be deduced that $A_i + D_i K_i(t)$ is a Metzler matrix. According to Lemma 1, system (31) is positive.

Next, we prove that system (31) is exponentially stable. We construct a discretized time-varying Lyapunov function that is identical to (10). From (33) there exists $\mu > 1$

such that

$$\begin{aligned}
&\frac{L}{\tau_{i1}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q}^\top A_i + (\beta_i\mu + \alpha)\xi_{i,q}^\top + \varepsilon_{i,q}^\top \prec 0,\\
&\frac{L}{\tau_{i2}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q}^\top A_i + (\beta_i\mu + \alpha)\xi_{i,q}^\top + \varepsilon_{i,q}^\top \prec 0,\\
&\frac{L}{\tau_{i1}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q+1}^\top A_i + (\beta_i\mu + \alpha)\xi_{i,q+1}^\top + \varepsilon_{i,q+1}^\top \prec 0,\\
&\frac{L}{\tau_{i2}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q+1}^\top A_i + (\beta_i\mu + \alpha)\xi_{i,q+1}^\top + \varepsilon_{i,q+1}^\top \prec 0.
\end{aligned} \tag{37}$$

There exist two positive constants $\overline{\xi}$ and $\underline{\xi}$ such that $\overline{\xi}\mathbf{1} \succcurlyeq \xi_{i,q}$ and $\underline{\xi}\mathbf{1} \preccurlyeq \xi_{i,q}$ for all $i \in \mathcal{P}$ and $q \in \mathbb{L}$ with $\mathbf{1} = (1, 1, \dots, 1)^\top$ being a vector of the same dimension as $\xi_{i,q}$. Next, we aim to prove that

$$V\big(t, x(t)\big) < \mu\overline{\xi}\|\rho\|_{\hat{\omega}}\theta^k, \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{N}. \tag{38}$$

The proof is divided into two steps based on mathematical derivations.

(i) For the time interval $t \in [0, t_1)$, we verify that (38) holds.

At $t = 0$, it is evident that $V(t, x(t)) < \mu\overline{\xi}\|\rho\|_{\hat{\omega}}$. Next, we will demonstrate that

$$V\big(t, x(t)\big) < \mu\overline{\xi}\|\rho\|_{\hat{\omega}},\ t \in (0, t_1).$$

Supposed that there exists a $t \in (0, t_1)$ such that $V(t, x(t)) \geqslant \mu\overline{\xi}\|\rho\|_{\hat{\omega}}$. Set

$$t^* = \inf\big\{t \in (0, t_1)\colon V(t, x(t)) \geqslant \mu\overline{\xi}\|\rho\|_{\hat{\omega}}\big\}$$

and

$$t^\circ = \sup\big\{t \in [0, t^*)\colon V(t, x(t)) \leqslant \overline{\xi}\|\rho\|_{\hat{\omega}}\big\}.$$

Given that $V(t, x(t))$ is continuous on $t \in [0, t_1)$, we obtain $V(t^*, x(t^*)) = \mu\overline{\xi}\|\rho\|_{\hat{\omega}}$ and $V(t^\circ, x(t^\circ)) = \overline{\xi}\|\rho\|_{\hat{\omega}}$. Consequently, for any $t \in [t^\circ, t^*]$, it yields that

$$V\big(t + \delta,\ x(t + \delta)\big) \leqslant \mu V\big(t, x(t)\big), \quad \delta \in [-\hat{\omega}, 0].$$

By differentiating $V(t, x(t))$ with respect to t, $t \in [t^\circ, t^*]$, and considering any positive constant β_i, we have

$$\begin{aligned}
&\dot{V}\big(t, x(t)\big)\\
&\quad = \mathrm{e}^{\alpha t}\big[\dot{\xi}_i^\top(t)x(t) + \xi_i^\top(t)\big(\big(A_i + D_iK_i(t)\big)x(t) + B_ix\big(t - \omega(t)\big)\big) + \alpha\xi_i^\top(t)x(t)\big]\\
&\quad = \mathrm{e}^{\alpha t}\big[\dot{\xi}_i^\top(t) + \xi_i^\top(t)A_i + \varepsilon_i(t) + \alpha\xi_i^\top(t)\big]x(t) + \mathrm{e}^{\alpha t}\xi_i^\top(t)B_ix\big(t - \omega(t)\big)\\
&\quad \leqslant \mathrm{e}^{\alpha t}\big[\dot{\xi}_i^\top(t) + \xi_i^\top(t)A_i + \varepsilon_i(t) + \alpha\xi_i^\top(t)\big]x(t) + \mathrm{e}^{\alpha t}\xi_i^\top(t)B_ix\big(t - \omega(t)\big)\\
&\qquad + \beta_i\big[\mu V\big(t, x(t)\big) - V\big(t - \omega(t),\ x\big(t - \omega(t)\big)\big)\big]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \mathrm{e}^{\alpha t}\big[\dot{\xi}_i^\top(t)+\xi_i^\top(t)A_i+\varepsilon_i(t)+\alpha\xi_i^\top(t)\big]x(t)+\mathrm{e}^{\alpha t}\xi_i^\top(t)B_i x\big(t-\omega(t)\big)\\
&\quad+\beta_i\mathrm{e}^{\alpha t}\big[\mu\xi_i(t)x(t)-\mathrm{e}^{-\alpha\hat{\omega}}\xi_j\big(t-\omega(t)\big)x\big(t-\omega(t)\big)\big]\\
&\leqslant \mathrm{e}^{\alpha t}\big[\dot{\xi}_i^\top(t)+\xi_i^\top(t)A_i+\varepsilon_i(t)+(\alpha+\beta_i\mu)\xi_i^\top(t)\big]x(t)\\
&\quad+\mathrm{e}^{\alpha t}\big[\xi_i^\top(t)B_i-\beta_i\mathrm{e}^{-\alpha\hat{\omega}}\xi^*\big]x\big(t-\omega(t)\big). \qquad (39)
\end{aligned}
$$

From (2)–(4) and (39) we get that

$$
\begin{aligned}
&\dot{V}\big(t,x(t)\big)\\
&\quad\leqslant \mathrm{e}^{\alpha t}\Bigg[f(t)g(t)\bigg(\frac{L}{\tau_{i2}}\big(\xi_{i,q+1}^\top-\xi_{i,q}^\top\big)+\xi_{i,q+1}^\top A_i+(\alpha+\beta_i\mu)\xi_{i,q+1}^\top+\varepsilon_{i,q+1}^\top\bigg)\\
&\qquad+f(t)\tilde{g}(t)\bigg(\frac{L}{\tau_{i1}}\big(\xi_{i,q+1}^\top-\xi_{i,q}^\top\big)+\xi_{i,q+1}^\top A_i+(\alpha+\beta_i\mu)\xi_{i,q+1}^\top+\varepsilon_{i,q+1}^\top\bigg)\\
&\qquad+\tilde{f}(t)g(t)\bigg(\frac{L}{\tau_{i2}}\big(\xi_{i,q+1}^\top-\xi_{i,q}^\top\big)+\xi_{i,q}^\top A_i+(\alpha+\beta_i\mu)\xi_{i,q}^\top+\varepsilon_{i,q}^\top\bigg)\\
&\qquad+\tilde{f}(t)\tilde{g}(t)\bigg(\frac{L}{\tau_{i1}}\big(\xi_{i,q+1}^\top-\xi_{i,q}^\top\bigg)+\xi_{i,q}^\top A_i+(\alpha+\beta_i\mu)\xi_{i,q}^\top+\varepsilon_{i,q}^\top\bigg)\Bigg]x(t)\\
&\qquad+\mathrm{e}^{\alpha t}\big[f(t)\big(\xi_{i,q+1}^\top B_i-\beta_i\mathrm{e}^{-\alpha\hat{\omega}}\xi^*\big)+\tilde{f}(t)\big(\xi_{i,q}^\top B_i-\beta_i\mathrm{e}^{-\alpha\hat{\omega}}\xi^*\big)\big]\\
&\qquad\times x\big(t-\omega(t)\big).
\end{aligned}
$$

From (34) and (37) we obtain

$$
\dot{V}\big(t,x(t)\big)<0,\quad t\in[t^\circ,t^*].
$$

The remaining steps are identical to those in the proof of Theorem 1 and are therefore omitted here. This completes the proof. □

Theorem 4. *Given an integer $L>0$, suppose there exist vectors $\xi_{i,q}\succ 0$, $\xi_{i,q+1}\succ 0$, $\vartheta_i\succ 0$, $\varepsilon_{i,q}$, $\varepsilon_{i,q+1}$, $q\in\mathbb{L}^-$, $i\in\mathcal{P}$, a vector $\xi^*\succ 0$, and constants $\beta_i>0$, $0<\theta<1$, $\alpha>0$, and $\hbar$ such that*

$$
\xi_{i,q}^\top D_i\vartheta_i A_i+D_i\vartheta_i\varepsilon_{i,q}^\top+\hbar I\geqslant 0,\quad q\in\mathbb{L}, \qquad (40)
$$

$$
\begin{aligned}
&\frac{L}{\tau_{i1}}\big(\xi_{i,q+1}^\top-\xi_{i,q}^\top\big)+\xi_{i,q}^\top A_i+\big(\beta_i-\alpha\big)\xi_{i,q}^\top+\varepsilon_{i,q}^\top\prec 0,\\
&\frac{L}{\tau_{i2}}\big(\xi_{i,q+1}^\top-\xi_{i,q}^\top\big)+\xi_{i,q}^\top A_i+\big(\beta_i-\alpha\big)\xi_{i,q}^\top+\varepsilon_{i,q}^\top\prec 0,\\
&\frac{L}{\tau_{i1}}\big(\xi_{i,q+1}^\top-\xi_{i,q}^\top\big)+\xi_{i,q+1}^\top A_i+\big(\beta_i-\alpha\big)\xi_{i,q+1}^\top+\varepsilon_{i,q+1}^\top\prec 0,\\
&\frac{L}{\tau_{i2}}\big(\xi_{i,q+1}^\top-\xi_{i,q}^\top\big)+\xi_{i,q+1}^\top A_i+(\beta_i-\alpha)\xi_{i,q+1}^\top+\varepsilon_{i,q+1}^\top\prec 0,
\end{aligned} \qquad (41)
$$

$$
\begin{gathered}
\xi_{i,q}^\top B_i-\beta_i\xi^*\prec 0,\quad q\in\mathbb{L}, \qquad (42)\\
\xi^*\preccurlyeq\xi_{i,q},\quad q\in\mathbb{L},\\
\xi_{j,0}\preccurlyeq\theta\xi_{i,L},\quad i\neq j,\ j\in\mathcal{P}.
\end{gathered}
$$

Then system (31) *is positive and exponentially stable under the MDIDT switching signal with* $\tau_2 > -\ln(\theta)/\alpha$*, where* $\tau_2 = \max_{i\in\mathcal{P}}\{\tau_{i2}\}$*. Moreover, the controller can be designed as follows*:

$$u(t) = K_i(t)x(t) = \frac{\vartheta_i \varepsilon_i^\top(t)}{\xi_i^\top(t) D_i \vartheta_i} x(t), \quad i \in \mathcal{P}. \tag{43}$$

Proof. Based on Theorem 3, it can be inferred that conditions (40) and (43) guarantee that system (31) is positive.

Next, we prove that system (31) is exponentially stable. We construct a discretized time-varying Lyapunov function that is identical to (22). From (41) there exists $0 < \mu < 1$ such that

$$\begin{aligned}
&\frac{L}{\tau_{i1}}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q}^\top A_i + \left(\beta_i \frac{1}{\mu} - \alpha\right)\xi_{i,q}^\top + \varepsilon_{i,q}^\top \prec 0,\\
&\frac{L}{\tau_{i2}}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q}^\top A_i + \left(\beta_i \frac{1}{\mu} - \alpha\right)\xi_{i,q}^\top + \varepsilon_{i,q}^\top \prec 0,\\
&\frac{L}{\tau_{i1}}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q+1}^\top A_i + \left(\beta_i \frac{1}{\mu} - \alpha\right)\xi_{i,q+1}^\top + \varepsilon_{i,q+1}^\top \prec 0,\\
&\frac{L}{\tau_{i2}}\left(\xi_{i,q+1}^\top - \xi_{i,q}^\top\right) + \xi_{i,q+1}^\top A_i + \left(\beta_i \frac{1}{\mu} - \alpha\right)\xi_{i,q+1}^\top + \varepsilon_{i,q+1}^\top \prec 0.
\end{aligned} \tag{44}$$

There exist two positive constants $\overline{\xi}$ and $\underline{\xi}$ such that $\overline{\xi}\mathbf{1} \succcurlyeq \xi_{i,q}$ and $\underline{\xi}\mathbf{1} \preccurlyeq \xi_{i,q}$ for all $i \in \mathcal{P}$ and $q \in \mathbb{L}$ with $\mathbf{1} = (1, 1, \dots, 1)^\top$ being a vector of the same dimension as $\xi_{i,q}$. Next, we aim to prove that

$$V\big(t, x(t)\big) < \frac{1}{\mu}\overline{\xi}\|\rho\|_{\hat{\omega}}\theta^k, \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{N}. \tag{45}$$

The proof is divided into two steps based on mathematical derivations.

(i) For the time interval $t \in [0, t_1)$, we verify that (45) is true.

When $t = 0$, $V(t, x(t)) < \overline{\xi}\|\rho\|_{\hat{\omega}}/\mu$ holds. Next, we provide a proof that

$$V\big(t, x(t)\big) < \frac{1}{\mu}\overline{\xi}\|\rho\|_{\hat{\omega}}, \quad t \in (0, t_1).$$

It is assumed that there exists a $t \in (0, t_1)$ such that $V(t, x(t)) \geqslant \overline{\xi}\|\rho\|_{\hat{\omega}}/\mu$, and we set

$$t^* = \inf\left\{t \in (0, t_1)\colon\ V\big(t, x(t)\big) \geqslant \frac{1}{\mu}\overline{\xi}\|\rho\|_{\hat{\omega}}\right\}$$

and

$$t^\circ = \sup\big\{t \in [0, t^*)\colon\ V\big(t, x(t)\big) \leqslant \overline{\xi}\|\rho\|_{\hat{\omega}}\big\}.$$

As the function $V(t, x(t))$ is continuous on $[0, t_1)$, we have $V(t^*, x(t^*)) = \overline{\xi}\|\rho\|_{\hat{\omega}}/\mu$ and $V(t^\circ, x(t^\circ)) = \overline{\xi}\|\rho\|_{\hat{\omega}}$. Hence, for any $t \in [t^\circ, t^*]$, we arrive at

$$V\big(t + \delta,\, x(t + \delta)\big) \leqslant \frac{1}{\mu}V\big(t, x(t)\big), \quad \delta \in [-\hat{\omega}, 0].$$

By differentiating $V(t, x(t))$ with respect to t, $t \in [t^\circ, t^*]$, and considering any positive constant β_i, we have

$$\begin{aligned}
\dot V\big(t, x(t)\big) &= \mathrm{e}^{-\alpha t}\big[\dot\xi_i^\top(t)x(t) + \xi_i^\top(t)\big(\big(A_i + D_iK_i(t)\big)x(t) + B_ix\big(t-\omega(t)\big)\big) \\
&\quad - \alpha\xi_i^\top(t)x(t)\big] \\
&= \mathrm{e}^{-\alpha t}\big[\dot\xi_i^\top(t)x(t) + \xi_i^\top(t)\big(A_ix(t) + \varepsilon_i(t) + B_ix\big(t-\omega(t)\big)\big) \\
&\quad - \alpha\xi_i^\top(t)x(t)\big] \\
&\leqslant \mathrm{e}^{-\alpha t}\big[\dot\xi_i^\top(t) + \xi_i^\top(t)A_i + \varepsilon_i(t) - \alpha\xi_i^\top(t)\big]x(t) \\
&\quad + \mathrm{e}^{-\alpha t}\xi_i^\top(t)B_ix\big(t-\omega(t)\big) \\
&\quad + \beta_i\left[\frac{1}{\mu}V\big(t, x(t)\big) - V\big(t-\omega(t), x\big(t-\omega(t)\big)\big)\right] \\
&\leqslant \mathrm{e}^{-\alpha t}\big[\dot\xi_i^\top(t) + \xi_i^\top(t)A_i + \varepsilon_i(t) - \alpha\xi_i^\top(t)\big]x(t) \\
&\quad + \mathrm{e}^{-\alpha t}\xi_i^\top(t)B_ix\big(t-\omega(t)\big) \\
&\quad + \beta_i\mathrm{e}^{-\alpha t}\left[\frac{1}{\mu}\xi_i(t)x(t) - \xi_j\big(t-\omega(t)\big)x\big(t-\omega(t)\big)\right] \\
&\leqslant \mathrm{e}^{-\alpha t}\left[\dot\xi_i^\top(t) + \xi_i^\top(t)A_i + \varepsilon_i(t) + \left(\beta_i\frac{1}{\mu} - \alpha\right)\xi_i^\top(t)\right]x(t) \\
&\quad + \mathrm{e}^{-\alpha t}\big[\xi_i^\top(t)B_i - \beta_i\xi^*\big]x\big(t-\omega(t)\big).
\end{aligned} \tag{46}$$

In light of (2)–(4) and (46), the following is established:

$$\begin{aligned}
&\dot V\big(t, x(t)\big) \\
&\quad\leqslant \mathrm{e}^{-\alpha t}\left[f(t)g(t)\left(\frac{L}{\tau_{i2}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q+1}^\top A_i + \left(\beta_i\frac{1}{\mu} - \alpha\right)\xi_{i,q+1}^\top + \varepsilon_{i,q+1}^\top\right)\right. \\
&\qquad + f(t)\tilde g(t)\left(\frac{L}{\tau_{i1}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q+1}^\top A_i + \left(\beta_i\frac{1}{\mu} - \alpha\right)\xi_{i,q+1}^\top + \varepsilon_{i,q+1}^\top\right) \\
&\qquad + \tilde f(t)g(t)\left(\frac{L}{\tau_{i2}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q}^\top A_i + \left(\beta_i\frac{1}{\mu} - \alpha\right)\xi_{i,q}^\top + \varepsilon_{i,q}^\top\right) \\
&\qquad \left. + \tilde f(t)\tilde g(t)\left(\frac{L}{\tau_{i1}}\big(\xi_{i,q+1}^\top - \xi_{i,q}^\top\big) + \xi_{i,q}^\top A_i + \left(\beta_i\frac{1}{\mu} - \alpha\right)\xi_{i,q}^\top + \varepsilon_{i,q}^\top\right)\right]x(t) \\
&\qquad + \mathrm{e}^{-\alpha t}\big[f(t)\big(\xi_{i,q+1}^\top B_i - \beta_i\xi^*\big) + \tilde f(t)(\xi_{i,q}^\top B_i - \beta_i\xi^*)\big]x\big(t-\omega(t)\big).
\end{aligned}$$

From (42) and (44) we get that

$$\dot V\big(t, x(t)\big) < 0, \quad t \in [t^\circ, t^*].$$

The remaining steps are identical to those in the proof of Theorem 2 and are therefore omitted here. This completes the proof. □

5 Numerical examples

In this section, several numerical examples are presented to verify the effectiveness of all the derived results.

Example 1. Consider system (5) with

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0.5 & -1.1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0.02 & 0.05 \\ 0 & 0.04 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -3 & 2 \\ 0.8 & -3 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.03 & 0.02 \\ 0.11 & 0.06 \end{bmatrix},$$

and $\omega(t) = 0.1 - 0.1 \sin t$. We deduce that $\hat{\omega} = 0.2$.

It is evident that both subsystems are stable as $A_1 + B_1$ and $A_2 + B_2$ are Hurwitz matrices. Given $L = 1$, $\theta = 1$, $\beta_1 = \beta_2 = 0.9$, $\alpha = 0.1$, we consider the case of the IDT switching signal. Set $\tau_1 = 0.77$, and we can obtain the feasible solutions that satisfy Corollary 1 by utilizing the SQP algorithm as follows:

$$\xi_{10} = \begin{bmatrix} 0.6232 \\ 1.4504 \end{bmatrix}, \qquad \xi_{11} = \begin{bmatrix} 0.5009 \\ 0.8766 \end{bmatrix}, \qquad \xi_{20} = \begin{bmatrix} 0.5009 \\ 0.8766 \end{bmatrix},$$
$$\xi_{21} = \begin{bmatrix} 0.7304 \\ 1.4504 \end{bmatrix}, \qquad \xi^* = \begin{bmatrix} 0.4735 \\ 0.8644 \end{bmatrix}, \qquad \tau_2 = 1.20.$$

The appropriate switching signal is selected as shown in Fig. 1, and the initial condition is set to $x(0) = [3, 4]^\top$. The state trajectory of the system under the given switching signal is depicted in Fig. 2.

Subsequently, by varying the value of the integer L, different stability regions are obtained using Algorithm 1, as illustrated in Fig. 3. In this figure, $\cup_{i=1}^{s} R_i$ represents the stability region of permissible dwell times when $L = s$, where $s \in \{1, 2, 3, 4, 5\}$. From Fig. 3 it can be observed that as the value of L increases, the range of the stability region expands, indicating a reduction in the conservativeness of the stability conclusion.

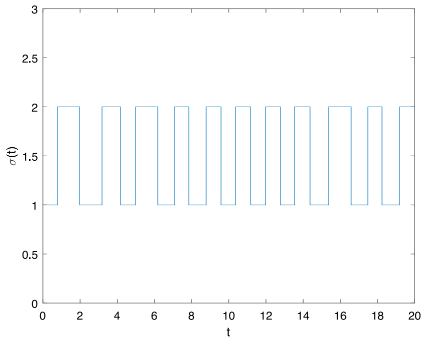

Figure 1. The design switching signal of system (5).

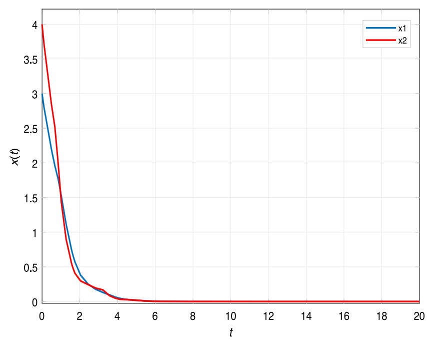

Figure 2. The state trajectories of system (5).

Algorithm 1. Stability region algorithm via IDT

Step 1: Input system matrices A_i, B_i, given an integer $L > 0$, and constants θ, β_i, α;

Step 2: Establish the objective function

$$\min_{\xi_{i,q},\,\xi^*} \tau_1 \quad \text{s.t. (15)}-\text{(18) ((27)}-\text{(30))},\ q \in \mathbb{L}.$$

Calculate τ_1 using the Sequential Quadratic Programming (SQP) algorithm to solve the optimization problem, then record τ_1;

Step 3: By treating τ_1 as a variable, plot the trajectory of τ_2 as it varies with τ_1 to obtain the stability region;

Step 4: Enter the parameters τ_1 and τ_2, solve (15)–(18) ((27)–(30)) and record $\xi_{i,q}$, ξ^* for $q \in \mathbb{L}$.

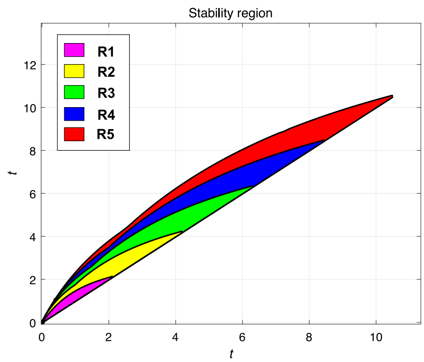

Figure 3. Stability region for admissible dwell time.

Example 2. Consider system (5) with

$$A_1 = \begin{bmatrix} 0.4 & 0.35 \\ 0 & -1.3 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0.01 & 0.05 \\ 0 & 0.03 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -1.4 & 0.2 \\ 0 & 0.33 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.2 & 0.12 \\ 0.06 & 0.01 \end{bmatrix},$$

and $\omega(t) = 0.1 - 0.1 \sin t$. We deduce that $\hat{\omega} = 0.2$.

It is evident that both subsystems are unstable, as $A_1 + B_1$ and $A_2 + B_2$ are not Hurwitz matrices. Given $L = 1$, $\theta = 0.8$, $\beta_1 = \beta_2 = 0.1$, $\alpha = 0.11$, and $\tau_1 = 0.95$, for convenience, we consider the case of IDT switching signal. We obtain vectors that satisfy Corollary 2 by utilizing the SQP algorithm as follows:

$$\xi_{10} = \begin{bmatrix} 0.1581 \\ 0.4720 \end{bmatrix}, \qquad \xi_{11} = \begin{bmatrix} 0.1019 \\ 1.0473 \end{bmatrix}, \qquad \xi_{20} = \begin{bmatrix} 0.0815 \\ 0.8378 \end{bmatrix},$$
$$\xi_{21} = \begin{bmatrix} 0.1976 \\ 0.5901 \end{bmatrix}, \qquad \xi^* = \begin{bmatrix} 0.0749 \\ 0.4205 \end{bmatrix}, \qquad \tau_2 = 1.18.$$

The appropriate switching signal is selected as shown in Fig. 4, and the initial condition is set to $x(0) = [2, 4]^\top$. The state trajectory of the system under the given switching signal is depicted in Fig. 5.

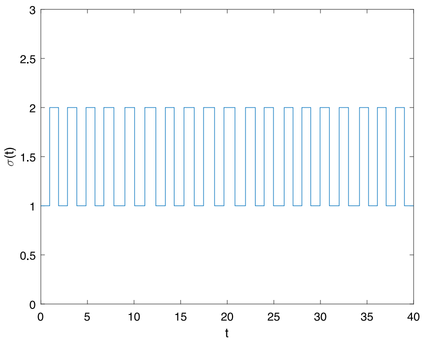

Figure 4. The design switching signal of system (5).

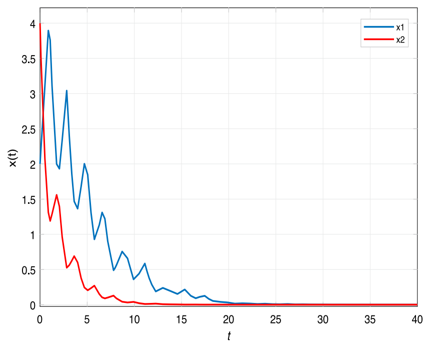

Figure 5. The state trajectories of system (5).

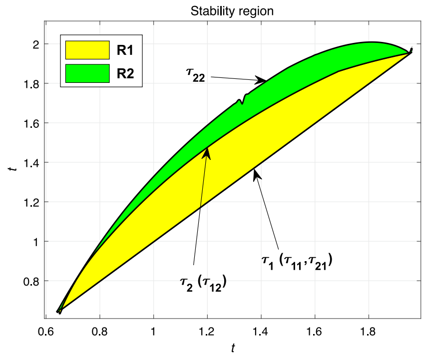

Figure 6. Stability region for admissible dwell time.

Finally, we demonstrate that Theorem 2 derived under the MDIDT switching signal is less conservative compared to Corollary 2 derived under the IDT switching signal. In Fig. 6, $R1$ denotes the stable domain of permissible dwell time under the IDT switching signal, and $R1 \cup R2$ denotes the stable domain of permissible dwell time under the MDIDT switching signal.

Example 3. Consider system (31) with the state feedback control as follows:

$$A_1 = \begin{bmatrix} 0.1 & 1 \\ 0.1 & -1.25 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0.01 & 0.02 \\ 0.02 & 0.01 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 1.5 \\ 0.8 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -1.1 & 1 \\ 1 & -0.8 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.02 & 0.03 \\ 0.03 & 0.02 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 1 \\ 1.2 \end{bmatrix},$$

and $\omega(t) = 0.1 - 0.1 \sin t$. We deduce that $\hat{\omega} = 0.2$.

It is evident that both subsystems are unstable without a control input, as $A_1 + B_1$ and $A_2 + B_2$ are not Hurwitz matrices. With initial values $x(0) = [2, 4]^\top$ and parameters $\tau_1 = 3$, $\tau_2 = 4$, the design of the switching signal without state feedback control is depicted in Fig. 7. The corresponding state trajectories are shown in Fig. 8, which suggest that the system is not exponentially stable.

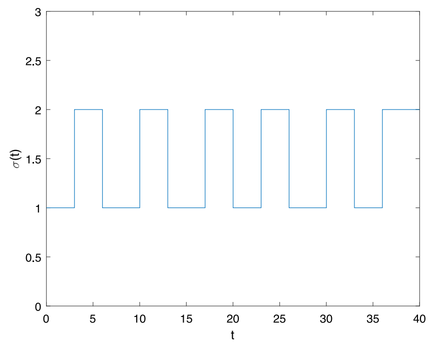

Figure 7. The design switching signal of system (31).

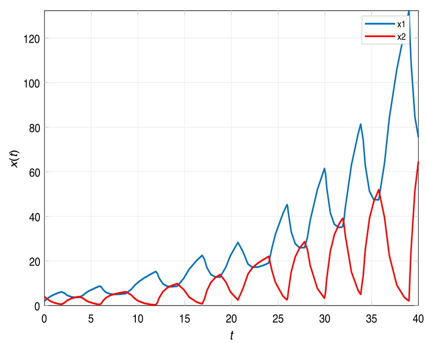

Figure 8. The state trajectories of system (31) without state feedback control.

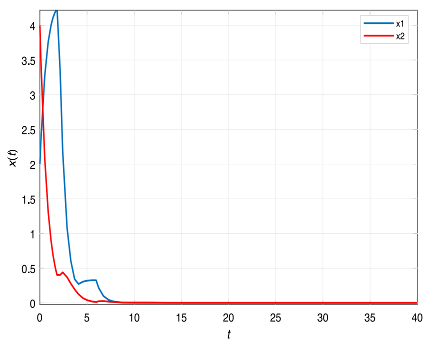

Figure 9. The state trajectories of system (31) with the state feedback control.

Utilizing the SQP algorithm in MATLAB, when $\vartheta_1 = \vartheta_2 = 1$, $\beta_1 = \beta_2 = 0.08$, $\alpha = 0.02$, $\theta = 2$, the following feasible solution are obtained:

$$\xi_{10} = \begin{bmatrix} 2.5986 \\ 2.1653 \end{bmatrix}, \qquad \xi_{11} = \begin{bmatrix} 0.9324 \\ 1.0169 \end{bmatrix}, \qquad \xi_{20} = \begin{bmatrix} 1.8122 \\ 2.0338 \end{bmatrix},$$

$$\xi_{21} = \begin{bmatrix} 1.2993 \\ 1.0827 \end{bmatrix}, \qquad \xi^* = \begin{bmatrix} 0.9315 \\ 1.0000 \end{bmatrix},$$

$$\varepsilon_{10} = \begin{bmatrix} -0.3531 \\ -0.2214 \end{bmatrix}, \quad \varepsilon_{11} = \begin{bmatrix} -0.2566 \\ -0.1541 \end{bmatrix}, \quad \varepsilon_{20} = \begin{bmatrix} -1.3711 \\ -1.2935 \end{bmatrix}, \quad \varepsilon_{21} = \begin{bmatrix} -1.5679 \\ -1.4463 \end{bmatrix}.$$

Besides, from (35) the state feedback gain matrix is designed as follows:

$$u(t) = K_{\sigma(t)}(t)x(t).$$

Under the same switching signal depicted in Fig. 7, Fig. 9 illustrates the system trajectories with the aforementioned state feedback control. It is evident that the unstable system can be positively stabilized through the designed state feedback control.

Example 4. Consider system (31) with the state feedback control as follows:

$$A_1 = \begin{bmatrix} 0.1 & 1 \\ 0.5 & -0.6 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0.05 & 0.01 \\ 0.02 & 0.03 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.6 & 0.5 \\ 1 & -0.5 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.02 & 0.02 \\ 0.02 & 0.02 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and $\omega(t) = 0.1 - 0.1\sin t$. We deduce that $\hat{\omega} = 0.2$.

It is evident that both subsystems are unstable in the absence of a control input, as $A_1 + B_1$ and $A_2 + B_2$ are not Hurwitz matrices. With initial values $x(0) = [2, 4]^\top$ and parameters $\tau_1 = 0.5$, $\tau_2 = 0.9$, the design of the switching signal without state feedback control is depicted in Fig, 10. The corresponding state trajectories are shown in Fig. 11, which indicates that the system is not exponentially stable.

Utilizing the SQP algorithm in MATLAB, when $L = 2$, $\vartheta_1 = \vartheta_2 = 1$, $\beta_1 = \beta_2 = 0.07$, $\alpha = 0.08$, $\theta = 0.93$, the following feasible solution are obtained:

$$\xi_{10} = \begin{bmatrix} 0.2452 \\ 0.1131 \end{bmatrix}, \qquad \xi_{11} = \begin{bmatrix} 0.2959 \\ 0.1184 \end{bmatrix}, \qquad \xi_{20} = \begin{bmatrix} 0.2751 \\ 0.1101 \end{bmatrix},$$

$$\xi_{21} = \begin{bmatrix} 0.2637 \\ 0.1217 \end{bmatrix}, \qquad \xi^* = \begin{bmatrix} 0.2452 \\ 0.1101 \end{bmatrix},$$

$$\varepsilon_{10} = \begin{bmatrix} -0.2816 \\ -0.2933 \end{bmatrix}, \quad \varepsilon_{11} = \begin{bmatrix} -0.3511 \\ -0.3307 \end{bmatrix}, \quad \varepsilon_{20} = \begin{bmatrix} -0.2476 \\ -0.1908 \end{bmatrix}, \quad \varepsilon_{21} = \begin{bmatrix} -0.2535 \\ -0.2024 \end{bmatrix}.$$

Besides, from (35) the state feedback gain matrix is designed as follows:

$$u(t) = K_{\sigma(t)}(t)x(t).$$

Under the same switching signal depicted in Fig. 10, Fig. 12 illustrates the system trajectories with the aforementioned state feedback control. It is evident that the unstable system can be positively stabilized through the designed state feedback control.

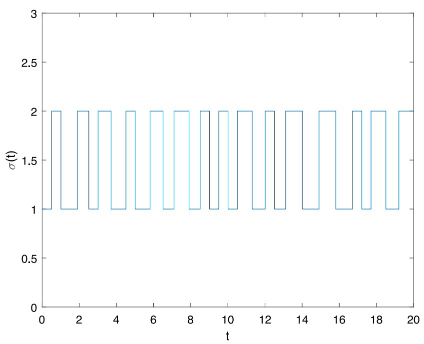

Figure 10. The design switching signal of system (31).

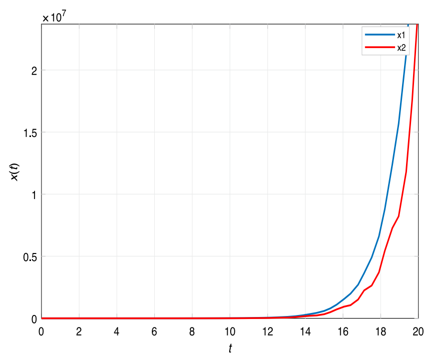

Figure 11. The state trajectories of system (31) without the state feedback control.

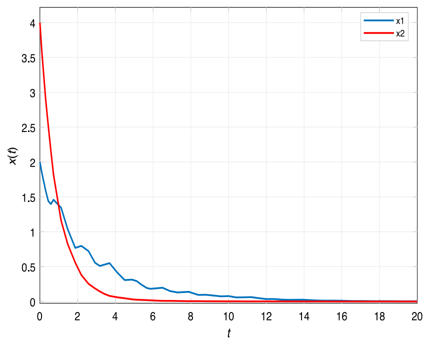

Figure 12. The state trajectories of system (31) with the state feedback control.

6 Conclusions

This paper addresses the issues of exponential stability and positive stabilization for positive switched system with time-varying delay. The switching behaviors include both nondecreasing switching and decreasing switching. Initially, sufficient conditions for the exponential stability of the system under the MDIDT switching signal are derived by utilizing the Lyapunov–Razumikhin technique and a discretized Lyapunov function method. Following this, the paper addresses the issue of positive stabilization for the system, using the established stability conditions as a foundation. Finally, two numerical examples are presented to verify the effectiveness of the derived conclusions.

Conflicts of interest. The authors declare no conflicts of interest.

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