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Positive solutions for a fourth-order integral boundary value problem involving impulses and the p-Laplacian*

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Abstract. In this paper, we investigate the existence of positive solutions for a fourth-order integral boundary value problem involving impulses and the p-Laplacian operator. First, we construct a linear operator that incorporates impulsive effects, and then analyze its spectral radius properties. Subsequently, under suitable conditions on the spectral radius, we apply the fixed point index theory to establish existence theorems for the problem. Our results cover cases where the nonlinear term exhibits $(p-1)$-superlinear/sublinear growth, and the impulsive term satisfies superlinear/sublinear growth conditions.

Keywords: fourth-order integral boundary value problem, positive solutions, fixed point index.

1 Introduction

In this paper, we will study the following fourth-order integral boundary value problem involving impulses and the p-Laplacian:

$$\begin{aligned}
&\bigl(\varphi_p\bigl(z''(t)\bigr)\bigr)'' = f\bigl(t, z(t)\bigr), \quad t \in J,\ t \neq t_k,\ k = 1, 2, \dots, m,\\
&\Delta z'|_{t=t_k} = -I_k\bigl(z(t_k)\bigr), \quad k = 1, 2, \dots, m,\\
&z(0) = z(1) = \int_0^1 z(t)\,\mathrm{d}\eta(t), \qquad \varphi_p\bigl(z''(0)\bigr) = \varphi_p\bigl(z''(1)\bigr) = \int_0^1 \varphi_p\bigl(z''(t)\bigr)\,\mathrm{d}\gamma(t),
\end{aligned} \tag{1}$$

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where $J = [0,1]$, t_k ($k = 1,2,\dots,m$, m is a given positive integer) are fixed points with $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$, $\varphi_p(t)$ is the p-Laplace operator, i.e., $\varphi_p(t) = |t|^{p-2}t$, $p > 1$, $\Delta z'|_{t=t_k} = z'(t_k^+) - z'(t_k^-)$, and $z'(t_k^+)$ and $z'(t_k^-)$ represent the right-hand limit and left-hand limit of $z'(t)$ at $t = t_k$, respectively. The functions f, I_k, η, γ satisfy the conditions:

(H_1) η, γ are nondecreasing and nonconstant functions on $[0,1]$ with $\int_0^1 \mathrm{d}\eta(t)$, $\int_0^1 \mathrm{d}\gamma(t) \in [0,1)$,
(H_2) $f \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0,+\infty)$.

In recent decades, the study of boundary value problems (BVPs) involving p-Laplacian operators and impulsive effects has become a focal point in nonlinear functional analysis due to their extensive applications in modeling real-world phenomena in physics, engineering, biology, and economics. The strong nonlinearity of the p-Laplacian operator can characterize nonNewtonian fluid flow, the elastic mechanics of anisotropic materials, and population dynamics with density-dependent diffusion. Impulsive effects are ubiquitous in practical systems, such as chemical reactions with instantaneous catalyst injection, neural networks with sudden voltage spikes, and ecological systems affected by occasional natural disasters (see [13, 21]).

In research on impulsive p-Laplacian boundary value problems (BVPs), there has been significant progress in existence theory using multiple tools, such as the upper and lower solution method, the monotone iterative technique, variational methods, and topological degree theory (see [2–8, 10, 11, 15–20, 22–28]). For example, in [5] the authors investigated the existence and uniqueness of solutions for three-point integral boundary-value problems of piecewise fractional impulsive differential equations involving the p-Laplacian operator and delay

$$\begin{aligned}
&\varphi_p\bigl({}^H D_c^{\alpha,v} x(t)\bigr) = \lambda h_k\bigl(t,\, x(t),\, x\bigl(t + (-1)^k \tau\bigr)\bigr),\\
&\Delta x(t_k) = P_k\bigl(x(t_k)\bigr), \qquad \Delta x'(t_k) = Q_k\bigl(x(t_k)\bigr),\\
&x(0) = x'(0), \qquad x'(1) = \int_0^\eta g(s, x(s))\,\mathrm{d}s,
\end{aligned}$$

where $1 < \alpha \leqslant 2$, and ${}^H D_{t_k}^{\alpha,\psi}$ is the ψ-Hadamard fractional derivative of order α. The authors employed the ψ-Hadamard fractional definition along with three fixed point theorems – the Leray–Schauder nonlinear alternative, Krasnosel'skii fixed point theorem, and the Banach fixed point theorem – to prove the existence and uniqueness of solutions.

In [16] the authors focused on the existence of solutions for noninstantaneous impulsive fractional differential equations with the p-Laplacian operator

$$\begin{aligned}
&\bigl(\phi_p\bigl({}^c D_0^\alpha u(t)\bigr)\bigr)' = f\bigl(t, u(t)\bigr), \quad t \in (s_k, t_{k+1}],\ k = 0,1,\dots,m,\\
&u(t) = g_k\bigl(t, u(t)\bigr), \quad u'(t) = h_k\bigl(t, u(t)\bigr), \quad t \in (t_k, s_k],\ k = 1,2,\dots,m,\\
&u(0) = u'(0) = 0, \qquad \phi_p\bigl({}^c D_0^\alpha u(s_k)\bigr) = 0, \quad k = 0,1,\dots,m,
\end{aligned}$$

where ${}^cD_0^\alpha$ denotes the Caputo fractional derivative with $1 < \alpha \leqslant 2$. By applying the Schauder fixed point theorem, the authors derived sufficient conditions for the existence of solutions.

In [23] the authors investigated the following fourth-order nonhomogeneous impulsive differential system:

$$\begin{aligned}
&u^{(4)}(t) + 2h(t)u'''(t) + \bigl(h^2(t) + h'(t) + \beta \mathrm{e}^{-\int_0^t h(\tau)\,\mathrm{d}\tau}\bigr)u''(t) = f_i\bigl(t, u(t)\bigr), \\
&\quad t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \dots, N, \\
&u^{(4)}(t) + 2h(t)u'''(t) + \bigl(h^2(t) + h'(t) + \beta \mathrm{e}^{-\int_0^t h(\tau)\,\mathrm{d}\tau}\bigr)u''(t) = 0, \\
&\quad t \in (t_i, s_i],\ i = 1, 2, \dots, N, \\
&u''(t_i^+) = u''(t_i^-), \quad u''(s_i^+) = u''(s_i^-), \quad u'''(s_i^+) = u'''(s_i^-), \quad i = 1, 2, \dots, N, \\
&\Delta u'''(t_i) = I_i(u(t_i)), \quad i = 1, 2, \dots, N, \\
&u(0) = u'(0) = u(T) = u'(T) = 0,
\end{aligned}$$

where f_i and I_i ($i = 1, 2, \dots, N$) satisfy some growth conditions. Under these assumptions, the authors established several existence theorems for the proposed system.

In [25] the authors investigated the existence of positive solutions for a second-order impulsive m-point boundary value problem on the half-line within the time-scale framework

$$\begin{aligned}
&\bigl[\varphi_p\bigl(u^\Delta(t)\bigr)\bigr]^\nabla + h(t)f\bigl(t, u(t), u^\Delta(t)\bigr) = 0, \quad t \in (0, \infty),\ t \neq t_k, \\
&u(0) = \sum_{i=1}^{m-2} \alpha_i u^\Delta(\eta_i), \qquad u^\Delta(\infty) = \sum_{i=1}^{m-2} \beta_i u(\eta_i), \\
&u(t_k^+) - u(t_k^-) = I_k\bigl(u(t_k)\bigr), \quad k \in \mathbb{N} \\
&\varphi_p\bigl(u^\Delta(t_k^+)\bigr) - \varphi_p\bigl(u^\Delta(t_k^-)\bigr) = -\bar{I}_k\bigl(u(t_k)\bigr), \quad k \in \mathbb{N}.
\end{aligned}$$

By employing the four functionals fixed point theorem, the authors established several existence results for positive solutions.

For (1), when η and γ are differentiable, in [8] the authors used fixed point theory to provide some sufficient conditions for the existence of one or two positive solutions. However, our method differs from the one adopted in that paper. First, we construct a linear operator with impulsive effects, obtain its adjoint operator by means of the Riemann–Stieltjes integral. Then we use the Krein–Rutman theorem to study the properties of its spectral radius. Under growth conditions related to the spectral radius, when the nonlinear term grows $(p-1)$-superlinearly and $(p-1)$-sublinearly, the impulsive term grows superlinearly and sublinearly, we apply the fixed point index to establish the existence of positive solutions. It is noteworthy that our conclusion can be directly applied to cases without the p-Laplacian operator and impulsive effects.

2 Preliminaries

In this section, we first present some basic knowledge and notations, which will be used in our paper. Let $E := C[0,1]$, $\|z\| := \sup_{t\in[0,1]}|z(t)|$. Then $(E, \|\cdot\|)$ is a real Banach space. Let $J' := J \setminus \{t_1, t_2, \dots, t_m\}$, and we introduce the space

$$PC'[0,1] := \bigl\{z \in C[0,1]\colon\ z'|_{(t_k,t_{k+1})} \in C(t_k, t_{k+1}),\ z'(t_k^-) = z'(t_k),\ \exists z'(t_k^+),\ k = 1,2,\dots,m\bigr\}$$

with the norm $\|z\|_{PC'} = \max\{\|z\|, \|z'\|\}$. Then $(PC'[0,1], \|\cdot\|_{PC'})$ is a Banach space. Let

$$P = \bigl\{z \in E\colon\ z(t) \geqslant 0,\ t \in [0,1]\bigr\}.$$

Then P is a cone on E.

In this paper, our aim is to establish the solvability for (1) by using the fixed point index theory. First, we consider the following auxiliary boundary value problem:

$$\begin{aligned}
&\bigl(\varphi_p\bigl(z''(t)\bigr)\bigr)'' = g(t), \quad t \in J,\ t \neq t_k,\ k = 1,2,\dots,m,\\
&\Delta z'|_{t=t_k} = -I_k\bigl(z(t_k)\bigr), \quad k = 1,2,\dots,m,\\
&z(0) = z(1) = \int_0^1 z(t)\,\mathrm{d}\eta(t), \qquad \varphi_p\bigl(z''(0)\bigr) = \varphi_p\bigl(z''(1)\bigr) = \int_0^1 \varphi_p\bigl(z''(t)\bigr)\,\mathrm{d}\gamma(t),
\end{aligned} \tag{2}$$

where $g \in L^1[0,1]$. Now, we will consider the Green's functions associated with (2), and obtain the following lemma. Let

$$\varphi_p\bigl(z''(t)\bigr) = -x(t), \quad t \in J,$$

then from (2) we have

$$-x''(t) = g(t), \quad t \in J, \qquad x(0) = x(1) = \int_0^1 x(t)\,\mathrm{d}\gamma(t), \tag{3}$$

and

$$\begin{aligned}
&z''(t) = -\varphi_q\bigl(x(t)\bigr), \quad t \in J,\ t \neq t_k,\ k = 1,2,\dots,m,\\
&\Delta z'|_{t=t_k} = -I_k\bigl(z(t_k)\bigr), \quad k = 1,2,\dots,m,\\
&z(0) = z(1) = \int_0^1 z(t)\,\mathrm{d}\eta(t),
\end{aligned} \tag{4}$$

where $\varphi_p^{-1} = \varphi_q$, $1/p + 1/q = 1$.

Lemma 1. *Suppose that* (H_1) *holds. Then problem* (3) *is equivalent to the following integral equation*:

$$x(t) = \int_0^1 G(t,s)g(s)\,\mathrm{d}s, \tag{5}$$

where

$$G(t,s) = G_1(t,s) + \frac{1}{1-\int_0^1 \mathrm{d}\gamma(t)} \int_0^1 G_1(t,s)\,\mathrm{d}\gamma(t),$$
$$G_1(t,s) = \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{6}$$

Proof. Note that (3) can be transformed into the following integral equation:

$$x(t) = \int_0^t (s-t)g(s)\,\mathrm{d}s + c_1 t + c_2,$$

where $c_i \in \mathbb{R}$, $i = 1, 2$. From the boundary conditions we obtain

$$c_1 = \int_0^1 (1-s)g(s)\,\mathrm{d}s, \qquad c_2 = \int_0^1 x(t)\,\mathrm{d}\gamma(t).$$

Therefore, we have

$$\begin{aligned} x(t) &= \int_0^t (s-t)g(s)\,\mathrm{d}s + \int_0^1 t(1-s)g(s)\,\mathrm{d}s + \int_0^1 x(t)\,\mathrm{d}\gamma(t) \\ &= \int_0^1 G_1(t,s)g(s)\,\mathrm{d}s + \int_0^1 x(t)\,\mathrm{d}\gamma(t). \end{aligned}$$

Also, we obtain

$$\int_0^1 x(t)\,\mathrm{d}\gamma(t) = \int_0^1 \int_0^1 G_1(t,s)g(s)\,\mathrm{d}s\,\mathrm{d}\gamma(t) = \int_0^1 \mathrm{d}\gamma(t) \int_0^1 x(t)\,\mathrm{d}\gamma(t)$$

and

$$\int_0^1 x(t)\,\mathrm{d}\gamma(t) = \frac{1}{1-\int_0^1 \mathrm{d}\gamma(t)} \int_0^1 \int_0^1 G_1(t,s)g(s)\,\mathrm{d}s\,\mathrm{d}\gamma(t).$$

Therefore, we have

$$\begin{aligned} x(t) &= \int_0^1 G_1(t,s)g(s)\,\mathrm{d}s + \frac{1}{1-\int_0^1 \mathrm{d}\gamma(t)} \int_0^1 \int_0^1 G_1(t,s)g(s)\,\mathrm{d}s\,\mathrm{d}\gamma(t) \\ &= \int_0^1 G(t,s)g(s)\,\mathrm{d}s. \end{aligned}$$

This completes the proof. □

Lemma 2. (*See* [8, Lemma 2.6].) *Suppose that* (H_1) *holds. Then problem* (4) *is equivalent to the following integral equation*:

$$z(t) = \int_0^1 H(t,s)\varphi_q\bigl(x(s)\bigr)\,\mathrm{d}s + \sum_{k=1}^m H(t,t_k)I_k\bigl(z(t_k)\bigr),$$

where

$$H(t,s) = G_1(t,s) + \frac{1}{1-\int_0^1 \mathrm{d}\eta(t)} \int_0^1 G_1(t,s)\,\mathrm{d}\eta(t),$$

and $G_1(t,s)$ *is defined in* (6).

Proof. From [8, (2.25)] we have

$$z(t) = \int_0^1 G_1(t,s)\varphi_q\bigl(x(s)\,\mathrm{d}s + \int_0^1 z(t)\,\mathrm{d}\eta(t) + \sum_{k=1}^m G_1(t,t_k)I_k\bigl(z(t_k)\bigr).$$

Also, we obtain

$$\begin{aligned}\int_0^1 z(t)\,\mathrm{d}\eta(t) &= \int_0^1\int_0^1 G_1(t,s)\varphi_q\bigl(x(s)\bigr)\,\mathrm{d}s\,\mathrm{d}\eta(t) + \int_0^1 \mathrm{d}\eta(t)\int_0^1 z(t)\,\mathrm{d}\eta(t)\\ &\quad + \int_0^1 \sum_{k=1}^m G_1(t,t_k)I_k\bigl(z(t_k)\bigr)\,\mathrm{d}\eta(t)\end{aligned}$$

and

$$\begin{aligned}\int_0^1 z(t)\,\mathrm{d}\eta(t) &= \frac{1}{1-\int_0^1 \mathrm{d}\eta(t)}\Biggl[\int_0^1\int_0^1 G_1(t,s)\varphi_q\bigl(x(s)\bigr)\,\mathrm{d}s\,\mathrm{d}\eta(t)\\ &\quad + \int_0^1 \sum_{k=1}^m G_1(t,t_k)I_k\bigl(z(t_k)\bigr)\,\mathrm{d}\eta(t)\Biggr].\end{aligned}$$

Therefore, we have

$$\begin{aligned}z(t) &= \int_0^1 G_1(t,s)\varphi_q\bigl(x(s)\bigr)\,\mathrm{d}s + \frac{1}{1-\int_0^1 \mathrm{d}\eta(t)}\int_0^1\int_0^1 G_1(t,s)\varphi_q\bigl(x(s)\bigr)\,\mathrm{d}s\,\mathrm{d}\eta(t)\\ &\quad + \sum_{k=1}^m G_1(t,t_k)I_k\bigl(z(t_k)\bigr) + \frac{1}{1-\int_0^1 \mathrm{d}\eta(t)}\int_0^1 \sum_{k=1}^m G_1(t,t_k)I_k\bigl(z(t_k)\bigr)\,\mathrm{d}\eta(t),\end{aligned}$$

which implies that

$$z(t)=\int_0^1 H(t,s)\varphi_q\bigl(x(s)\bigr)\,\mathrm{d}s+\sum_{k=1}^{m}H(t,t_k)I_k\bigl(z(t_k)\bigr).$$

This completes the proof. □

From Lemmas 1–2 we obtain that (2) can be transformed into the following integral equation:

$$z(t)=\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)g(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s+\sum_{k=1}^{m}H(t,t_k)I_k\bigl(z(t_k)\bigr). \tag{7}$$

Lemma 3. *The functions G, G_1, and H has the following properties*:

(i) $G, G_1, H \in C([0,1]\times[0,1],\mathbb{R}^+)$;
(ii) *G, G_1, and H are strictly greater than* 0 *on* $(0,1)\times(0,1)$;
(iii) *G and H satisfy the inequalities*

$$t(1-t)s(1-s)\left[1+\frac{\int_0^1 t(1-t)\,\mathrm{d}\gamma(t)}{1-\int_0^1 \mathrm{d}\gamma(t)}\right]\leqslant G(t,s)\leqslant\frac{s(1-s)}{1-\int_0^1 \mathrm{d}\gamma(t)}$$

and

$$t(1-t)s(1-s)\left[1+\frac{\int_0^1 t(1-t)\,\mathrm{d}\eta(t)}{1-\int_0^1 \mathrm{d}\eta(t)}\right]\leqslant H(t,s)\leqslant\frac{s(1-s)}{1-\int_0^1 \mathrm{d}\eta(t)}$$

for $(t,s)\in[0,1]\times[0,1]$.

Proof. The proof follows from routine calculations. □

From Lemmas 1–2 and (7) we can see that (1) is equivalent to the following integral equation:

$$\begin{aligned}z(t)&=\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\bigl(\tau,z(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s+\sum_{k=1}^{m}H(t,t_k)I_k\bigl(z(t_k)\bigr)\\&:=(Tz)(t),\quad t\in[0,1].\end{aligned}$$

Note that Lemma 3(i) and assumption (H_2) imply that $T:P\to P$. Moreover, if there exists a $z^*\in P\setminus\{0\}$ such that $Tz^*=z^*$, then this z^* is a positive solution for (1). Therefore, in what follows, we study the existence of positive fixed points for the operator T.

Lemma 4. *Let* $\omega_0 = 1 - \int_0^1 \mathrm{d}\eta(t) + \int_0^1 t(1-t)\,\mathrm{d}\eta(t) > 0$. *Then* $T(P) \subset P_0$, *where* $P_0 = \{z \in P\colon z(t) \geqslant \omega_0 t(1-t)\|z\|,\ t \in [0,1]\}$.

Proof. Note that if $z \in P$, from Lemma 3(iii) we have

$$\begin{aligned}(Tz)(t) \leqslant{}& \int_0^1 \frac{s(1-s)}{1-\int_0^1 \mathrm{d}\eta(t)} \varphi_q\left(\int_0^1 G(s,\tau) f\big(\tau, z(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &+ \sum_{k=1}^m \frac{t_k(1-t_k)}{1-\int_0^1 \mathrm{d}\eta(t)} I_k\big(z(t_k)\big), \quad t \in [0,1],\end{aligned}$$

and

$$\begin{aligned}(Tz)(t) \geqslant{}& \int_0^1 t(1-t)s(1-s)\left[1 + \frac{\int_0^1 t(1-t)\,\mathrm{d}\eta(t)}{1-\int_0^1 \mathrm{d}\eta(t)}\right]\varphi_q\left(\int_0^1 G(s,\tau) f\big(\tau, z(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &+ \sum_{k=1}^m t(1-t)t_k(1-t_k)\left[1 + \frac{\int_0^1 t(1-t)\,\mathrm{d}\eta(t)}{1-\int_0^1 \mathrm{d}\eta(t)}\right] I_k\big(z(t_k)\big) \\ ={}& t(1-t)\left[1 + \frac{\int_0^1 t(1-t)\,\mathrm{d}\eta(t)}{1-\int_0^1 \mathrm{d}\eta(t)}\right]\left[1 - \int_0^1 \mathrm{d}\eta(t)\right] \\ &\times \left[\int_0^1 \frac{s(1-s)}{1-\int_0^1 \mathrm{d}\eta(t)} \varphi_q\left(\int_0^1 G(s,\tau) f\big(\tau, z(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \right. \\ &\left. + \sum_{k=1}^m \frac{t_k(1-t_k)}{1-\int_0^1 \mathrm{d}\eta(t)} I_k\big(z(t_k)\big)\right] \\ \geqslant{}& \omega_0 t(1-t)\|Tz\|.\end{aligned}$$

This completes the proof. □

Let $\alpha, \beta \geqslant 0$ such that $\alpha^2 + \beta^2 \neq 0$, and define the linear operator $L_{\alpha,\beta} : P \to P$ as follows:

$$\begin{aligned}(L_{\alpha,\beta} z)(t) ={}& \alpha \int_0^1 \int_0^1 H(t,s) G(s,\tau)\,\mathrm{d}s\, z(\tau)\,\mathrm{d}\tau \\ &+ \beta \sum_{k=1}^m H(t,t_k) z(t_k), \quad z \in P,\ t \in [0,1].\end{aligned} \tag{8}$$

Now we prove that the spectral radius of $L_{\alpha,\beta}$, denoted by $r(L_{\alpha,\beta})$, is positive.

Lemma 5. $r(L_{\alpha,\beta}) > 0$.

Proof. Let

$$(L_\alpha z)(t) = \alpha \int_0^1 \int_0^1 H(t,s)G(s,\tau)\,\mathrm{d}s\, z(\tau)\,\mathrm{d}\tau, \qquad (L_\beta z)(t) = \beta \sum_{k=1}^m H(t,t_k)z(t_k),$$

and

$$(L_{G_1,\alpha} z)(t) = \alpha \int_0^1 \int_0^1 G_1(t,s)G_1(s,\tau)\,\mathrm{d}s\, z(\tau)\,\mathrm{d}\tau.$$

For $n \in \mathbb{N}^+$, we have

$$\bigl(L_\beta^n z\bigr)(t) = \beta^n \sum_{k_1=1}^m \sum_{k_2=1}^m \cdots \sum_{k_n=1}^m H(t,t_{k_1})H(t_{k_1},t_{k_2})\cdots H(t_{k_{n-1}},t_{k_n})z(t_{k_n}),$$

and Lemma 3(iii) implies that

$$\begin{aligned}
\bigl\|L_\beta^n\bigr\| &= \max_{t\in[0,1],\,\|z\|=1} \bigl(L_\beta^n z\bigr)(t) \geqslant \max_{t\in[0,1]} \bigl(\mathbf{1}L_\beta^n\bigr)(t) \\
&\geqslant \beta^n \left[1 + \frac{\int_0^1 t(1-t)\,\mathrm{d}\eta(t)}{1-\int_0^1 \mathrm{d}\eta(t)}\right]^n \\
&\quad \times \max_{t\in[0,1]} \sum_{k_1=1}^m \sum_{k_2=1}^m \cdots \sum_{k_n=1}^m t(1-t)t_{k_1}(1-t_{k_1})t_{k_1}(1-t_{k_1})t_{k_2}(1-t_{k_2})\cdots \\
&\qquad \times t_{k_{n-1}}(1-t_{k_{n-1}})t_{k_n}(1-t_{k_n}) \\
&= \frac{1}{4}\beta^n \left[1 + \frac{\int_0^1 t(1-t)\,\mathrm{d}\eta(t)}{1-\int_0^1 \mathrm{d}\eta(t)}\right]^n \left[\sum_{k=1}^m t_k^2(1-t_k)^2\right]^{n-1} \sum_{k=1}^m t_k(1-t_k),
\end{aligned}$$

where $\mathbf{1}(t) \equiv 1$, $t \in [0,1]$, $k_j = k$, $k = 1,2,\dots,m$, $j = 1,2,\dots,n$. Let $r(L_\beta)$ denote the spectral radius of L_β. Gelfand's theorem implies that

$$r(L_\beta) = \liminf_{n\to+\infty} \sqrt[n]{\bigl\|L_\beta^n\bigr\|} \geqslant \beta \left[1 + \frac{\int_0^1 t(1-t)\,\mathrm{d}\eta(t)}{1-\int_0^1 \mathrm{d}\eta(t)}\right] \sum_{k=1}^m t_k^2(1-t_k)^2.$$

Now we consider the following boundary value problem:

$$z^{(4)}(t) = \lambda z(t), \qquad z(0) = z(1) = z''(0) = z''(1) = 0. \tag{9}$$

If there exists a nonidentically zero function z^* such that (9) holds, then the parameter λ is called an eigenvalue associated with z^*. As is well known, (9) has nontrivial solutions if and only if the parameter $\lambda > 0$, and its solutions can be expressed as

$$z^*(t) = C_1\mathrm{e}^{\mu t} + C_2\mathrm{e}^{-\mu t} + C_3\cos\mu t + C_4\sin\mu t, \quad \mu^4 = \lambda,\ \mu > 0.$$

In view of the boundary conditions, we have $C_1 = C_2 = C_3 = 0$. If we want the solution z^* to be nontrivial, we need $C_4 \neq 0$. Note that $z^*(1) = C_4\sin\mu = 0$, and thus $\mu = n\pi$, i.e., $\lambda = (n\pi)^4$ $(n = 1, 2, \dots)$. Moreover, we can also obtain the nontrivial solution $z^*(t) = \sin n\pi t$, $t \in [0, 1]$.

On the other hand, if we let $z''(t) = -x(t)$, then (9) can be decomposed into the following two boundary value problems:

$$z''(t) = -x(t), \qquad z(0) = z(1) = 0$$

and

$$-x''(t) = \lambda z(t), \qquad x(0) = x(1) = 0.$$

Solving the boundary value problems, we obtain

$$z(t) = \lambda \int_0^1 \int_0^1 G_1(t,s)G_1(s,\tau)z(\tau)\,\mathrm{d}s\,\mathrm{d}\tau.$$

Combining this with (9), we have

$$z^*(t) = (n\pi)^4 \int_0^1 \int_0^1 G_1(t,s)G_1(s,\tau)z^*(\tau)\,\mathrm{d}s\,\mathrm{d}\tau,$$

where G_1 is defined in (6). Therefore, the spectral radius of $L_{G_1,\alpha}$, denoted by $r(L_{G_1,\alpha})$, is equal to α/π^4. From the definitions of L_α and $L_{G_1,\alpha}$ we have

$$r(L_\alpha) \geqslant r(L_{G_1,\alpha}) = \frac{\alpha}{\pi^4},$$

where $r(L_\alpha)$ is the spectral radius of L_α.

Thus, we have

$$r(L_{\alpha,\beta}) \geqslant r(L_\alpha) \geqslant \frac{\alpha}{\pi^4}$$

and

$$r(L_{\alpha,\beta}) \geqslant r(L_\beta) \geqslant \beta\left[1 + \frac{\int_0^1 t(1-t)\,\mathrm{d}\eta(t)}{1 - \int_0^1 \mathrm{d}\eta(t)}\right] \sum_{k=1}^m t_k^2(1-t_k)^2.$$

Therefore, $r(L_{\alpha,\beta}) > 0$. This completes the proof. □

By Lemma 5 and the well-known Krein–Rutman theorem [12], there exists a $\zeta_{\alpha,\beta} \in P \setminus \{0\}$ such that

$$L_{\alpha,\beta}\zeta_{\alpha,\beta} = r(L_{\alpha,\beta})\zeta_{\alpha,\beta}, \tag{10}$$

i.e.,

$$\alpha \int_0^1 \int_0^1 H(t,s)G(s,\tau)\,\mathrm{d}s\,\zeta_{\alpha,\beta}(\tau)\,\mathrm{d}\tau + \beta \sum_{k=1}^m H(t,t_k)\zeta_{\alpha,\beta}(t_k) = r(L_{\alpha,\beta})\zeta_{\alpha,\beta}(t), \quad t \in [0,1].$$

From the definition of $L_{\alpha,\beta}$ and Lemma 4 we obtain

$$\zeta_{\alpha,\beta} \in P_0.$$

Remark 1. Let $\kappa_H = \max_{t,s\in[0,1]} H(t,s)$, $\kappa_G = \max_{t,s\in[0,1]} G(t,s)$. Now, we estimate the upper bound of the spectral radius of the operator $L_{\alpha,\beta}$. From Lemma 3(iii) we have

$$r(L_{\alpha,\beta}) \leqslant \|L_{\alpha,\beta}\| \leqslant \alpha\kappa_H\kappa_G + \frac{\beta}{1-\int_0^1 \mathrm{d}\eta(t)} \sum_{k=1}^m t_k(1-t_k).$$

Lemma 6. (*See* [9]). *Let $\Omega \subset E$ be a bounded open set, and let $T : \overline{\Omega} \cap P \to P$ be a continuous compact operator. If there exists a $z_0 \in P \setminus \{0\}$ such that $z - Tz \neq \mu z_0$ for all $\mu \geqslant 0$ and $z \in \partial\Omega \cap P$, then $i(T,\, \Omega \cap P,\, P) = 0$, where i denotes the fixed point index on P.*

Lemma 7. (*See* [9]). *Let $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Suppose that $T : \overline{\Omega} \cap P \to P$ is a continuous compact operator. If $z \neq \mu Tz$ for all $z \in \partial\Omega \cap P$ and $0 \leqslant \mu \leqslant 1$, then $i(T,\, \Omega \cap P,\, P) = 1$.*

Lemma 8 [Jensen's inequalities]. (*See* [6].) *Let $\theta > 0$, $n \geqslant 1$, $a_i \geqslant 0$ ($i = 1, 2, \dots, n$), and $\psi \in C([a,b], \mathbb{R}^+)$. Then for all $\theta \geqslant 1$,*

$$\left(\int_a^b \psi(t)\,\mathrm{d}t\right)^\theta \leqslant (b-a)^{\theta-1} \int_a^b \bigl(\psi(t)\bigr)^\theta\,\mathrm{d}t \quad \textit{and} \quad \left(\sum_{i=1}^n a_i\right)^\theta \leqslant n^{\theta-1} \sum_{i=1}^n a_i^\theta;$$

and for all $0 < \theta \leqslant 1$,

$$\left(\int_a^b \psi(t)\,\mathrm{d}t\right)^\theta \geqslant (b-a)^{\theta-1} \int_a^b (\psi(t))^\theta\,\mathrm{d}t \quad \textit{and} \quad \left(\sum_{i=1}^n a_i\right)^\theta \geqslant n^{\theta-1} \sum_{i=1}^n a_i^\theta.$$

3 Main results

Let $p_* = \min\{1, p-1\}$, $p^* = \max\{1, p-1\}$, and $B_\rho = \{z \in P\colon \|z\| < \rho\}$, $\rho > 0$. Note that B_ρ is an open ball in P, and

$$\overline{B}_\rho = \bigl\{z \in P\colon \|z\| \leqslant \rho\bigr\}, \qquad \partial B_\rho = \bigl\{z \in P\colon \|z\| = \rho\bigr\}.$$

Now, we list our assumptions:

(H_3) There exist $a_1, b_1 \geqslant 0$ with $a_1^2 + b_1^2 \neq 0$ such that $r(L_{a_1,b_1}) > 1$ and

$$\liminf_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} \geqslant \left(\frac{a_1}{(2\kappa_H)^{p_*-1}\kappa_G^{p_*/(p-1)-1}}\right)^{(p-1)/p_*} \quad \text{uniformly on } t \in [0,1],$$

$$\liminf_{z\to 0^+} \frac{I_k(z)}{z} \geqslant \frac{b_1^{1/p_*}}{(2m\kappa_H)^{1-1/p_*}} \quad \text{uniformly on } k = 1, 2, \dots, m.$$

(H$_4$) There exist $a_2, b_2 \geqslant 0$ with $a_2^2 + b_2^2 \neq 0$ such that $r(L_{a_2,b_2}) < 1$ and

$$\limsup_{z\to+\infty} \frac{f(t,z)}{z^{p-1}} \leqslant \left(\frac{a_2}{2^{p^*p/(p-1)-2}\kappa_H^{p^*-1}\kappa_G^{p^*/(p-1)-1}}\right)^{(p-1)/p^*}$$

uniformly on $t \in [0,1]$,

$$\limsup_{z\to+\infty} \frac{I_k(z)}{z} \leqslant \frac{b_2^{1/p^*}}{(4m\kappa_H)^{1-1/p^*}} \quad \text{uniformly on } k = 1,2,\dots,m.$$

(H$_5$) There exist $a_3, b_3 \geqslant 0$ with $a_3^2 + b_3^2 \neq 0$ such that $r(L_{a_3,b_3}) < 1$ and

$$\limsup_{z\to0^+} \frac{f(t,z)}{z^{p-1}} \leqslant \left(\frac{a_3}{(2\kappa_H)^{p^*-1}\kappa_G^{p^*/(p-1)-1}}\right)^{(p-1)/p^*} \quad \text{uniformly on } t\in[0,1],$$

$$\limsup_{z\to0^+} \frac{I_k(z)}{z} \leqslant \frac{b_3^{1/p^*}}{(2m\kappa_H)^{1-1/p^*}} \quad \text{uniformly on } k = 1,2,\dots,m.$$

(H$_6$) There exist $a_4, b_4 \geqslant 0$ with $a_4^2 + b_4^2 \neq 0$ such that $r(L_{a_4,b_4}) > 1$ and

$$\liminf_{z\to+\infty} \frac{f(t,z)}{z^{p-1}} \geqslant \left(\frac{a_4}{(2\kappa_H)^{p_*-1}\kappa_G^{p_*/(p-1)-1}}\right)^{(p-1)/p_*} \quad \text{uniformly on } t\in[0,1],$$

$$\liminf_{z\to+\infty} \frac{I_k(z)}{z} \geqslant \frac{b_4^{1/p_*}}{(2m\kappa_H)^{1-1/p_*}} \quad \text{uniformly on } k = 1,2,\dots,m.$$

Theorem 1. *Suppose that* (H$_1$)–(H$_4$) *hold. Then* (1) *has at least one positive solution.*

Proof. From (H$_3$) there exists a sufficiently small $r_1 > 0$ such that

$$f(t,z) \geqslant \left(\frac{a_1}{(2\kappa_H)^{p_*-1}\kappa_G^{p_*/(p-1)-1}}\right)^{(p-1)/p_*} z^{p-1}, \quad z \in [0,r_1],\ t \in [0,1], \tag{11}$$

and

$$I_k(z) \geqslant \frac{b_1^{1/p_*}}{(2m\kappa_H)^{1-1/p_*}} z, \quad z \in [0,r_1],\ k = 1,2,\dots,m. \tag{12}$$

For a_1, b_1, by (8) and (10), we obtain that there exists a $\zeta_{a_1,b_1} \in P \setminus \{0\}$ such that

$$L_{a_1,b_1}\zeta_{a_1,b_1} = r(L_{a_1,b_1})\zeta_{a_1,b_1}. \tag{13}$$

Now we prove that

$$z - Tz \neq \nu\zeta_{a_1,b_1}, \quad z \in \partial B_{r_1} \cap P,\ \nu \geqslant 0. \tag{14}$$

We argue by contradiction. Suppose there exist $z_1 \in \partial B_{r_1} \cap P$, $\nu_1 \geqslant 0$ such that

$$z_1 - Tz_1 = \nu_1 \zeta_{a_1,b_1}. \tag{15}$$

Note that we only need to consider the case where $\nu_1 > 0$ (since if $\nu_1 = 0$, then $Tz_1 = z_1$, and z_1 is a positive solution for (1), so the theorem is proved). As a result, we can obtain

$$z_1(t) \geqslant (Tz_1)(t), \quad t \in [0,1].$$

Therefore, we have

$$\begin{aligned}
z_1^{p_*}(t) &\geqslant \big[(Tz_1)(t)\big]^{p_*} \\
&= \Bigg[\int_0^1 \kappa_H \frac{H(t,s)}{\kappa_H}\varphi_q\bigg(\int_0^1 \kappa_G \frac{G(s,\tau)}{\kappa_G} f\big(\tau, z_1(\tau)\big)\,\mathrm{d}\tau\bigg)\,\mathrm{d}s \\
&\quad + \sum_{k=1}^m \kappa_H \frac{H(t,t_k)}{\kappa_H} I_k\big(z_1(t_k)\big)\Bigg]^{p_*} \\
&\geqslant 2^{p_*-1}\Bigg[\int_0^1 \kappa_H \frac{H(t,s)}{\kappa_H}\varphi_q\bigg(\int_0^1 \kappa_G \frac{G(s,\tau)}{\kappa_G} f\big(\tau, z_1(\tau)\big)\,\mathrm{d}\tau\bigg)\,\mathrm{d}s\Bigg]^{p_*} \\
&\quad + 2^{p_*-1}\Bigg[\sum_{k=1}^m \kappa_H \frac{H(t,t_k)}{\kappa_H} I_k\big(z_1(t_k)\big)\Bigg]^{p_*} \\
&\geqslant 2^{p_*-1}\int_0^1 \kappa_H^{p_*}\bigg[\frac{H(t,s)}{\kappa_H}\bigg]^{p_*}\int_0^1 \kappa_G^{p_*/(p-1)}\bigg[\frac{G(s,\tau)}{\kappa_G}\bigg]^{p_*/(p-1)} f^{p_*/(p-1)}\big(\tau, z_1(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + (2m)^{p_*-1}\sum_{k=1}^m \kappa_H^{p_*}\bigg[\frac{H(t,t_k)}{\kappa_H}\bigg]^{p_*} I_k^{p_*}\big(z_1(t_k)\big) \\
&\geqslant (2\kappa_H)^{p_*-1}\kappa_G^{p_*/(p-1)-1}\int_0^1\int_0^1 H(t,s)G(s,\tau) f^{p_*/(p-1)}\big(\tau, z_1(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + (2m\kappa_H)^{p_*-1}\sum_{k=1}^m H(t,t_k) I_k^{p_*}\big(z_1(t_k)\big).
\end{aligned} \tag{16}$$

By substituting (11)–(12) into (16), we obtain

$$\begin{aligned}
&\big[(Tz_1)(t)\big]^{p_*} \\
&\quad \geqslant (2\kappa_H)^{p_*-1}\kappa_G^{p_*/(p-1)-1} \\
&\quad\quad \times \int_0^1\int_0^1 H(t,s)G(s,\tau)\Bigg[\bigg(\frac{a_1}{(2\kappa_H)^{p_*-1}\kappa_G^{p_*/(p-1)-1}}\bigg)^{(p-1)/p_*} z_1^{p-1}(\tau)\Bigg]^{p_*/(p-1)}\,\mathrm{d}\tau\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ (2m\kappa_H)^{p_*-1} \sum_{k=1}^{m} H(t,t_k)\left[\frac{b_1^{1/p_*}}{(2m\kappa_H)^{1-1/p_*}} z_1(t_k)\right]^{p_*} \\
&= a_1 \int_0^1\int_0^1 H(t,s)G(s,\tau) z_1^{p_*}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + b_1 \sum_{k=1}^{m} H(t,t_k) z_1^{p_*}(t_k) \\
&= (L_{a_1,b_1} z_1^{p_*})(t), \quad t \in [0,1]. \qquad (17)
\end{aligned}
$$

Note that from (15) we have $z_1 \geqslant \nu_1 \zeta_{a_1,b_1}$, and then

$$z_1^{p_*} \geqslant \nu_1^{p_*} \zeta_{a_1,b_1}^{p_*} \geqslant \nu_1^{p_*} \|\zeta_{a_1,b_1}\|^{p_*-1} \zeta_{a_1,b_1}.$$

Let $\nu^* = \sup\{\nu\colon\ z_1^{p_*} \geqslant \nu\zeta_{a_1,b_1}\}$. Then $\nu^* \geqslant \nu_1^{p_*}\|\zeta_{a_1,b_1}\|^{p_*-1} > 0$ and $z_1^{p_*} \geqslant \nu^*\zeta_{a_1,b_1}$. From (17) and (13) we have

$$z_1^{p_*} \geqslant L_{a_1,b_1} z_1^{p_*} \geqslant L_{a_1,b_1} \nu^* \zeta_{a_1,b_1} = \nu^* r(L_{a_1,b_1}) \zeta_{a_1,b_1}.$$

Note that $r(L_{a_1,b_1}) > 1$, and so $\nu^* r(L_{a_1,b_1}) > \nu^*$, which contradicts the fact that ν^* is the supremum. Therefore, (14) holds, and Lemma 6 implies that

$$i(T,\ B_{r_1} \cap P,\ P) = 0. \qquad (18)$$

From (H_4) there exists a $d_1 > 0$ such that

$$
\begin{aligned}
f(t,z) \leqslant{}& \left(\frac{a_2}{2^{p^* p/(p-1)-2} \kappa_H^{p^*-1} \kappa_G^{p^*/(p-1)-1}}\right)^{(p-1)/p^*} z^{p-1} \\
&+ d_1, \quad z \in \mathbb{R}^+,\ t \in [0,1], \qquad (19)
\end{aligned}
$$

and

$$I_k(z) \leqslant \frac{b_2^{1/p^*}}{(4m\kappa_H)^{1-1/p^*}} z + d_1, \quad z \in \mathbb{R}^+,\ k = 1,2,\dots,m. \qquad (20)$$

Now, we prove that the following set is bounded:

$$W = \{z \in P\colon\ z = \nu T z,\ 0 \leqslant \nu \leqslant 1\}.$$

Suppose there exist $z_2 \in W$, $\nu_2 \in [0,1]$ such that $z_2 = \nu_2 T z_2$. Then

$$
\begin{aligned}
z_2^{p^*}(t) &\leqslant \big[(Tz_2)(t)\big]^{p^*} \\
&= \Bigg[\int_0^1 \kappa_H \frac{H(t,s)}{\kappa_H} \varphi_q\Bigg(\int_0^1 \kappa_G \frac{G(s,\tau)}{\kappa_G} f\big(\tau, z_2(\tau)\big)\,\mathrm{d}\tau\Bigg)\,\mathrm{d}s \\
&\quad + \sum_{k=1}^{m} \kappa_H \frac{H(t,t_k)}{\kappa_H} I_k\big(z_2(t_k)\big)\Bigg]^{p^*}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant 2^{p^*-1}\Biggl[\int_0^1 \kappa_H \frac{H(t,s)}{\kappa_H}\varphi_q\Biggl(\int_0^1 \kappa_G \frac{G(s,\tau)}{\kappa_G} f\bigl(\tau, z_2(\tau)\bigr)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s\Biggr]^{p^*} \\
&\quad + 2^{p^*-1}\Biggl[\sum_{k=1}^{m} \kappa_H \frac{H(t,t_k)}{\kappa_H} I_k\bigl(z_2(t_k)\bigr)\Biggr]^{p^*} \\
&\leqslant (2\kappa_H)^{p^*-1}\kappa_G^{p^*/(p-1)-1}\int_0^1\int_0^1 H(t,s)G(s,\tau) f^{p^*/(p-1)}\bigl(\tau, z_2(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + (2m\kappa_H)^{p^*-1}\sum_{k=1}^{m} H(t,t_k) I_k^{p^*}\bigl(z_2(t_k)\bigr).
\end{aligned} \tag{21}
$$

By substituting (19)–(20) into (21), we have

$$
\begin{aligned}
z_2^{p^*}(t) &\leqslant (2\kappa_H)^{p^*-1}\kappa_G^{p^*/(p-1)-1} \\
&\quad \times \int_0^1\int_0^1 H(t,s)G(s,\tau)\Biggl[\Biggl(\frac{a_2}{2^{p^*p/(p-1)-2}\kappa_H^{p^*-1}\kappa_G^{p^*/(p-1)-1}}\Biggr)^{(p-1)/p^*} z_2^{p-1}(\tau) \\
&\qquad + d_1\Biggr]^{p^*/(p-1)}\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + (2m\kappa_H)^{p^*-1}\sum_{k=1}^{m} H(t,t_k)\Biggl[\frac{b_2^{1/p^*}}{(4m\kappa_H)^{1-1/p^*}} z_2(t_k) + d_1\Biggr]^{p^*} \\
&\leqslant 2^{p^*p/(p-1)-2}\kappa_H^{p^*-1}\kappa_G^{p^*/(p-1)-1} \\
&\quad \times \int_0^1\int_0^1 H(t,s)G(s,\tau)\Biggl(\frac{a_2}{2^{p^*p/(p-1)-2}\kappa_H^{p^*-1}\kappa_G^{p^*/(p-1)-1}} z_2^{p^*}(\tau) \\
&\qquad + d_1^{p^*/(p-1)}\Biggr)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + (4m\kappa_H)^{p^*-1}\sum_{k=1}^{m} H(t,t_k)\Biggl(\frac{b_2}{(4m\kappa_H)^{p^*-1}} z_2^{p^*}(t_k) + d_1^{p^*}\Biggr) \\
&= a_2\int_0^1\int_0^1 H(t,s)G(s,\tau) z_2^{p^*}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + b_2\sum_{k=1}^{m} H(t,t_k) z_2^{p^*}(t_k) \\
&\quad + 2^{p^*p/(p-1)-2}\kappa_H^{p^*-1}\kappa_G^{p^*/(p-1)-1} d_1^{p^*/(p-1)} \\
&\quad \times \int_0^1\int_0^1 H(t,s)G(s,\tau)\,\mathrm{d}\tau\,\mathrm{d}s + (4m\kappa_H)^{p^*-1} d_1^{p^*}\sum_{k=1}^{m} H(t,t_k).
\end{aligned}
$$

Let

$$\vartheta(t) = 2^{p^* p/(p-1)-2} \kappa_H^{p^*-1} \kappa_G^{p^*/(p-1)-1} d_1^{p^*/(p-1)} \int_0^1 \int_0^1 H(t,s) G(s,\tau)\, \mathrm{d}\tau\, \mathrm{d}s + (4m\kappa_H)^{p^*-1} d_1^{p^*} \sum_{k=1}^{m} H(t,t_k), \quad t \in [0,1].$$

Then we have $\|\vartheta\| < +\infty$ and

$$z_2^{p^*}(t) \leqslant \big(L_{a_2,b_2} z_2^{p^*}\big)(t) + \vartheta(t), \quad t \in [0,1].$$

After performing n iterations of the above inequality, we obtain

$$z_2^{p^*}(t) \leqslant \big(L_{a_2,b_2}^n z_2^{p^*}\big)(t) + \big(L_{a_2,b_2}^{n-1} \vartheta\big)(t) + \cdots + \big(L_{a_2,b_2} \vartheta\big)(t) + \vartheta(t), \quad t \in [0,1],\ n \in \mathbb{N}^+.$$

Note that from $r(L_{a_2,b_2}) < 1$ we have

$$\lim_{n\to+\infty} \big(L_{a_2,b_2}^n z_2^{p^*}\big)(t) = 0$$

and

$$\lim_{n\to+\infty} \big[\big(L_{a_2,b_2}^{n-1}\vartheta\big)(t) + \cdots + (L_{a_2,b_2}\vartheta)(t) + \vartheta(t)\big] = \big((I - L_{a_2,b_2})^{-1}\vartheta\big)(t), \quad t \in [0,1].$$

Note that $(I - L_{a_2,b_2})^{-1} : P \to P$, and hence we have

$$\big\|z_2^{p^*}\big\| \leqslant \big\|(I - L_{a_2,b_2})^{-1}\vartheta\big\|.$$

Thus, W is a bounded set in P. Now, we choose $R_1 > \sup W$ and $R_1 > r_1$ such that

$$z \neq \nu T z, \quad z \in \partial B_{R_1} \cap P,\ 0 \leqslant \nu \leqslant 1.$$

Therefore, Lemma 7 implies that

$$i(T,\ B_{R_1} \cap P,\ P) = 1. \tag{22}$$

From (18) and (22) we have

$$i\big(T,\ (B_{R_1} \setminus \overline{B}_{r_1}) \cap P,\ P\big) = i(T,\ B_{R_1} \cap P,\ P) - i(T,\ B_{r_1} \cap P,\ P) = 1.$$

This implies that T has a fixed point in $(B_{R_1} \setminus \overline{B}_{r_1}) \cap P$, and thus (1) has at least one positive solution. The proof is completed. □

Theorem 2. *Suppose that* (H_1)–(H_2) *and* (H_5)–(H_6) *hold. Then* (1) *has at least one positive solution.*

Proof. From (H_4) there exists a sufficiently small $r_2 > 0$ such that

$$f(t,z) \leqslant \left(\frac{a_3}{(2\kappa_H)^{p^*-1}\kappa_G^{p^*/(p-1)-1}}\right)^{(p-1)/p^*} z^{p-1}, \quad z \in [0,r_2],\ t \in [0,1], \tag{23}$$

and

$$I_k(z) \leqslant \frac{b_3^{1/p^*}}{(2m\kappa_H)^{1-1/p^*}} z, \quad z \in [0,r_2],\ k = 1,2,\dots,m. \tag{24}$$

Now, we claim that

$$z \neq \nu T z, \quad z \in \partial B_{r_2} \cap P,\ 0 \leqslant \nu \leqslant 1.$$

If this claim does not hold, then there exist $z_3 \in \partial B_{r_2} \cap P, \nu_3 \in [0,1]$ such that

$$z_3 = \nu_3 T z_3.$$

By (21), (23), and (24) we have

$$\begin{aligned} z_3^{p^*}(t) &\leqslant \big[(Tz_3)(t)\big]^{p^*} \\ &\leqslant (2\kappa_H)^{p^*-1}\kappa_G^{p^*/(p-1)-1} \\ &\quad \times \int_0^1\int_0^1 H(t,s)G(s,\tau)\left[\left(\frac{a_3}{(2\kappa_H)^{p^*-1}\kappa_G^{p^*/(p-1)-1}}\right)^{(p-1)/p^*} z_3^{p-1}(\tau)\right]^{p^*/(p-1)} \mathrm{d}\tau\,\mathrm{d}s \\ &\quad + (2m\kappa_H)^{p^*-1}\sum_{k=1}^m H(t,t_k)\left[\frac{b_3^{1/p^*}}{(2m\kappa_H)^{1-1/p^*}} z_3(t_k)\right]^{p^*} \\ &= a_3 \int_0^1\int_0^1 H(t,s)G(s,\tau) z_3^{p^*}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + b_3 \sum_{k=1}^m H(t,t_k) z_3^{p^*}(t_k). \end{aligned}$$

This implies that

$$\big((I - L_{a_3,b_3}) z_3^{p^*}\big)(t) \leqslant 0, \quad t \in [0,1].$$

Note that $r(L_{a_3,b_3}) < 1$, and similarly to the proof of Theorem 1, we have

$$\big\|z_3^{p^*}\big\| \leqslant \big\|(I - L_{a_3,b_3})^{-1}\mathbf{0}\big\| = 0,$$

where $\mathbf{0}(t) \equiv 0$, $t \in [0,1]$. Note that $z_3 \in P$, and thus $z_3(t) \equiv 0$, $t \in [0,1]$, which contradicts $\|z_3\| = r_2$. Therefore, Lemma 7 implies that

$$i(T,\ B_{r_2} \cap P,\ P) = 1. \tag{25}$$

In what follows, we will rewrite $\sum_{k=1}^{m} H(t,t_k)z(t_k)$ as a Riemann–Stieltjes integral. First, define a function $\xi_t(\tau)$ (depending on t) on $[0,1]$ such that $\xi_t(\tau)$ has jump discontinuities at $t_1, t_2, \dots, t_m$, and the size of the jump at t_k is $H(t,t_k)$, i.e.,

$$\xi_t(\tau) = \sum_{k:\, t_k \leqslant \tau} H(t,t_k), \quad \tau \in [0,1],\ k = 1,2,\dots,m.$$

Then, by the definition of the Riemann–Stieltjes integral, we have

$$\sum_{k=1}^{m} H(t,t_k)z(t_k) = \int_0^1 z(\tau)\,\mathrm{d}\xi_t(\tau), \quad t \in [0,1].$$

Therefore, the operator $L_{\alpha,\beta}$ in (8) can be rewritten by

$$(L_{\alpha,\beta}z)(t) = \alpha \int_0^1 \int_0^1 H(t,s)G(s,\tau)\,\mathrm{d}s\, z(\tau)\,\mathrm{d}\tau + \beta \int_0^1 z(\tau)\,\mathrm{d}\xi_t(\tau), \quad z \in P,\ t \in [0,1].$$

From [14] we obtain that the conjugate space of E, denoted by E^*, is $E^* = \{\xi\colon \xi$ has bounded variation on $[0,1]\}$. Moreover, the dual cone of P and the bounded linear functional on E can be expressed by

$$P^* := \bigl\{\xi \in E^*\colon\ \xi \text{ is nondecreasing on } [0,1]\bigr\}$$

and

$$\langle \xi, z\rangle = \int_0^1 z(t)\,\mathrm{d}\xi(t), \quad z \in E,\ \xi \in E^*.$$

Note that $r(L_{\alpha,\beta}) > 0$ in Lemma 5, and there exists a $\zeta^*_{\alpha,\beta} \in P^* \setminus \{0\}$ such that

$$L^*_{\alpha,\beta}\zeta^*_{\alpha,\beta} = r(L_{\alpha,\beta})\zeta^*_{\alpha,\beta}, \tag{26}$$

where $L^*_{\alpha,\beta}$ is the conjugate operator of $L_{\alpha,\beta}$, defined by

$$(L^*_{\alpha,\beta}\theta)(\tau) := a_4 \int_0^\tau \mathrm{d}x \int_0^1 \int_0^1 H(t,s)G(s,x)\,\mathrm{d}s\,\mathrm{d}\theta(t) + b_4 \int_0^1 \xi_t(\tau)\,\mathrm{d}\theta(t), \quad \theta \in E^*.$$

From (H$_6$) there exist a $d_2 > 0$ such that

$$f(t,z) \geqslant \left(\frac{a_4}{(2\kappa_H)^{p_*-1}\kappa_G^{p_*/(p-1)-1}}\right)^{(p-1)/p_*} z^{p-1} - d_2, \quad z \in \mathbb{R}^+,\ t \in [0,1],$$

and

$$I_k(z) \geqslant \frac{b_4^{1/p_*}}{(2m\kappa_H)^{1-1/p_*}} z - d_2, \quad z \in \mathbb{R}^+,\ k = 1, 2, \dots, m.$$

Noting that $p_*, p_*/(p-1) \in (0, 1]$, we have

$$\bigl[f(t, z) + d_2\bigr]^{p_*/(p-1)} \leqslant f^{p_*/(p-1)}(t, z) + d_2^{p_*/(p-1)}$$

and

$$[I_k(z) + d_2]^{p_*} \leqslant \bigl[I_k(z)\bigr]^{p_*} + d_2^{p_*}.$$

Consequently, we find

$$\begin{aligned}
&f^{p_*/(p-1)}(t, z) && (27)\\
&\quad \geqslant \bigl[f(t, z) + d_2\bigr]^{p_*/(p-1)} - d_2^{p_*/(p-1)}\\
&\quad \geqslant \biggl[\biggl(\frac{a_4}{(2\kappa_H)^{p_*-1}\kappa_G^{p_*/(p-1)-1}}\biggr)^{(p-1)/p_*} z^{p-1}\biggr]^{p_*/(p-1)} - d_2^{p_*/(p-1)}\\
&\quad = \frac{a_4}{(2\kappa_H)^{p_*-1}\kappa_G^{p_*/(p-1)-1}} z^{p_*} - d_2^{p_*/(p-1)}, \quad z \in \mathbb{R}^+,\ t \in [0, 1], && (28)
\end{aligned}$$

and

$$\begin{aligned}
\bigl[I_k(z)\bigr]^{p_*} &\geqslant \bigl[I_k(z) + d_2\bigr]^{p_*} - d_2^{p_*} \geqslant \biggl[\frac{b_4^{1/p_*}}{(2m\kappa_H)^{1-1/p_*}} z\biggr]^{p_*} - d_2^{p_*}\\
&= \frac{b_4}{(2m\kappa_H)^{p_*-1}} z^{p_*} - d_2^{p_*}, \quad z \in \mathbb{R}^+,\ k = 1, 2, \dots, m. \qquad (29)
\end{aligned}$$

Let

$$S = \{z \in P\colon\ z - Tz = \nu\widehat{z},\ \nu \geqslant 0\},$$

where $\widehat{z}$ is a given element in P_0. Now we prove that S is a bounded set in P. Suppose there exist $z_4 \in S$ and $\nu_4 \geqslant 0$ such that

$$z_4 = Tz_4 + \nu_4\widehat{z}.$$

Lemma 4 implies that

$$z_4 \in P_0. \qquad (30)$$

Similarly to (16), from (28)–(29) we have

$$\begin{aligned}
z_4^{p_*}(t) \geqslant{}& (2\kappa_H)^{p_*-1}\kappa_G^{p_*/(p-1)-1} \int_0^1\!\!\int_0^1 H(t, s)G(s, \tau) f^{p_*/(p-1)}\bigl(\tau, z_4(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s\\
&+ (2m\kappa_H)^{p_*-1} \sum_{k=1}^m H(t, t_k) I_k^{p_*}\bigl(z_4(t_k)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&\geqslant (2\kappa_H)^{p_*-1}\kappa_G^{p_*/(p-1)-1}\\
&\quad\times \int_0^1\int_0^1 H(t,s)G(s,\tau)\left[\frac{a_4}{(2\kappa_H)^{p_*-1}\kappa_G^{p_*/(p-1)-1}} z_4^{p_*}(\tau) - d_2^{p_*/(p-1)}\right]\mathrm{d}\tau\,\mathrm{d}s\\
&\quad+ (2m\kappa_H)^{p_*-1}\sum_{k=1}^m H(t,t_k)\left[\frac{b_4}{(2m\kappa_H)^{p_*-1}} z_4^{p_*}(t_k) - d_2^{p_*}\right]\\
&\geqslant a_4\int_0^1\int_0^1 H(t,s)G(s,\tau)z_4^{p_*}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + b_4\sum_{k=1}^m H(t,t_k)z_4^{p_*}(t_k)\\
&\quad- 2^{p_*-1}\left[\kappa_H^{p_*}\kappa_G^{p_*/(p-1)}d_2^{p_*/(p-1)} + m^{p_*}\kappa_H^{p_*}d_2^{p_*}\right].
\end{aligned}
$$

Note that $r(L_{a_4,b_4}) > 1$, and similarly to (26), there exists a $\zeta^*_{a_4,b_4} \in P^* \setminus \{0\}$ such that

$$
L^*_{a_4,b_4}\zeta^*_{a_4,b_4} = r(L_{a_4,b_4})\zeta^*_{a_4,b_4}. \tag{31}
$$

From (31) we obtain

$$
\begin{aligned}
&\int_0^1 z_4^{p_*}(t)d\zeta^*_{a_4,b_4}(t) + 2^{p_*-1}\left[\kappa_H^{p_*}\kappa_G^{p_*/(p-1)}d_2^{p_*/(p-1)} + m^{p_*}\kappa_H^{p_*}d_2^{p_*}\right]\int_0^1 \mathrm{d}\zeta^*_{a_4,b_4}(t)\\
&\quad\geqslant \int_0^1\left[a_4\int_0^1\int_0^1 H(t,s)G(s,\tau)z_4^{p_*}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + b_4\int_0^1 z_4^{p_*}(\tau)\,\mathrm{d}\xi_t(\tau)\right]\mathrm{d}\zeta^*_{a_4,b_4}(t)\\
&\quad= \int_0^1 z_4^{p_*}(\tau)\,\mathrm{d}\left(a_4\int_0^\tau \mathrm{d}x\int_0^1\int_0^1 H(t,s)G(s,x)\,\mathrm{d}s\,\mathrm{d}\zeta^*_{a_4,b_4}(t) + b_4\int_0^1 \xi_t(\tau)\,\mathrm{d}\zeta^*_{a_4,b_4}(t)\right)\\
&\quad= \langle L^*_{a_4,b_4}\zeta^*_{a_4,b_4}, z_4^{p_*}\rangle = r(L_{a_4,b_4})\langle\zeta^*_{a_4,b_4}, z_4^{p_*}\rangle = r(L_{a_4,b_4})\int_0^1 z_4^{p_*}(t)\,\mathrm{d}\zeta^*_{a_4,b_4}(t).
\end{aligned}
$$

Note that $r(L_{a_4,b_4}) > 1$, and we have

$$
\int_0^1 z_4^{p_*}(t)\,\mathrm{d}\zeta^*_{a_4,b_4}(t) \leqslant \frac{2^{p_*-1}[\kappa_H^{p_*}\kappa_G^{p_*/(p-1)}d_2^{p_*/(p-1)} + m^{p_*}\kappa_H^{p_*}d_2^{p_*}]\int_0^1 \mathrm{d}\zeta^*_{a_4,b_4}(t)}{r(L_{a_4,b_4}) - 1}.
$$

From (30) we have

$$
\int_0^1 z_4^{p_*}(t)\,\mathrm{d}\zeta^*_{a_4,b_4}(t) \geqslant \int_0^1 z_4(t)\|z_4\|^{p_*-1}\,\mathrm{d}\zeta^*_{a_4,b_4}(t) \geqslant \omega_0\|z_4\|^{p_*}\int_0^1 t(1-t)\,\mathrm{d}\zeta^*_{a_4,b_4}(t),
$$

and then we obtain

$$\|z_4\|^{p_*} \leqslant \frac{2^{p_*-1}[\kappa_H^{p_*}\kappa_G^{p_*/(p-1)} d_2^{p_*/(p-1)} + m^{p_*}\kappa_H^{p_*} d_2^{p_*}] \int_0^1 \mathrm{d}\zeta^*_{a_4,b_4}(t)}{\omega_0[r(L_{a_4,b_4})-1]\int_0^1 t(1-t)\,\mathrm{d}\zeta^*_{a_4,b_4}(t)}.$$

This implies that S is a bounded set in P. Now, we choose $R_2 > \sup S$ and $R_2 > r_2$ such that

$$z - Tz \neq \nu\widehat{z}, \quad z \in \partial B_{R_2} \cap P,\ \nu \geqslant 0.$$

Lemma 6 implies that

$$i(T,\ B_{R_2}\cap P,\ P) = 0. \tag{32}$$

From (25) and (32) we find

$$i\bigl(T,\ (B_{R_2}\setminus \overline{B}_{r_2})\cap P,\ P\bigr) = i(T,\ B_{R_2}\cap P,\ P) - i(T,\ B_{r_2}\cap P,\ P) = -1.$$

This implies that T has a fixed point in $(B_{R_2}\setminus \overline{B}_{r_2})\cap P$, and thus (1) has at least one positive solution. The proof is completed. □

Remark 2. In [1] the authors established the existence of positive solutions for the fourth-order boundary value problem

$$\begin{aligned} &u^{(4)}(t) - \lambda f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1,\\ &u(0) = u(1) = u''(0) = u''(1) = 0, \end{aligned} \tag{33}$$

where $\lambda > 0$ is a constant, and the nonlinearity f satisfies:

($\mathrm{H_{B1}}$) sublinear growth condition

$$\liminf_{u\to 0^+} \frac{f(t,u)}{u} > \pi^4,\ \limsup_{u\to+\infty} \frac{f(t,u)}{u} < \pi^4$$

uniformly on $t \in [0,1]$;

($\mathrm{H_{B2}}$) superlinear growth condition

$$\liminf_{u\to +\infty} \frac{f(t,u)}{u} > \pi^4,\ \limsup_{u\to 0^+} \frac{f(t,u)}{u} < \pi^4$$

uniformly on $t \in [0,1]$.

If the considered problem (1) is without impulses and the p-Laplacian and $\eta(t) = \gamma(t) \equiv 0$, $t \in [0,1]$, then our conditions (H_3)–(H_6) reduce to the following conditions:

$(\mathrm{H}_3)'$ There exist $a_1 \geqslant 0$ such that $r(L_{G_1,a_1}) > 1$ and

$$\liminf_{z\to 0^+} \frac{f(t,z)}{z} \geqslant a_1 \quad \text{uniformly on } t \in [0,1];$$

$(\mathrm{H}_4)'$ There exist $a_2 \geqslant 0$ such that $r(L_{G_1,a_2}) < 1$ and

$$\limsup_{z\to+\infty} \frac{f(t,z)}{z} \leqslant a_2 \quad \text{uniformly on } t\in[0,1];$$

$(\mathrm{H}_5)'$ There exist $a_3 \geqslant 0$ such that $r(L_{G_1,a_3}) < 1$ and

$$\limsup_{z\to 0^+} \frac{f(t,z)}{z} \leqslant a_3 \quad \text{uniformly on } t\in[0,1];$$

$(\mathrm{H}_6)'$ There exist $a_4 \geqslant 0$ such that $r(L_{G_1,a_4}) > 1$ and

$$\liminf_{z\to+\infty} \frac{f(t,z)}{z} \geqslant a_4 \quad \text{uniformly on } t\in[0,1].$$

Here

$$(L_{G_1,a_i}z)(t) = a_i \int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\,\mathrm{d}s\,z(\tau)\,\mathrm{d}\tau \quad (i=1,2,3,4).$$

Note that $r(L_{G_1,a_i}) = a_i/\pi^4$, and we have $a_1, a_4 > \pi^4$, $a_2, a_3 < \pi^4$. Consequently, (33) is a special case of (1), and our conclusions can be applied to (33).

In what follows, we offer some examples to verify our main results. Let $\eta(t) = t/2$, $\gamma(t) = t/4$, $t\in[0,1]$, and $m=3$, $t_1 = 1/4$, $t_2 = 1/3$, $t_3 = 1/2$. Then $\kappa_H = \max_{t,s\in[0,1]} H(t,s) = 1/2$, $\kappa_G = \max_{t,s\in[0,1]} G(t,s) = 1/3$. Note that from Lemma 5 and Remark 1 we have

$$r(L_{\alpha,\beta}) \geqslant \frac{\alpha}{\pi^4}, \qquad r(L_{\alpha,\beta}) \geqslant 0.172\beta,$$

and

$$r(L_{\alpha,\beta}) \leqslant \frac{\alpha}{6} + 1.32\beta, \quad \alpha,\beta\geqslant 0.$$

Example 1. Let $p = 3/2$, $a_1 = 2\pi^4$, $b_1 = 10$, $a_2 = 2$, $b_2 = 0.3$. Then $p_* = 1/2$, $p^* = 1$, $r(L_{a_1,b_1}) > 1$, $r(L_{a_2,b_2}) < 1$. If we choose

$$f(t,z) = \chi_1(t)z^{0.4}, \quad I_1(z) = \sqrt[4]{z}, \quad I_2(z) = \sqrt[3]{z}, \quad I_3(z) = \sqrt{z}, \quad z\in\mathbb{R}^+,\ t\in[0,1],$$

where $\chi_1(t)$ is a continuous positive function on $[0,1]$, then we have

$$\liminf_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} = \liminf_{z\to 0^+} \frac{\chi_1(t)z^{0.4}}{z^{0.5}} = +\infty, \qquad \limsup_{z\to+\infty} \frac{f(t,z)}{z^{p-1}} = \limsup_{z\to+\infty} \frac{\chi_1(t)z^{0.4}}{z^{0.5}} = 0$$

uniformly on $t\in[0,1]$; and

$$\liminf_{z\to 0^+} \frac{I_k(z)}{z} = \liminf_{z\to 0^+} \frac{\sqrt[5-k]{z}}{z} = +\infty, \qquad \limsup_{z\to+\infty} \frac{I_k(z)}{z} = \limsup_{z\to+\infty} \frac{\sqrt[5-k]{z}}{z} = 0.$$

uniformly on $k = 1,2,3$.

Thus, all the conditions of Theorem 1 are satisfied. Then (1) has at least one positive solution.

Example 2. Let $p = 5/2$, $a_3 = 1$, $b_3 = 0.2$, $a_4 = 3\pi^4$, $b_4 = 15$. Then $p_* = 1$, $p^* = 3/2$, $r(L_{a_3,b_3}) < 1$, $r(L_{a_4,b_4}) > 1$. If we choose

$$f(t,z) = \chi_2(t)z^{1.6}, \quad I_1(z) = z^4, \quad I_2(z) = z^3, \quad I_3(z) = z^2, \quad z \in \mathbb{R}^+,\ t \in [0,1],$$

where $\chi_2(t)$ is a continuous positive function on $[0,1]$, then we have

$$\limsup_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} = \limsup_{z\to 0^+} \frac{\chi_2(t)z^{1.6}}{z^{1.5}} = 0, \qquad \liminf_{z\to +\infty} \frac{f(t,z)}{z^{p-1}} = \liminf_{z\to +\infty} \frac{\chi_2(t)z^{1.6}}{z^{1.5}} = +\infty$$

uniformly on $t \in [0,1]$; and

$$\limsup_{z\to 0^+} \frac{I_k(z)}{z} = \limsup_{z\to 0^+} \frac{z^{5-k}}{z} = 0, \qquad \liminf_{z\to +\infty} \frac{I_k(z)}{z} = \liminf_{z\to +\infty} \frac{z^{5-k}}{z} = +\infty$$

uniformly on $k = 1, 2, 3$.

Thus, all the conditions of Theorem 2 are satisfied. Then (1) has at least one positive solution.

Conflicts of interest. The authors declare no conflicts of interest.

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