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# Existence of positive solutions for tempered fractional differential equations with $p$-Laplacian operator*

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**Abstract.** This paper investigates the existence of positive solutions for a specific category of $p$-Laplacian tempered fractional differential equations in which the nonlinear term $f$ contains an integral operator $\theta$. By employing fixed point theorems for sum operators in partially ordered Banach spaces, together with Krasnosel'skii fixed point theorem, the existence of positive solutions is established. Moreover, iterative sequences are constructed to approximate the unique positive solution of the problem. Finally, three examples are presented to illustrate the main results.

**Keywords:** $p$-Laplacian operator, tempered fractional derivative, fractional differential equation, Krasnosel'skii fixed point theorem, fixed point theorems for a sum operator.

## 1 Introduction

Fractional-order differential equations can be extensively applied in a variety of fields, such as engineering, biophysics, chemical physics, fluid flow, economics, and so on. In addition, fractional differential equations are capable of more accurately identifying the essence of real-world problems compared to the integer-order differential equations in many research areas of science and engineering. For example, in materials science, viscoelastic mechanics, fractional-order models are more suitable for describing the properties of viscoelastic materials over a broad frequency range; see [5, 22]. In the field of

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mechanical engineering, introducing fractional-order terms into dynamic systems can form fractional-order controllers, which effectively improve the control performance of the systems; see [4, 10]. For the other related applications and details about fractional differential equation, see [1,9,11,13,18,23,29]. In fractional differential equation studies, boundary value problems stand out as an essential research area, and lots of excellent results have been obtained by means of fixed point theorems, such as Guo–Krasnosel'skii fixed point theorem [26, 27], Krasnosel'skii fixed point theorem [3, 6, 14, 25], Avery–Peterson fixed point theorem [17], Leggett–Williams fixed point theorem [2,19], Schauder fixed point theorem and contraction mapping principle [8, 15], monotone iterative technique [7, 12, 21, 30], upper and lower solutions technique [16, 19, 28], and so forth.

In [3], the authors studied the following nonlinear fractional boundary value problem:

$$\begin{aligned}&{}^{C}D^{\alpha}_{0+}\bigl({}^{C}D^{\beta}_{0+}u\bigr)(t) = f\bigl(t, u(t), \varphi u(t), \psi u(t)\bigr), \quad 0 < t < 1,\\ &u(1) = u(0) = u'(1) = 0,\end{aligned}$$

where $1 < \alpha \leqslant 2$, $0 < \beta \leqslant 1$, $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and

$$\varphi u(t) = \int_0^t \gamma(t,s)u(s)\,\mathrm{d}s, \qquad \psi u(t) = \int_0^t \lambda(t,s)u(s)\,\mathrm{d}s,$$

where $\gamma, \lambda : [0,1] \times [0,1] \to \mathbb{R}^+$ with the properties $\sup_{t\in[0,1]}(\int_0^1 \gamma(t,s)\,\mathrm{d}s) < \infty$ and $\sup_{t\in[0,1]}(\int_0^1 \lambda(t,s)\,\mathrm{d}s) < \infty$. By using Krasnosel'skii fixed point theorem, the author obtained the existence of solutions for nonlinear fractional integro-differential equations.

In [20], the authors considered the following Hadamard-type fractional $p$-Laplacian integral boundary value problem:

$$\begin{aligned}&D^{\alpha}_{1+}\bigl(\varphi_p\bigl(D^{\beta}_{1+}\chi(t)\bigr)\bigr) = f\bigl(t, \chi(t)\bigr), \quad 1 < t < \mathrm{e},\\ &D^{\beta}_{1+}\chi(1) = D^{\beta}_{1+}\chi(\mathrm{e}) = 0, \qquad \chi(1) = \delta\chi(1) = 0,\\ &\delta\chi(\mathrm{e}) = \int_1^{\mathrm{e}} \eta(t)g\bigl(t, \chi(t)\bigr)\frac{\mathrm{d}t}{t},\end{aligned}$$

where $D^{\alpha}$, $D^{\beta}$ are the Hadamard-type fractional derivatives with $\alpha \in (1,2]$, $\beta \in (2,3]$, $\delta\chi(t) = t\,\mathrm{d}\chi/\mathrm{d}t$, and $\varphi_p(s) = |s|^{p-2}s$ is the $p$-Laplacian operator, $p > 1$, $s \in \mathbb{R}$. The functions $f, g \in C([1,\mathrm{e}] \times \mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0,\infty)$, $\eta$ is a nonnegative continuous function on $[1,e]$ with $\eta(t) \not\equiv 0$. By using fixed point index, the author obtained the existence of positive solutions.

In [24], the author discussed the following Hadamard-type fractional boundary value problem on an infinite interval:

$$\begin{aligned}&{}^{H}D^{\alpha}_{1+}x(t) + a(t)f\bigl(t, x(t)\bigr) + b(t)g\bigl(t, x(t)\bigr) = 0, \quad t \in (1, +\infty),\\ &x(1) = x'(1) = 0, \qquad {}^{H}D^{\alpha-1}_{1+}x(+\infty) = \sum_{i=1}^{m} \alpha_i\,{}^{H}I^{\beta_i}_{1+}x(\eta) + c\sum_{j=1}^{n}\sigma_j x(\xi_j),\end{aligned}$$

where ${}^{H}D^{\alpha}_{1+}$ is the Hadamard-type fractional derivative, $2 < \alpha < 3$; ${}^{H}I^{\beta_i}_{1+}$ is the Hadamard-type fractional integral; $1 < \eta < \xi_1 < \xi_2 < \cdots < \xi_n < +\infty$; $c, \alpha_i, \sigma_j \geqslant 0$ ($i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$) are some given constants. $a, b \in C(J, \mathbb{R}^+)$, $f, g \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $J = [1, \infty)$. By two fixed point theorems of a sum operator in partial ordering Banach spaces, the author obtained the existence and uniqueness of positive solutions.

In [27], the author explored the following singular tempered fractional equation:

$$\begin{aligned} &-D^{\alpha,\lambda}_{0+}u(t) = p(t)h\bigl(\mathrm{e}^{\lambda t}u(t), D^{\beta,\lambda}_{0+}u(t)\bigr), \quad 0 < t < 1, \\ &D^{\beta,\lambda}_{0+}u(0) = 0, \qquad D^{\beta,\lambda}_{0+}u(1) = 0, \end{aligned}$$

where $\alpha \in (1, 2]$, $\beta \in (0, 1)$ with $\alpha - \beta > 1$, $h \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $p \in L^1((0, 1), \mathbb{R}^+_0)$ with $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^+_0 = (0, \infty)$. By using the Guo–Krasnosel'skii fixed point theorem, the author obtained the existence of the multiple positive solutions. $D^{\alpha,\lambda}_{0+}$ and $D^{\beta,\lambda}_{0+}$ are the tempered fractional derivatives, which are actually obtained by multiplying an exponential factor in the Riemann–Liouville fractional derivative, i.e., the following relationship exists between tempered fractional derivative and Riemann–Liouville fractional derivative:

$$D^{\alpha,\lambda}_{0+}u(t) = \mathrm{e}^{-\lambda t} D^{\alpha}_{0+}\bigl(\mathrm{e}^{\lambda t}u(t)\bigr), \quad \lambda \geqslant 0. \tag{1}$$

For the definition of the standard Riemann–Liouville fractional integral and derivative, we refer the reader to [13]. From this relation we can obtain that when $\lambda = 0$, the tempered fractional-order derivative becomes the Riemann–Liouville fractional derivative, thus the tempered fractional-order derivative is an exponential optimisation of the Riemann–Liouville fractional derivative. In contrast to the classical Riemann–Liouville fractional derivative, the tempered fractional derivative incorporates an exponential tempering factor that effectively truncates the heavy-tailed power-law kernel. This modification preserves the essential nonlocal and memory effects of fractional operators, while avoiding unrealistic infinite moments and excessively long-range interactions. Therefore, tempered fractional derivatives provide a more accurate and physically meaningful framework for modeling practical phenomena in which anomalous behavior exists only within finite spatial or temporal scales, such as truncated Lévy flights, bounded diffusion processes, and transport in complex media. This distinctive capability to bridge theoretical generality and empirical realism constitutes the core motivation of the present study.

Motivated by the excellent work above, in this paper, we consider the $p$-Laplacian tempered fractional differential equation

$$\begin{aligned} &-D^{\alpha,\lambda}_{0+}\bigl(\varphi_p\bigl(-D^{\beta,\lambda}_{0+}u(t)\bigr)\bigr) = f\bigl(t, u(t), \theta u(t)\bigr), \quad 0 < t < 1, \\ &u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D^{\beta,\lambda}_{0+}u(0) = 0, \\ &D^{\gamma,\lambda}_{0+}u(1) = \int_0^1 \mathrm{e}^{-\lambda} a(s)\psi\bigl(s, u(s), \theta u(s)\bigr)\,\mathrm{d}s, \\ &\varphi_p\bigl(-D^{\beta,\lambda}_{0+}u(1)\bigr) = \varrho \int_0^1 \mathrm{e}^{-\lambda(1-s)}\bigl(\varphi_p\bigl(-D^{\beta,\lambda}_{0+}u(s)\bigr)\bigr)\,\mathrm{d}s, \end{aligned} \tag{2}$$

where $1 < \alpha \leqslant 2$, $n-1 < \beta \leqslant n$, $\varrho < \alpha$, $0 < \gamma < \beta - 1$, $n \geqslant 3$, $D_{0+}^{\alpha,\lambda}$, $D_{0+}^{\beta,\lambda}$, and $D_{0+}^{\gamma,\lambda}$ are the tempered fractional derivatives. $\varphi_p$ is the $p$-Laplacian operator, that is, $\varphi_p(s) = |s|^{p-2}s$ with $p > 1$, $\varphi_p^{-1} = \varphi_q$, $1/p + 1/q = 1$ and $\theta u(t) = \int_0^t H(t,s)u(s)\,\mathrm{d}s$, where $H : [0,1] \times [0,1] \to \mathbb{R}^+$ with the property $\sigma = \sup_{t\in[0,1]}(\int_0^t H(t,s)\,\mathrm{d}s) < \infty$ and $\mathbb{R}^+ = [0,\infty)$.

The purpose of this paper is to investigate the existence and uniqueness of positive solutions for the $p$-Laplacian tempered fractional differential equation (2). Compared with [20, 24, 27], problem (2) we study contains an integral operator $\theta$ in the nonlinear term, which leads to more difficulties in the analysis of existence results of solutions. Compared with [20], the derivative we used is the tempered fractional derivative $D_{0+}^{\alpha,\lambda}$, this is more general than $D_{0+}^{\alpha}$. We can obtain that when $\lambda = 0$, $D_{0+}^{\alpha,\lambda}$ is equivalent $D_{0+}^{\alpha}$. Compared with [3], the nonlinearity we study may be singular at $t = 0$ or $t = 1$ when using Krasnosel'skii fixed point theorem. Compared with [24], the construction of our sum operator is different, and the boundary value conditions involve integral, which is more general than multipoint boundary value conditions. Compared with [3, 20, 27], our boundary conditions are more complicated, and we not only obtain the existence of positive solution, but also construct a Cauchy sequence to approximate the unique positive solution.

Throughout this paper, we suppose that the following conditions are satisfied.

(H0) $a$ is a nonnegative continuous function on $[0,1]$ with $a(t) \not\equiv 0$, $t \in [0,1]$ and $a^* = \int_0^1 a(t)\,\mathrm{d}t$.

## 2 Preliminaries and lemmas

In this section, for the convenience of reader, we introduce some notations and definitions that will be used in the proof of our main results.

**Lemma 1.** (*See* [13].) *Let $y(t) \in (0,1) \cap L^1(0,1)$, $\alpha > 0$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} y(t) = y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

*where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, and $n - 1 < \alpha < n$.*

**Lemma 2.** (*See* [13].)

(i) *If $u \in L^1(0,1)$, $v > \sigma > 0$, then*

$$I_{0+}^{v} I_{0+}^{\sigma} u(t) = I_{0+}^{v+\sigma} u(t), \qquad D_{0+}^{\sigma} I_{0+}^{v} u(t) = I_{0+}^{v-\sigma} u(t),$$
$$D_{0+}^{\sigma} I_{0+}^{\sigma} u(t) = u(t).$$

(ii) *If $\rho > 0$, $\mu > 0$, then*

$$D_{0+}^{\rho} t^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu-\rho)} t^{\mu-\rho-1}.$$

**Lemma 3.** *Let $g \in C(0,1) \cap L^1(0,1)$, $n-1 < \beta \leqslant n$, $0 < \gamma < \beta - 1$, $n \geqslant 3$, then the tempered boundary value problem*

$$\begin{aligned} &D_{0+}^{\beta,\lambda} u(t) + g(t) = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ &D_{0+}^{\gamma,\lambda} u(1) = \int_0^1 \mathrm{e}^{-\lambda} a(s) \psi\big(s, u(s), \theta u(s)\big)\,\mathrm{d}s \end{aligned} \tag{3}$$

*has the unique solution*

$$u(t) = \int_0^1 G(t,s) g(s)\,\mathrm{d}s + \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t} t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\big(s, u(s), \theta u(s)\big)\,\mathrm{d}s,$$

*where*

$$G(t,s) = \frac{\mathrm{e}^{\lambda(s-t)}}{\Gamma(\beta)} \begin{cases} t^{\beta-1}(1-s)^{\beta-\gamma-1} - (t-s)^{\beta-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\beta-1}(1-s)^{\beta-\gamma-1}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{4}$$

*Proof.* For problem (3), by (1) and Lemma 1, we get

$$\mathrm{e}^{\lambda t} u(t) = -I_{0+}^{\beta}\big(\mathrm{e}^{\lambda t} g(t)\big) + c_1 t^{\beta-1} + c_2 t^{\beta-2} + \cdots + c_n t^{\beta-n},$$

where $c_1, c_2, \dots, c_n$ represent real constants. By employing the boundary condition provided in (3), we can obtain $c_n = c_{n-1} = \cdots = c_2 = 0$. Thus,

$$\begin{aligned} u(t) &= -\mathrm{e}^{-\lambda t} I_{0+}^{\beta}\big(\mathrm{e}^{\lambda t} g(t)\big) + c_1 \mathrm{e}^{-\lambda t} t^{\beta-1} \\ &= -\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \mathrm{e}^{-\lambda t} \mathrm{e}^{\lambda s} g(s)\,\mathrm{d}s + c_1 \mathrm{e}^{-\lambda t} t^{\beta-1}. \end{aligned} \tag{5}$$

By using the tempered fractional derivative operator $D_{0+}^{\gamma,\lambda}$ on both sides of (5), we get

$$\begin{aligned} D_{0+}^{\gamma,\lambda} u(t) &= -D_{0+}^{\gamma,\lambda}\big(\mathrm{e}^{-\lambda t} I_{0+}^{\beta}\big(\mathrm{e}^{\lambda t} g(t)\big)\big) + c_1 D_{0+}^{\gamma,\lambda}\big(\mathrm{e}^{-\lambda t} t^{\beta-1}\big) \\ &= -\mathrm{e}^{-\lambda t} D_{0+}^{\gamma} I_{0+}^{\beta}\big(\mathrm{e}^{\lambda t} g(t)\big) + c_1 \mathrm{e}^{-\lambda t} D_{0+}^{\gamma} t^{\beta-1} \\ &= -\mathrm{e}^{-\lambda t} I_{0+}^{\beta-\gamma}\big(\mathrm{e}^{\lambda t} g(t)\big) + c_1 \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)} \mathrm{e}^{-\lambda t} t^{\beta-\gamma-1} \\ &= -\int_0^t \frac{(t-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \mathrm{e}^{-\lambda t} \mathrm{e}^{\lambda s} g(s)\,\mathrm{d}s + c_1 \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)} \mathrm{e}^{-\lambda t} t^{\beta-\gamma-1}. \end{aligned} \tag{6}$$

Combining (6) with $(3)_3$, we obtain

$$c_1 = \int_0^1 \frac{(1-s)^{\beta-\gamma-1}}{\Gamma(\beta)} \mathrm{e}^{\lambda s} g(s)\,\mathrm{d}s + \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)} \int_0^1 a(s)\psi\big(s, u(s), \theta u(s)\big)\,\mathrm{d}s. \tag{7}$$

Substituting (7) into (5), we have

$$\begin{aligned} u(t) &= \int_0^1 \frac{(1-s)^{\beta-\gamma-1}t^{\beta-1}}{\Gamma(\beta)} \mathrm{e}^{-\lambda t}\mathrm{e}^{\lambda s} g(s)\,\mathrm{d}s - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \mathrm{e}^{-\lambda t}\mathrm{e}^{\lambda s} g(s)\,\mathrm{d}s \\ &\quad + \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\big(s, u(s), \theta u(s)\big)\,\mathrm{d}s \\ &= \int_0^1 G(t,s)g(s)\,\mathrm{d}s + \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\big(s, u(s), \theta u(s)\big)\,\mathrm{d}s. \end{aligned}$$

This completes the proof. □

**Lemma 4.** *Let* $\widetilde{g} \in C(0,1) \cap L^1(0,1)$, $1 < \alpha < 2$, $\varrho < \alpha$, $n-1 < \beta \leqslant n$, $n \geqslant 3$. *Then the tempered boundary value problem*

$$\begin{aligned} &-D_{0^+}^{\alpha,\lambda}\big(\varphi_p\big(-D_{0^+}^{\beta,\lambda}u(t)\big)\big) = \widetilde{g}(t), \quad 0<t<1, \\ &u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0^+}^{\beta,\lambda}u(0) = 0, \\ &D_{0^+}^{\gamma,\lambda}u(1) = \int_0^1 \mathrm{e}^{-\lambda} a(s)\psi\big(s, u(s), \theta u(s)\big)\,\mathrm{d}s, \\ &\varphi_p\big(-D_{0^+}^{\beta,\lambda}u(1)\big) = \varrho \int_0^1 \mathrm{e}^{-\lambda(1-s)}\big(\varphi_p\big(-D_{0^+}^{\beta,\lambda}u(s)\big)\big)\,\mathrm{d}s \end{aligned} \tag{8}$$

*has the unique solution*

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau)\widetilde{g}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\quad + \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\big(s, u(s), \theta u(s)\big)\,\mathrm{d}s, \end{aligned}$$

*where* $G(t,s)$ *is given as* (4), $K(t,s)$ *is a Green function, and*

$$K(t,s) = \mathrm{e}^{\lambda(s-t)} \begin{cases} \frac{\alpha(1-s)^{\alpha-1}(\alpha-\varrho+\varrho s)t^{\alpha-1} - \alpha(\alpha-\varrho)(t-s)^{\alpha-1}}{(\alpha-\varrho)\Gamma(\alpha+1)}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \frac{\alpha(1-s)^{\alpha-1}(\alpha-\varrho+\varrho s)t^{\alpha-1}}{(\alpha-\varrho)\Gamma(\alpha+1)}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{9}$$

*Proof.* Using (1) and Lemma 1, we can express the solution to (8) as

$$\mathrm{e}^{\lambda t}\varphi_p\big(-D_{0^+}^{\beta,\lambda}u(t)\big) = -I_{0^+}^{\alpha}\big(\mathrm{e}^{\lambda t}\widetilde{g}(t)\big) + d_1 t^{\alpha-1} + d_2 t^{\alpha-2},$$

where $d_1$ and $d_2$ are real constants. By employing the boundary condition $D_{0^+}^{\beta,\lambda}u(0)=0$ provided in (8), we can obtain $d_2 = 0$, that is,

$$\mathrm{e}^{\lambda t}\varphi_p\big(-D_{0^+}^{\beta,\lambda}u(t)\big) = -I_{0^+}^{\alpha}\big(\mathrm{e}^{\lambda t}\widetilde{g}(t)\big) + d_1 t^{\alpha-1}. \tag{10}$$

Moreover, we have

$$\begin{aligned}\int_0^1 \mathrm{e}^{\lambda t}\varphi_p\big(-D_{0^+}^{\beta,\lambda}u(t)\big)\,\mathrm{d}t &= -\int_0^1\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{e}^{\lambda s}\widetilde{g}(s)\,\mathrm{d}s\right)\mathrm{d}t + d_1\int_0^1 t^{\alpha-1}\,\mathrm{d}t\\ &= -\int_0^1 \frac{\mathrm{e}^{\lambda s}\widetilde{g}(s)}{\Gamma(\alpha)}\int_s^1 (t-s)^{\alpha-1}\,\mathrm{d}t\,\mathrm{d}s + \frac{d_1}{\alpha}\\ &= -\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}\mathrm{e}^{\lambda s}\widetilde{g}(s)\,\mathrm{d}s + \frac{d_1}{\alpha}.\end{aligned}$$

By the boundary condition $(8)_4$, we can get

$$d_1 = \int_0^1 \frac{\alpha^2(1-s)^{\alpha-1} - \alpha\varrho(1-s)^{\alpha}}{(\alpha-\varrho)\Gamma(\alpha+1)}\mathrm{e}^{\lambda s}\widetilde{g}(s)\,\mathrm{d}s. \tag{11}$$

Substituting (11) into (10), we have

$$\begin{aligned}\varphi_p\big(-D_{0^+}^{\beta,\lambda}u(t)\big) = &-\int_0^t \frac{\alpha(\alpha-\varrho)(t-s)^{\alpha-1}}{(\alpha-\varrho)\Gamma(\alpha+1)}\mathrm{e}^{-\lambda t}\mathrm{e}^{\lambda s}\widetilde{g}(s)\,\mathrm{d}s\\ &+ t^{\alpha-1}\int_0^1 \frac{\alpha^2(1-s)^{\alpha-1} - \alpha\varrho(1-s)^{\alpha}}{(\alpha-\varrho)\Gamma(\alpha+1)}\mathrm{e}^{-\lambda t}\mathrm{e}^{\lambda s}\widetilde{g}(s)\,\mathrm{d}s\\ = &\int_0^1 K(t,s)\widetilde{g}(s)\,\mathrm{d}s.\end{aligned} \tag{12}$$

By employing the $p$-Laplacian operator $\varphi_q$ on both sides of (12), we have

$$D_{0^+}^{\beta,\lambda}u(t) + \varphi_q\left(\int_0^1 K(t,s)\widetilde{g}(s)\,\mathrm{d}s\right) = 0.$$

Setting $g(t) \triangleq \varphi_q(\int_0^1 K(t,s)\widetilde{g}(s)\,\mathrm{d}s)$, the $p$-Laplacian tempered fractional boundary value problem (8) is equivalent to the following fractional boundary value problem:

$$\begin{aligned}&D_{0^+}^{\beta,\lambda}u(t) + g(t) = 0, \quad 0 < t < 1,\\ &u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,\\ &D_{0^+}^{\gamma,\lambda}u(1) = \int_0^1 \mathrm{e}^{-\lambda}a(s)\psi\big(s,u(s),\theta u(s)\big)\,\mathrm{d}s.\end{aligned}$$

By Lemma 3, we can obtain that the above problem has a unique integral solution

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)g(s)\,\mathrm{d}s + \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\bigl(s,u(s),\theta u(s)\bigr)\,\mathrm{d}s \\ &= \int_0^1 G(t,s)\varphi_q\Biggl(\int_0^1 K(s,\tau)\widetilde{g}(\tau)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \\ &\quad + \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\bigl(s,u(s),\theta u(s)\bigr)\,\mathrm{d}s, \end{aligned}$$

where the Green function $G(t,s)$ and $K(t,s)$ are given by (4) and (9), respectively. □

**Lemma 5.** *The functions $G(t,s)$ and $K(t,s)$ given by* (4) *and* (9)*, respectively, admit the following properties*:

(i) *$G(t,s)$, $K(t,s)$ are continuous and nonnegative for $(t,s)\in[0,1]\times[0,1]$.*

(ii) *$m(s)\mathrm{e}^{-\lambda t}t^{\beta-1} \leqslant G(t,s) \leqslant M(s)\mathrm{e}^{-\lambda t}t^{\beta-1}$ with*

$$m(s) = \frac{[(1-s)^{\beta-\gamma-1}-(1-s)^{\beta-1}]\mathrm{e}^{\lambda s}}{\Gamma(\beta)}, \qquad M(s) = \frac{(1-s)^{\beta-\gamma-1}\mathrm{e}^{\lambda s}}{\Gamma(\beta)}.$$

(iii) *$n(s)\mathrm{e}^{-\lambda t}t^{\alpha-1} \leqslant K(t,s) \leqslant N(s)\mathrm{e}^{-\lambda t}t^{\alpha-1}$ with*

$$n(s) = \frac{\alpha\varrho s(1-s)^{\alpha-1}\mathrm{e}^{\lambda s}}{(\alpha-\varrho)\Gamma(\alpha+1)}, \qquad N(s) = \frac{\alpha(1-s)^{\alpha-1}(\alpha-\varrho+\varrho s)\mathrm{e}^{\lambda s}}{(\alpha-\varrho)\Gamma(\alpha+1)}.$$

*Proof.* It is easy to check that $G(t,s)$ and $H(t,s)$ are continuous, $G(t,s) \leqslant M(s)\mathrm{e}^{-\lambda t} \times t^{\beta-1}$, and $K(t,s) \leqslant N(s)\mathrm{e}^{-\lambda t}t^{\alpha-1}$. So we only need to prove the left sides of the inequalities and $G(t,s) \geqslant 0$, $K(t,s) \geqslant 0$.

For $0\leqslant s\leqslant t\leqslant 1$, we get $0 \leqslant t-s \leqslant t-ts = (1-s)t$, then $(t-s)^{\beta-1} \leqslant (1-s)^{\beta-1}t^{\beta-1}$. Hence, we have

$$\begin{aligned} G(t,s) &= \frac{t^{\beta-1}(1-s)^{\beta-\gamma-1}-(t-s)^{\beta-1}}{\Gamma(\beta)}\mathrm{e}^{\lambda s}\mathrm{e}^{-\lambda t} \\ &\geqslant \frac{t^{\beta-1}(1-s)^{\beta-\gamma-1}-(1-s)^{\beta-1}t^{\beta-1}}{\Gamma(\beta)}\mathrm{e}^{\lambda s}\mathrm{e}^{-\lambda t} \\ &= \frac{\bigl[(1-s)^{\beta-\gamma-1}-(1-s)^{\beta-1}\bigr]\mathrm{e}^{\lambda s}}{\Gamma(\beta)}\mathrm{e}^{-\lambda t}t^{\beta-1} \geqslant 0. \end{aligned}$$

For $0\leqslant t\leqslant s\leqslant 1$, we can easily see that

$$\begin{aligned} G(t,s) &= \frac{t^{\beta-1}(1-s)^{\beta-\gamma-1}}{\Gamma(\beta)}\mathrm{e}^{\lambda s}\mathrm{e}^{-\lambda t} \\ &\geqslant \frac{\bigl[(1-s)^{\beta-\gamma-1}-(1-s)^{\beta-1}\bigr]\mathrm{e}^{\lambda s}}{\Gamma(\beta)}\mathrm{e}^{-\lambda t}t^{\beta-1} \geqslant 0. \end{aligned}$$

For $0 \leqslant s \leqslant t \leqslant 1$, we get $0 \leqslant t-s \leqslant t-ts = (1-s)t$, then $(t-s)^{\alpha-1} \leqslant (1-s)^{\alpha-1}t^{\alpha-1}$. Hence, we have

$$\begin{aligned} K(t,s) &= \frac{[\alpha^2(1-s)^{\alpha-1} - \alpha\varrho(1-s)^{\alpha}]t^{\alpha-1} - \alpha(\alpha-\varrho)(t-s)^{\alpha-1}}{(\alpha-\varrho)\Gamma(\alpha+1)}\mathrm{e}^{\lambda s}\mathrm{e}^{-\lambda t} \\ &\geqslant \frac{[\alpha^2(1-s)^{\alpha-1} - \alpha\varrho(1-s)^{\alpha}]t^{\alpha-1} - \alpha(\alpha-\varrho)(1-s)^{\alpha-1}t^{\alpha-1}}{(\alpha-\varrho)\Gamma(\alpha+1)}\mathrm{e}^{\lambda s}\mathrm{e}^{-\lambda t} \\ &= \frac{\alpha\varrho s(1-s)^{\alpha-1}\mathrm{e}^{\lambda s}}{(\alpha-\varrho)\Gamma(\alpha+1)}\mathrm{e}^{-\lambda t}t^{\alpha-1} \geqslant 0. \end{aligned}$$

For $0 \leqslant t \leqslant s \leqslant 1$,

$$\begin{aligned} K(t,s) &= \frac{[\alpha^2(1-s)^{\alpha-1} - \alpha\varrho(1-s)^{\alpha}]t^{\alpha-1}}{(\alpha-\varrho)\Gamma(\alpha+1)}\mathrm{e}^{\lambda s}\mathrm{e}^{-\lambda t} \\ &\geqslant \frac{[\alpha^2(1-s)^{\alpha-1} - \alpha\varrho(1-s)^{\alpha}]t^{\alpha-1} - \alpha(\alpha-\varrho)(1-s)^{\alpha-1}t^{\alpha-1}}{(\alpha-\varrho)\Gamma(\alpha+1)}\mathrm{e}^{\lambda s}\mathrm{e}^{-\lambda t} \\ &= \frac{\alpha\varrho s(1-s)^{\alpha-1}\mathrm{e}^{\lambda s}}{(\alpha-\varrho)\Gamma(\alpha+1)}\mathrm{e}^{-\lambda t}t^{\alpha-1} \geqslant 0. \end{aligned}$$

This completes the proof. □

Let $(E, \|\cdot\|)$ be a real Banach space, and let $\theta$ be the zero element of $E$. $E$ is partially ordered by a cone $P \subset E$, i.e., $x \leqslant y$ if and only if $y - x \in P$. A cone $P$ is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$. In this case, $N$ is called the normality constant of $P$. We say that an operator $A : E \to E$ is increasing (decreasing) if $x \leqslant y$ implies $Ax \leqslant Ay$ ($Ax \geqslant Ay$).

For $x, y \in E$, the notation $x \sim y$ denotes that there exist $l > 0$ and $\mu > 0$ such that $lx \leqslant y \leqslant \mu x$. Clearly, $\sim$ is an equivalence relation. For $h > \theta$ (i.e., $h \geqslant \theta$ and $h \neq \theta$), define $P_h = \{x \in E\colon x \sim h\}$. It is easy to see that $P_h \subset P$.

**Definition 1.** (See [9].) Let $0 < \delta < 1$. An operator $A : P \to P$ is said to be $\delta$-concave if $A(tx) \geqslant t^{\delta}Ax$ for $t \in (0,1)$, $x \in P$. An operator $A : P \to P$ is called to be subhomogeneous if $A(tx) \geqslant tAx$ for $t > 0$, $x \in P$.

**Lemma 6.** (*See* [23].) *Let $E$ be a real Banach space. $P$ is a normal cone in $E$, $A, B : P \to P$ are increasing operators, $A$ is $\delta$-concave, and $B$ is subhomogeneous. Suppose that*

(i) *there is $h > \theta$ such that $Ah \in P_h$ and $Bh \in P_h$;*
(ii) *there exists a constant $\delta_0 > 0$ such that $Ax \geqslant \delta_0 Bx$ for all $x \in P$.*

*Then the operator equation*

$$Ax + Bx = x \tag{13}$$

*has a unique solution $x^* \in P_h$. Further, making the sequence*

$$x_n = Ax_{n-1} + Bx_{n-1}, \quad n = 1, 2, \dots,$$

*for any initial value $x_0 \in P_h$, one has $x_n \to x^*$ as $n \to \infty$.*

**Lemma 7.** (*See* [21].) *Let $E$ be a real Banach space. $P$ is a normal cone in $E$, $A: P \to P$ is an increasing operator, and $B: P \to P$ is a decreasing operator. In addition,*

(i) *for $x \in P$ and $t \in (0,1)$, there exist $\phi_i(t) \in (t,1)$, $i = 1,2$, such that $A(tx) \geqslant \phi_1(t)Ax$, $B(tx) \leqslant Bx/\phi_2(t)$;*

(ii) *there is $h_0 \in P_h$ such that $Ah_0 + Bh_0 \in P_h$.*

*Then the operator equation* (13) *has a unique solution $x^* \in P_h$. Further, for any initial values $x_0, y_0 \in P_h$, making the sequences*

$$x_n = Ax_{n-1} + By_{n-1}, \quad y_n = Ay_{n-1} + Bx_{n-1}, \quad n = 1,2,\dots,$$

*one has $x_n \to x^*$, $y_n \to x^*$ as $n \to \infty$.*

**Remark 1.** If $B$ is a null operator, the conclusions in Lemmas 6 and 7 are still true.

**Lemma 8 [Krasnosel'skii fixed point theorem].** (*See* [14].) *Let $X$ be a closed convex and nonempty subset of a Banach space $E$. Let $A$ and $B$ be two operators such that*

(i) *$Ax + By \in X$, whenever $x, y \in X$;*

(ii) *$A$ is a contraction;*

(iii) *$B$ is compact and continuous.*

*Then there exists $z \in X$ such that $z = Az + Bz$.*

## 3 Main results

In this section, we work in a Banach space $E = C[0,1]$ equipped with the norm $\|x\| = \max\{|x(t)|, t \in [0,1]\}$. Let $P = \{x \in C[0,1]\colon x(t) \geqslant 0, t \in [0,1]\}$, then it is a normal cone in $C[0,1]$. This space is equipped with a partial order

$$x \leqslant y, \quad x, y \in C[0,1] \iff x(t) \leqslant y(t), \quad t \in [0,1].$$

**Theorem 1.** *Assume that* (H0) *and the following conditions hold true:*

(H1) *$f, \psi: [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and increasing with respect to the second and third arguments, $f(t,0,0) \not\equiv 0$, $t \in [0,1]$.*

(H2) *There exist constants $\varkappa \in (0, 1/(q-1))$ and $\delta \in (0,1)$ such that $f(t, \kappa u, \kappa v) \geqslant \kappa^{\varkappa} f(t,u,v)$, $\psi(t, \kappa u, \kappa v) \geqslant \kappa^{\delta} \psi(t,u,v)$ for any $\kappa \in (0,1)$, $t \in [0,1]$, $u, v \in \mathbb{R}^+$.*

(H3) *There exists a constant $\delta_0 = \int_0^1 a(t)\psi(t,0,0)\,\mathrm{d}t > 0$ such that $f(t,u,v) \leqslant \delta_0$, $t \in [0,1]$, $u, v \in \mathbb{R}^+$.*

*Then problem* (2) *has a unique positive solution $u^* \in P_h$ with $h(t) = \mathrm{e}^{-\lambda t} t^{\beta-1}$ for $t \in [0,1]$. For any initial value $u_0 \in P_h$, define the iterative sequence*

$$\begin{aligned} u_{n+1}(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau) f\big(\tau, u_n(\tau), \theta u_n(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\quad + \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t} t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\big(s, u_n(s), \theta u_n(s)\big)\,\mathrm{d}s, \quad n = 0,1,2,\dots. \end{aligned}$$

*The sequence $u_n(t)$ converges uniformly to the unique positive solution $u^*(t)$ of problem* (2) *on $[0,1]$ as $n \to \infty$, where $G(t,s)$ and $K(s,\tau)$ are given by* (4) *and* (9)*, respectively.*

*Proof.* From Lemma 4 we can deduce that the $p$-Laplacian tempered fractional boundary value problem (2) is equivalent to the integral formulation given by

$$u(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau)f\big(t,u(\tau),\theta u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s + \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\int_0^1 a(s)\psi\big(s,u(s),\theta u(s)\big)\,\mathrm{d}s.$$

Define two operators $A: P \to E$ and $B: P \to E$ by

$$Au(t) = \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\int_0^1 a(s)\psi\big(s,u(s),\theta u(s)\big)\,\mathrm{d}s,$$

$$Bu(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau)f\big(\tau,u(\tau),\theta u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Then we see that $u$ is the solution of problem (2) if and only if $u = Au + Bu$. From (H0), (H1), and Lemma 5 we can easily obtain that $A: P \to P$ and $B: P \to P$. Next, we show that all the conditions of Lemma 6 are satisfied and divide it into the following five steps.

*Step 1.* For $u, v \in P$ with $u \leqslant v$, we have $u(t) \leqslant v(t)$, $t \in [0,1]$. By (H1), we get

$$Au(t) = \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\int_0^1 a(s)\psi\big(s,u(s),\theta u(s)\big)\,\mathrm{d}s \leqslant \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\int_0^1 a(s)\psi\big(s,v(s),\theta v(s)\big)\,\mathrm{d}s = Av(t).$$

Since $\varphi_q(t)$ is increasing in $t$, we get

$$Bu(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau)f\big(\tau,u(\tau),\theta u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \leqslant \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau)f\big(\tau,v(\tau),\theta v(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s = Bv(t).$$

That is, $Au \leqslant Av$ and $Bu \leqslant Bv$.

*Step 2*. We prove that operator $A$ is $\delta$-concave. For any $\kappa \in (0,1)$ and $u \in P$, by (H2), we have

$$\begin{aligned} A(\kappa u)(t) &= \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\big(s,\kappa u(s),\kappa(\theta u)(s)\big)\,\mathrm{d}s \\ &\geqslant \kappa^{\delta} \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\big(s,u(s),\theta u(s)\big)\,\mathrm{d}s = \kappa^{\delta} Au(t), \end{aligned}$$

that is, $A(\kappa u) \geqslant \kappa^{\delta} Au$ for $\kappa \in (0,1)$, $u \in P$.

*Step 3*. We prove that operator $B$ is subhomogeneous. For any $\kappa \in (0,1)$ and $u \in P$, by (H2), we have

$$\begin{aligned} B(\kappa u)(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau) f\big(\tau,\kappa u(\tau),\kappa(\theta u)(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\geqslant \kappa^{\varkappa(q-1)} \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau) f\big(\tau,u(\tau),\theta u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &= \kappa^{\varkappa(q-1)} Bu(t) \geqslant \kappa Bu(t), \end{aligned}$$

that is, $B(\kappa u) \geqslant \kappa Bu$ for $\kappa \in (0,1)$, $u \in P$.

*Step 4*. We prove that operators $Ah, Bh \in P_h$. Let

$$l_1 = \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)} \int_0^1 a(s)\psi(s,h_{\max},\sigma h_{\max})\,\mathrm{d}s, \qquad l_2 = \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)} \int_0^1 a(s)\psi(s,0,0)\,\mathrm{d}s,$$

$$l_3 = \int_0^1 \frac{M(s)s^{(\alpha-1)(q-1)}}{\mathrm{e}^{\lambda s(q-1)}} \varphi_q\left(\int_0^1 N(\tau) f(\tau,h_{\max},\sigma h_{\max})\,\mathrm{d}\tau\right)\mathrm{d}s,$$

$$l_4 = \int_0^1 \frac{m(s)s^{(\alpha-1)(q-1)}}{\mathrm{e}^{\lambda s(q-1)}} \varphi_q\left(\int_0^1 n(\tau) f(\tau,0,0)\,\mathrm{d}\tau\right)\mathrm{d}s,$$

where $h_{\max} = \max\{h(t),\ t \in [0,1]\}$.

From (H0), (H1), and Lemma 5 we have

$$\begin{aligned} Ah(t) &= \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\big(s,h(s),\theta h(s)\big)\,\mathrm{d}s \\ &\leqslant \mathrm{e}^{-\lambda t}t^{\beta-1} \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)} \int_0^1 a(s)\psi(s,h_{\max},\sigma h_{\max})\,\mathrm{d}s = l_1 h(t), \end{aligned}$$

$$\begin{aligned}Ah(t) &= \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\int_0^1 a(s)\psi\bigl(s,h(s),\theta h(s)\bigr)\,\mathrm{d}s\\ &\geqslant \mathrm{e}^{-\lambda t}t^{\beta-1}\frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}\int_0^1 a(s)\psi(s,0,0)\,\mathrm{d}s = l_2h(t).\end{aligned}$$

By similar discussion, from (H1) and Lemma 5 we get

$$\begin{aligned}Bh(t) &\leqslant \mathrm{e}^{-\lambda t}t^{\beta-1}\int_0^1 M(s)\varphi_q\Biggl(\int_0^1 N(\tau)\mathrm{e}^{-\lambda s}s^{\alpha-1}f\bigl(\tau,h(\tau),\theta h(\tau)\bigr)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s\\ &\leqslant \mathrm{e}^{-\lambda t}t^{\beta-1}\int_0^1 \frac{M(s)s^{(\alpha-1)(q-1)}}{\mathrm{e}^{\lambda s(q-1)}}\varphi_q\Biggl(\int_0^1 N(\tau)f(\tau,h_{\max},\sigma h_{\max})\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s\\ &= l_3h(t),\\ Bh(t) &\geqslant \mathrm{e}^{-\lambda t}t^{\beta-1}\int_0^1 m(s)\varphi_q\Biggl(\int_0^1 n(\tau)\mathrm{e}^{-\lambda s}s^{\alpha-1}f\bigl(\tau,h(\tau),\theta h(\tau)\bigr)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s\\ &\geqslant \mathrm{e}^{-\lambda t}t^{\beta-1}\int_0^1 \frac{m(s)s^{(\alpha-1)(q-1)}}{\mathrm{e}^{\lambda s(q-1)}}\varphi_q\Biggl(\int_0^1 n(\tau)f(\tau,0,0)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s\\ &= l_4h(t).\end{aligned}$$

Since $f(t,0,0)\not\equiv 0$, it is easy to see that $l_1\geqslant l_2>0$ and $l_3\geqslant l_4>0$. Then $l_2h\leqslant Ah\leqslant l_1h$ and $l_4h\leqslant Bh\leqslant l_3h$. That is, $Ah, Bh\in P_h$.

*Step 5*. We prove that condition (ii) of Lemma 6 is also satisfied. For $u\in P$, by (H1) and (H3), we have

$$\begin{aligned}Bu(t) &\leqslant \int_0^1 M(s)\mathrm{e}^{-\lambda t}t^{\beta-1}\varphi_q\Biggl(\int_0^1 N(\tau)\mathrm{e}^{-\lambda s}s^{\alpha-1}f\bigl(\tau,u(\tau),\theta u(\tau)\bigr)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s\\ &\leqslant \frac{\mathrm{e}^{\lambda}\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\biggl(\frac{\alpha^2\mathrm{e}^{\lambda}}{(\alpha-\varrho)\Gamma(\alpha+1)}\biggr)^{q-1}\\ &\quad\times\int_0^1 \frac{s^{(\alpha-1)(q-1)}}{\mathrm{e}^{\lambda s(q-1)}}\varphi_q\Biggl(\int_0^1 f\bigl(\tau,u(\tau),\theta u(\tau)\bigr)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s\\ &\leqslant \frac{\mathrm{e}^{\lambda}}{\Gamma(\beta)}\biggl(\frac{\alpha^2\mathrm{e}^{\lambda}}{(\alpha-\varrho)\Gamma(\alpha+1)}\biggr)^{q-1}\delta_0^{q-1}\mathrm{e}^{-\lambda t}t^{\beta-1}\\ &= \frac{\mathrm{e}^{\lambda}}{\Gamma(\beta)}\biggl(\frac{\alpha^2\mathrm{e}^{\lambda}}{(\alpha-\varrho)\Gamma(\alpha+1)}\biggr)^{q-1}\delta_0^{q-2}\mathrm{e}^{-\lambda t}t^{\beta-1}\int_0^1 a(s)\psi(s,0,0)\,\mathrm{d}s\end{aligned}$$

$$\leqslant \frac{\mathrm{e}^{\lambda}}{\Gamma(\beta-\gamma)}\left(\frac{\alpha^2\mathrm{e}^{\lambda}}{(\alpha-\varrho)\Gamma(\alpha+1)}\right)^{q-1}$$
$$\times\,\delta_0^{q-2}\frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\int_0^1 a(s)\psi\bigl(s,u(s),\theta u(s)\bigr)\,\mathrm{d}s$$
$$=\frac{\mathrm{e}^{\lambda}}{\Gamma(\beta-\gamma)}\left(\frac{\alpha^2\mathrm{e}^{\lambda}}{(\alpha-\varrho)\Gamma(\alpha+1)}\right)^{q-1}\delta_0^{q-2}Au(t).$$

Let $\delta_0^* = [\mathrm{e}^{\lambda}/\Gamma(\beta-\gamma)]^{-1}(\alpha^2\mathrm{e}^{\lambda}/((\alpha-\varrho)\Gamma(\alpha+1)))^{1-q}\delta_0^{2-q}$. We can get $Au(t)\geqslant \delta_0^*Bu(t)$, $t\in[0,1]$. Thus, $Au\geqslant\delta_0^*Bu$ for all $u\in P$.

By the above discussion and Lemma 6, we can obtain that operator equation $Au+Bu=u$ has a unique solution $u^*$ in $P_h$. For any initial value $u_0\in P_h$, the sequence $u_n=Au_{n-1}+Bu_{n-1}$, $n=1,2,\dots$, satisfies $u_n\to u^*$ as $n\to\infty$. That is,

$$u_{n+1}(t)=\int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau)f\bigl(\tau,u_n(\tau),\theta u_n(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$+\frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\int_0^1 a(s)\psi\bigl(s,u_n(s),\theta u_n(s)\bigr)\,\mathrm{d}s,\quad n=0,1,2,\dots.$$

For any initial value $u_0\in P_h$, we have $u_n\to u^*$ as $n\to\infty$. □

**Corollary 1.** *Assume that* (H0) *holds,* $f$ *satisfies* (H1)*, and there exist constants* $\varkappa\in(0,1/(q-1))$ *and* $\delta\in(0,1)$ *such that* $f(t,\kappa u,\kappa v)\geqslant\kappa^{\varkappa\delta}f(t,u,v)$ *for* $\kappa\in(0,1)$*,* $t\in[0,1]$*,* $u,v\in\mathbb{R}^+$*. Then the problem*

$$\begin{aligned}
&-D_{0+}^{\alpha,\lambda}\bigl(\varphi_p\bigl(-D_{0+}^{\beta,\lambda}u(t)\bigr)\bigr)=f\bigl(t,u(t),\theta u(t)\bigr),\quad 0<t<1,\\
&u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\qquad D_{0+}^{\beta,\lambda}u(0)=0,\\
&D_{0+}^{\gamma,\lambda}u(1)=0,\\
&\varphi_p\bigl(-D_{0+}^{\beta,\lambda}u(1)\bigr)=\varrho\int_0^1\mathrm{e}^{-\lambda(1-s)}\bigl(\varphi_p\bigl(-D_{0+}^{\beta,\lambda}u(s)\bigr)\bigr)\,\mathrm{d}s
\end{aligned}\tag{14}$$

*has a unique positive solution* $u^*\in P_h$*, where* $h(t)=\mathrm{e}^{-\lambda t}t^{\beta-1}$*,* $t\in[0,1]$*. For any initial value* $u_0\in P_h$*, define the iterative sequence*

$$u_{n+1}(t)=\int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau)f\bigl(\tau,u_n(\tau),\theta u_n(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s,\quad n=0,1,2,\dots.$$

*The iterative sequence* $u_n(t)$ *converges uniformly to the unique positive solution* $u^*(t)$ *of problem* (14) *on* $[0,1]$ *as* $n\to\infty$*.*

*Proof.* From Remark 1 and Theorem 1, the conclusion holds. □

**Theorem 2.** *Assume that* (H0) *holds, and let* $f$ *satisfy* (H1) *and*

- (H4) $\psi : [0,1) \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and decreasing with respect to the second and third arguments,* $\psi(t, h_{\max}, \sigma h_{\max}) \not\equiv 0$, $t \in [0,1]$;
- (H5) *For* $\kappa \in (0,1)$, *there exist* $\bar{\varkappa} \in (0, 1/(q-1))$ *and* $\phi_i(\kappa) \in (\kappa, 1)$ $(i = 1,2)$ *such that* $f(t, \kappa u, \kappa v) \geqslant \phi_1^{\bar{\varkappa}}(\kappa) f(t,u,v)$, $\psi(t, \kappa u, \kappa v) \leqslant \psi(t,u,v)/\phi_2(\kappa)$ *for* $t \in [0,1]$, $u, v \in \mathbb{R}^+$.

*Then problem* (2) *has a unique positive solution* $u^* \in P_h$ *with* $h(t) = \mathrm{e}^{-\lambda t} t^{\beta-1}$ *for* $t \in [0,1]$. *For any initial value* $u_0, v_0 \in P_h$, *define iterative sequences*

$$u_{n+1}(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau) f\bigl(\tau, u_n(\tau), \theta u_n(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s + \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t} t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\bigl(s, v_n(s), \theta v_n(s)\bigr)\,\mathrm{d}s,$$

$$v_{n+1}(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau) f\bigl(\tau, v_n(\tau), \theta v_n(\tau)\,\mathrm{d}\tau\bigr)\right)\mathrm{d}s + \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t} t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\bigl(s, u_n(s), \theta u_n(s)\bigr)\,\mathrm{d}s,$$

$n = 0,1,2,\dots$. *Then we have* $u_n(t) \to u^*(t)$, $v_n(t) \to u^*(t)$ *as* $n \to \infty$.

*Proof.* Similarly to the proof of Theorem 1, we still consider two operators $\mathcal{A} : P \to E$ and $\mathcal{B} : P \to E$ by

$$\mathcal{A}u(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau) f\bigl(\tau, u(\tau), \theta u(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s,$$

$$\mathcal{B}u(t) = \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t} t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\bigl(s, u(s), \theta u(s)\bigr)\,\mathrm{d}s.$$

From (H0), (H1), (H4), and Lemma 5 we have that $\mathcal{A} : P \to P$ is increasing and $\mathcal{B} : P \to P$ is decreasing. Moreover, from (H5), for $\kappa \in (0,1)$, we have

$$\mathcal{A}(\kappa u)(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau) f\bigl(\tau, \kappa u(\tau), \kappa(\theta u)(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$\geqslant \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau)\phi_1^{\bar{\varkappa}}(\kappa) f\bigl(\tau, u(\tau), \theta u(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s$$

$$= \phi_1^{\bar{\varkappa}(q-1)}(\kappa) \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau) f(\tau, u(\tau), \theta u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$\geqslant \phi_1(\kappa)\mathcal{A}u(t)$$

and

$$\mathcal{B}(\kappa u)(t) = \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\bigl(s, \kappa u(s), \kappa(\theta u)(s)\bigr)\,\mathrm{d}s$$
$$\leqslant \frac{1}{\phi_2(\kappa)} \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\bigl(s, u(s), \theta u(s)\bigr)\,\mathrm{d}s$$
$$= \frac{1}{\phi_2(\kappa)}\mathcal{B}u(t),$$

that is, $\mathcal{A}$ and $\mathcal{B}$ satisfy the inequalities of Lemma 7(i). Next, we prove that $\mathcal{A}h+\mathcal{B}h \in P_h$. Let

$$l_5 = \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)} \int_0^1 a(s)\psi(s,0,0)\,\mathrm{d}s, \qquad l_6 = \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)} \int_0^1 a(s)\psi(s, h_{\max}, \sigma h_{\max})\,\mathrm{d}s.$$

From Lemma 5 we have

$$\mathcal{A}h(t) \leqslant \mathrm{e}^{-\lambda t}t^{\beta-1} \int_0^1 M(s)\varphi_q\left(\int_0^1 N(\tau)\mathrm{e}^{-\lambda s}s^{\alpha-1} f\bigl(\tau, h(\tau), \theta h(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$\leqslant \mathrm{e}^{-\lambda t}t^{\beta-1} \int_0^1 \frac{M(s)s^{(\alpha-1)(q-1)}}{\mathrm{e}^{\lambda s(q-1)}}\varphi_q\left(\int_0^1 N(\tau) f(\tau, h_{\max}, \sigma h_{\max})\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$= l_3 h(t),$$
$$\mathcal{A}h(t) \geqslant \mathrm{e}^{-\lambda t}t^{\beta-1} \int_0^1 m(s)\varphi_q\left(\int_0^1 n(\tau)\mathrm{e}^{-\lambda s}s^{\alpha-1} f\bigl(\tau, h(\tau), \theta h(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$\geqslant \mathrm{e}^{-\lambda t}t^{\beta-1} \int_0^1 \frac{m(s)s^{(\alpha-1)(q-1)}}{\mathrm{e}^{\lambda s(q-1)}}\varphi_q\left(\int_0^1 n(\tau) f(\tau,0,0)\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$= l_4 h(t).$$

Also,

$$\mathcal{B}h(t) = \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)} \int_0^1 a(s)\psi\bigl(s, h(s), \theta h(s)\bigr)\,\mathrm{d}s$$
$$\leqslant \mathrm{e}^{-\lambda t}t^{\beta-1} \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)} \int_0^1 a(s)\psi(s,0,0)\,\mathrm{d}s = l_5 h(t),$$

$$\begin{aligned}\mathcal{B}h(t) &= \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\int_0^1 a(s)\psi\big(s,h(s),\theta h(s)\big)\,\mathrm{d}s\\ &\geqslant \mathrm{e}^{-\lambda t}t^{\beta-1}\frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}\int_0^1 a(s)\psi\big(s,h_{\max},\sigma h_{\max}\big)\,\mathrm{d}s = l_6 h(t).\end{aligned}$$

By (H4), we have $l_5 \geqslant l_6 > 0$, then $\mathcal{A}h(t)+\mathcal{B}h(t) \leqslant (l_3+l_5)h(t)$ and $\mathcal{A}h(t)+\mathcal{B}h(t) \geqslant (l_4+l_6)h(t)$. Thus, $\mathcal{A}h+\mathcal{B}h \in P_h$.

Consequently, based on Lemma 7, the operator equation $\mathcal{A}u+\mathcal{B}u = u$ has a unique solution $u^*$ in $P_h$. For given initial values $u_0, v_0 \in P_h$, define the sequences

$$u_n = \mathcal{A}u_{n-1}+\mathcal{B}v_{n-1}, \quad v_n = \mathcal{A}v_{n-1}+\mathcal{B}u_{n-1}, \quad n = 1,2,\dots .$$

We have $u_n \to u^*$, $v_n \to u^*$ as $n \to \infty$. That is, $u^*$ is the unique solution of problem (2) in $P_h$, where $h(t) = \mathrm{e}^{-\lambda t}t^{\beta-1}$, $t \in [0,1]$. For given initial values $u_0, v_0 \in P_h$, define the following sequences:

$$\begin{aligned}u_{n+1}(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau)f\big(\tau,u_n(\tau),\theta u_n(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\\ &\quad + \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\int_0^1 a(s)\psi\big(s,v_n(s),\theta v_n(s)\big)\,\mathrm{d}s,\\ v_{n+1}(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau)f\big(\tau,v_n(\tau),\theta v_n(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\\ &\quad + \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\int_0^1 a(s)\psi\big(s,u_n(s),\theta u_n(s)\big)\,\mathrm{d}s,\end{aligned}$$

$n = 0,1,2,\dots$, and we have $u_n(t) \to u^*(t)$, $v_n(t) \to u^*(t)$ as $n \to \infty$. □

**Corollary 2.** *Assume that* (H0) *holds, $f$ satisfies* (H1) *and* (H5). *Then problem* (14) *has a unique positive solution $u^* \in P_h$, with $h(t) = \mathrm{e}^{-\lambda t}t^{\beta-1}$ for $t \in [0,1]$. For any initial value $u_0 \in P_h$, define the iterative sequence*

$$u_{n+1}(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau)f\big(\tau,u_n(\tau),\theta u_n(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s, \quad n = 0,1,2,\dots .$$

*Then the sequence $u_n(t)$ converges uniformly to the unique positive solution $u^*(t)$ of problem* (14) *on $[0,1]$ as $n \to \infty$, where $G(t,s)$ and $K(s,\tau)$ are given as in* (4) *and* (9), *respectively.*

*Proof.* From Remark 1 and Theorem 2 the conclusion holds. □

Before proving the Theorem 3, we set

$$\varpi_1 = \frac{\Gamma(\beta-\gamma)a^*}{\Gamma(\beta)}, \qquad \varpi_2 = \frac{\mathrm{e}^{\lambda}}{\Gamma(\beta)}\left(\frac{\alpha^2\mathrm{e}^{\lambda}}{(\alpha-\varrho)\Gamma(\alpha+1)}\right)^{q-1}.$$

**Theorem 3.** *Assume that* (H0) *and the following conditions hold true*:

(H6) $f \in C((0,1)\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$, *$f$ may be singular at $t=0$ or $t=1$, and* $\psi \in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$.

(H7) *There exists $L_1>0$ such that for any $t\in[0,1]$, $u,v,u',v'\in\mathbb{R}^+$,*

$$\left|\psi(t,u,v)-\psi(t,u',v')\right| \leqslant L_1\left(|u-u'|+|v-v'|\right).$$

(H8) *For each $k>0$, there exists $\vartheta_k\in L^1(0,1)$, $\vartheta_k(t)\geqslant 0$, such that for all $t\in[0,1]$, $|u|,|v|\leqslant(1+\sigma)k$,*

$$\left|f(t,u,v)\right| \leqslant \vartheta_k(t).$$

*Then problem* (2) *has at least one positive solution whenever*

$$\frac{\Gamma(\beta-\gamma)a^*L_1(1+\sigma)}{\Gamma(\beta)} < 1.$$

*Proof.* Let

$$\Delta = \max\left\{\widetilde{M}_0,\ L_1(1+\sigma)\right\}, \quad \widetilde{M}_0 = \max_{t\in[0,1]}\left|\psi(t,0,0)\right|.$$

Set $\Omega_R=\{u\in P\colon \|u\|\leqslant R\}$, where $\varpi_1 \vartriangle (R+1)+\varpi_2\varphi_q\left(\int_0^1\vartheta_R(\tau)\,\mathrm{d}\tau\right)\leqslant R$, then $\Omega_R$ is a nonempty bounded closed convex subset of $E$. Define two operators $\mathscr{A}: P\to E$ and $\mathscr{B}: P\to E$ by

$$\mathscr{A}u(t) = \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\int_0^1 a(s)\psi\left(s,u(s),\theta u(s)\right)\mathrm{d}s,$$

$$\mathscr{B}u(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K(s,\tau)f\left(\tau,u(\tau),\theta u(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}s.$$

For any $u\in\Omega_R$, $t\in[0,1]$, by (H7), we have

$$\begin{aligned}
\left|\mathscr{A}u(t)\right| &\leqslant \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\int_0^1 a(s)\left|\psi\left(s,u(s),\theta u(s)\right)-\psi(s,0,0)\right|\mathrm{d}s \\
&\quad + \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\int_0^1 a(s)\left|\psi(s,0,0)\right|\mathrm{d}s \\
&\leqslant \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}\int_0^1 a(s)L_1\left(|u(s)|+|\theta u(s)|\right)\mathrm{d}s + \frac{\Gamma(\beta-\gamma)\widetilde{M}_0}{\Gamma(\beta)}\int_0^1 a(s)\,\mathrm{d}s \\
&\leqslant \frac{\Gamma(\beta-\gamma)a^*}{\Gamma(\beta)}\left[L_1(1+\sigma)\|u\|+\widetilde{M}_0\right].
\end{aligned}$$

Hence, we have

$$\|\mathscr{A}u\| \leqslant \frac{\Gamma(\beta-\gamma)a^*}{\Gamma(\beta)}\big[L_1(1+\sigma)R+\widetilde{M_0}\big]. \tag{15}$$

Similarly, for $v \in \varOmega_R$, we have $\|v\|, \|\theta v\| \leqslant (1+\sigma)R$. By (H8), there exists $\vartheta_R \in L^1(0,1)$ such that

$$\big|f\big(\tau, v(\tau), \theta v(\tau)\big)\big| \leqslant \vartheta_R(\tau), \quad \tau \in [0,1].$$

Then we have

$$\begin{aligned}\big|\mathscr{B}v(t)\big| &\leqslant \int_0^1 \frac{\mathrm{e}^\lambda}{\Gamma(\beta)}\varphi_q\left(\int_0^1 \frac{\alpha^2\mathrm{e}^\lambda}{(\alpha-\varrho)\Gamma(\alpha+1)}\vartheta_R(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\leqslant \frac{\mathrm{e}^\lambda}{\Gamma(\beta)}\left(\frac{\alpha^2\mathrm{e}^\lambda}{(\alpha-\varrho)\Gamma(\alpha+1)}\right)^{q-1}\varphi_q\left(\int_0^1 \vartheta_R(\tau)\,\mathrm{d}\tau\right).\end{aligned}$$

Hence, we have

$$\|\mathscr{B}v\| \leqslant \frac{\mathrm{e}^\lambda}{\Gamma(\beta)}\left(\frac{\alpha^2\mathrm{e}^\lambda}{(\alpha-\varrho)\Gamma(\alpha+1)}\right)^{q-1}\varphi_q\left(\int_0^1 \vartheta_R(\tau)\,\mathrm{d}\tau\right). \tag{16}$$

Thus, by (15) and (16), we can get

$$\begin{aligned}\|\mathscr{A}u+\mathscr{B}v\| &\leqslant \|\mathscr{A}u\|+\|\mathscr{B}v\| \\ &\leqslant \frac{\Gamma(\beta-\gamma)a^*}{\Gamma(\beta)}\big[L_1(1+\sigma)R+\widetilde{M_0}\big] \\ &\quad+\frac{\mathrm{e}^\lambda}{\Gamma(\beta)}\left(\frac{\alpha^2\mathrm{e}^\lambda}{(\alpha-\varrho)\Gamma(\alpha+1)}\right)^{q-1}\varphi_q\left(\int_0^1 \vartheta_R(\tau)\,\mathrm{d}\tau\right) \\ &\leqslant \varpi_1 \vartriangle (R+1)+\varpi_2\varphi_q\left(\int_0^1 \vartheta_R(\tau)\,\mathrm{d}\tau\right) \leqslant R,\end{aligned}$$

that is, $\mathscr{A}u+\mathscr{B}v \in \varOmega_R$.

Next, we prove that $\mathscr{A}$ is a contraction. Let $u, v \in \varOmega_R$, $t \in [0,1]$, by (H7), we have

$$\begin{aligned}&\big|\mathscr{A}u(t)-\mathscr{A}v(t)\big| \\ &\quad\leqslant \frac{\Gamma(\beta-\gamma)\mathrm{e}^{-\lambda t}t^{\beta-1}}{\Gamma(\beta)}\int_0^1 a(s)\big|\psi\big(s,u(s),\theta u(s)\big)-\psi\big(s,v(s),\theta v(s)\big)\big|\,\mathrm{d}s \\ &\quad\leqslant \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}\int_0^1 a(s)L_1\big(\big|u(s)-v(s)\big|+\big|\theta u(s)-\theta v(s)\big|\big)\,\mathrm{d}s \\ &\quad\leqslant \frac{\Gamma(\beta-\gamma)a^*L_1(1+\sigma)}{\Gamma(\beta)}\|u-v\|,\end{aligned}$$

thus,

$$\|\mathscr{A}u - \mathscr{A}v\| \leqslant \frac{\Gamma(\beta-\gamma)a^* L_1(1+\sigma)}{\Gamma(\beta)}\|u-v\|.$$

By $\Gamma(\beta-\gamma)a^* L_1(1+\sigma)/\Gamma(\beta) < 1$, we can obtain that $\mathscr{A}$ is a contraction.

Finally, let us prove that $\mathscr{B}$ is compact and continuous.

Let $u_n, u \in P$, $n = 1, 2, 3, \dots$, and $\|u_n - u\| \to 0$ as $n \to \infty$, that is, for any $t \in [0,1]$, we have

$$u_n(t) \to u(t), \quad n \to \infty. \tag{17}$$

Thus, there exists $r > 0$ such that $\|u\|, \|\theta u\|, \|u_n\|, \|\theta u_n\| \leqslant (1+\sigma)r$. By (H8), we have

$$\left|f\left(\tau, u(\tau), \theta u(\tau)\right)\right| \leqslant \vartheta_r(\tau), \quad \left|f\left(\tau, u_n(\tau), \theta u_n(\tau)\right)\right| \leqslant \vartheta_r(\tau), \quad \tau \in [0,1]. \tag{18}$$

Thus,

$$\left|u_n(s) - u(s)\right| \leqslant 2(1+\sigma)r, \tag{19}$$

$$\left|f\left(\tau, u_n(\tau), \theta u_n(\tau)\right) - f\left(\tau, u(\tau), \theta u(\tau)\right)\right| \leqslant 2\vartheta_r(\tau). \tag{20}$$

By (17)–(20) and the Lebesgue dominated convergence theorem, we have

$$\left|\theta u_n(\tau) - \theta u(\tau)\right| = \left|\int_0^\tau H(\tau,s)u_n(s)\,\mathrm{d}s - \int_0^\tau H(\tau,s)u(s)\,\mathrm{d}s\right| \to 0, \tag{21}$$

$$\left|\varphi_q\left(\int_0^1 K(s,\tau)f\left(\tau, u_n(\tau), \theta u_n(\tau)\right)\mathrm{d}\tau\right) - \varphi_q\left(\int_0^1 K(s,\tau)f\left(\tau, u(\tau), \theta u(\tau)\right)\mathrm{d}\tau\right)\right| \to 0 \tag{22}$$

as $n \to \infty$, and

$$\left|\varphi_q\left(\int_0^1 K(s,\tau)f\left(\tau, u_n(\tau), \theta u_n(\tau)\right)\mathrm{d}\tau\right) - \varphi_q\left(\int_0^1 K(s,\tau)f\left(\tau, u(\tau), \theta u(\tau)\right)\mathrm{d}\tau\right)\right| \leqslant 2\varphi_q\left(\int_0^1 K(s,\tau)\vartheta_r(\tau)\,\mathrm{d}\tau\right). \tag{23}$$

Then, by (21)–(23) and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\left|\mathscr{B}u_n(t) - \mathscr{B}u(t)\right| \leqslant \frac{\mathrm{e}^\lambda}{\Gamma(\beta)}\int_0^1 \Bigg|\varphi_q\left(\int_0^1 K(s,\tau)f\left(\tau, u_n(\tau), \theta u_n(\tau)\right)\mathrm{d}\tau\right) \\ - \varphi_q\left(\int_0^1 K(s,\tau)f\left(\tau, u(\tau), \theta u(\tau)\right)\mathrm{d}\tau\right)\Bigg|\,\mathrm{d}s \\ \to 0\end{aligned}$$

as $n \to \infty$. That is, $\|\mathscr{B}u_n - \mathscr{B}u\| \to 0$ as $n \to \infty$. In consequence, $\mathscr{B}$ is continuous.

Let $U = \{u \in P\colon \|u\| \leqslant \Theta\}$ for some $\Theta > 0$. From (16) we have

$$\|\mathscr{B}u\| \leqslant \frac{\mathrm{e}^{\lambda}}{\Gamma(\beta)}\left(\frac{\alpha^2\mathrm{e}^{\lambda}}{(\alpha-\varrho)\Gamma(\alpha+1)}\right)^{q-1}\varphi_q\left(\int_0^1 \vartheta_\Theta(\tau)\,\mathrm{d}\tau\right) = \varpi_2\varphi_q\left(\int_0^1 \vartheta_\Theta(\tau)\,\mathrm{d}\tau\right).$$

Thus, $\mathscr{B}U$ is uniformly bounded.

On the other hand, since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it follows that $G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$. For $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|t_1 - t_2| < \delta,\ t_1, t_2 \in [0,1]$, and for $s \in [0,1]$, we have

$$\bigl|G(t_1,s) - G(t_2,s)\bigr| < \frac{\epsilon}{\int_0^1 \varphi_q(\int_0^1 \vartheta_\Theta(\tau)N(\tau)\mathrm{e}^{-\lambda s}s^{\alpha-1}\,\mathrm{d}\tau)\,\mathrm{d}s}. \tag{24}$$

For any $u \in U$, by (24), we can get

$$\begin{aligned}
&\bigl|\mathscr{B}u(t_1) - \mathscr{B}u(t_2)\bigr| \\
&\quad\leqslant \int_0^1 \bigl|G(t_1,s) - G(t_2,s)\bigr|\varphi_q\left(\int_0^1 K(s,\tau)f\bigl(\tau, u(\tau), \theta u(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad\leqslant \int_0^1 \bigl|G(t_1,s) - G(t_2,s)\bigr|\varphi_q\left(\int_0^1 \vartheta_\Theta(\tau)N(\tau)\mathrm{e}^{-\lambda s}s^{\alpha-1}\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad< \frac{\epsilon\int_0^1 \varphi_q(\int_0^1 \vartheta_\Theta(\tau)N(\tau)\mathrm{e}^{-\lambda s}s^{\alpha-1}\,\mathrm{d}\tau)\,\mathrm{d}s}{\int_0^1 \varphi_q(\int_0^1 \vartheta_\Theta(\tau)N(\tau)\mathrm{e}^{-\lambda s}s^{\alpha-1}\,\mathrm{d}\tau)\,\mathrm{d}s} = \epsilon.
\end{aligned}$$

Therefore, $\mathscr{B}U$ is equicontinuous. According to the Arzelà–Ascoli theorem, we have that $\mathscr{B}$ is compact operator. So we can easily obtain that operator $\mathscr{B}$ is completely continuous. By Lemma 8, we conclude that problem (2) has at least one positive solution in $\Omega_R$. □

## 4 Applications

*Example 1.* Consider the following fractional differential equation:

$$\begin{aligned}
&-D_{0^+}^{3/2,1}\bigl(\varphi_3\bigl(-D_{0^+}^{5/2,1}u(t)\bigr)\bigr) \\
&\qquad = \cos^2 t + \frac{u^{1/3}(t)}{1+u^{1/3}(t)} + \frac{(\int_0^t tsu(s)\,\mathrm{d}s)^{1/2}}{1+(\int_0^t tsu(s)\,\mathrm{d}s)^{1/2}}, \quad 0<t<1, \\
&u(0) = u'(0) = 0, \qquad D_{0^+}^{5/2,1}u(0) = 0, \\
&D_{0^+}^{1,1}u(1) = \int_0^1 \mathrm{e}^{t-1}\left[t\left(u^{1/4}(t) + \left(\int_0^t tsu(s)\,\mathrm{d}s\right)^{1/2}\right) + 3\right]\mathrm{d}t, \\
&\varphi_3\bigl(-D_{0^+}^{5/2,1}u(1)\bigr) = \int_0^1 \mathrm{e}^{-(1-t)}\bigl(\varphi_3\bigl(-D_{0^+}^{5/2,1}u(t)\bigr)\bigr)\,\mathrm{d}t,
\end{aligned} \tag{25}$$

where $\alpha = 3/2$, $\beta = 5/2$, $\gamma = 1$, $\lambda = 1$, $p = 3$, $q = 3/2$, $\varrho = 1$, $a(t) = \mathrm{e}^t$, and let

$$f(t,u,v) = \cos^2 t + \frac{u^{1/3}}{1+u^{1/3}} + \frac{v^{1/2}}{1+v^{1/2}}, \qquad \psi(t,u,v) = t\left(u^{1/4} + v^{1/2}\right) + 3.$$

Clearly, $f, \psi \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+)$ are increasing with respect to the second and third arguments, $f(t,0,0) \not\equiv 0$, thus (H0), (H1) are satisfied.

In addition, for $\kappa \in (0,1)$, $t \in [0,1]$, $u, v \in \mathbb{R}^+$,

$$\begin{aligned} f(t,\kappa u,\kappa v) &= \cos^2 t + \frac{(\kappa u)^{1/3}}{1+(\kappa u)^{1/3}} + \frac{(\kappa v)^{1/2}}{1+(\kappa v)^{1/2}} \\ &\geqslant \kappa^{3/2}\cos^2 t + \kappa^{3/2}\frac{u^{1/3}}{1+u^{1/3}} + \kappa^{3/2}\frac{v^{1/2}}{1+v^{1/2}} = \kappa^{3/2} f(t,u,v). \end{aligned}$$

On the other hand, take $\delta = 1/2$. For $\kappa \in (0,1)$, $t \in [0,1]$, $u, v \in \mathbb{R}^+$,

$$\begin{aligned} \psi(t,\kappa u,\kappa v) &= t\bigl((\kappa u)^{1/4} + (\kappa v)^{1/2}\bigr) + 3 \geqslant \kappa^{1/2}\left[t\left(u^{1/4} + v^{1/2}\right) + 3\right] \\ &= \kappa^{1/2}\psi(t,\kappa u,\kappa v). \end{aligned}$$

Thus, (H2) is satisfied.

Also, we can get $\delta_0 = \int_0^1 \mathrm{e}^t \psi(t,0,0)\,\mathrm{d}t = 3(\mathrm{e}-1) > 0$ and

$$f(t,u,v) = \cos^2 t + \frac{u^{1/3}}{1+u^{1/3}} + \frac{v^{1/2}}{1+v^{1/2}} \leqslant \delta_0.$$

Thus, (H3) is satisfied. So all the conditions of Theorem 1 are satisfied, then problem (25) has a unique positive solution in $P_h$, where $h(t) = \mathrm{e}^{-t}t^{3/2}$.

*Example 2.* Consider the following fractional differential equation:

$$\begin{aligned} &-D_{0^+}^{3/2,1}\bigl(\varphi_3\bigl(-D_{0^+}^{7/2,1}u(t)\bigr)\bigr) \\ &\quad = t^{1/3}\left(u^{1/8}(t) + \left(\int_0^t tsu(s)\,\mathrm{d}s\right)^{1/6}\right) + 3, \quad 0 < t < 1, \\ &u(0) = u'(0) = u''(0) = 0, \qquad D_{0^+}^{7/2,1}u(0) = 0, \\ &D_{0^+}^{2,1}u(1) = \int_0^1 \mathrm{e}^{t-1}\left[t^{1/3}\left(u^{1/3}(t) + \left(\int_0^t tsu(s)\,\mathrm{d}s\right)^{1/6}\right) + 2\right]^{-1}\mathrm{d}t, \\ &\varphi_3\bigl(-D_{0^+}^{7/2,1}u(1)\bigr) = \int_0^1 \mathrm{e}^{-(1-t)}\bigl(\varphi_3\bigl(-D_{0^+}^{7/2,1}u(t)\bigr)\bigr)\,\mathrm{d}t, \end{aligned} \tag{26}$$

where $\alpha = 3/2$, $\beta = 7/2$, $\gamma = 2$, $\lambda = 1$, $p = 3$, $q = 3/2$, $\varrho = 1$, $a(t) = \mathrm{e}^t$, $h(t) = \mathrm{e}^{-t}t^{5/2}$, and let

$$f(t,u,v) = t^{1/3}\bigl(u^{1/8} + v^{1/6}\bigr) + 3, \qquad \psi(t,u,v) = \bigl[t^{1/3}\bigl(u^{1/3} + v^{1/6}\bigr) + 2\bigr]^{-1}.$$

Clearly, $f, \psi \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+)$, $f$ is increasing with respect to the second and third arguments, $\psi$ is decreasing with respect to the second and third arguments, and $f(t,0,0) \not\equiv 0$, $\psi(t,h_{\max},\sigma h_{\max}) \not\equiv 0$, thus, $f$ satisfies (H1), (H0) and (H4) are satisfied.

In addition, take $\phi_1(\kappa) = \kappa^{1/9}$, $\phi_2(\kappa) = \kappa^{1/3}$. Then for $\kappa \in (0,1)$, $\phi_i(\kappa) \in (\kappa,1)$ $(i = 1,2)$,

$$\begin{aligned} f(t,\kappa u,\kappa v) &= t^{1/3}\bigl((\kappa u)^{1/8} + (\kappa v)^{1/6}\bigr) + 3 = t^{1/3}\bigl(\kappa^{1/8}u^{1/8} + \kappa^{1/6}v^{1/6}\bigr) + 3 \\ &\geqslant \kappa^{1/6}\bigl[t^{1/3}\bigl(u^{1/8} + v^{1/6}\bigr) + 3\bigr] = \bigl(\phi_1(\kappa)\bigr)^{3/2} f(t,u,v), \\ \psi(t,\kappa u,\kappa v) &= \bigl[t^{1/3}\bigl((\kappa u)^{1/3} + (\kappa v)^{1/6}\bigr) + 2\bigr]^{-1} = \bigl[t^{1/3}\bigl(\kappa^{1/3}u^{1/3} + \kappa^{1/6}v^{1/6}\bigr) + 2\bigr]^{-1} \\ &\leqslant \kappa^{-1/3}\bigl[t^{1/3}\bigl(u^{1/3} + v^{1/6}\bigr) + 2\bigr]^{-1} = \frac{1}{\phi_2(\kappa)}\psi(t,u,v). \end{aligned}$$

Thus, (H5) is satisfied. So all the conditions of Theorem 2 are satisfied, then problem (26) has a unique positive solution in $P_h$ with $h(t) = \mathrm{e}^{-t}t^{5/2}$ for $t \in [0,1]$.

*Example 3.* Consider the following fractional differential equation:

$$\begin{aligned} &-D_{0^+}^{3/2,1}\bigl(\varphi_3\bigl(-D_{0^+}^{5/2,1}u(t)\bigr)\bigr) \\ &\quad = \frac{1}{t\sqrt{(1-t)}}\left[u^2(t) + \left(\int_0^t 2tsu(s)\,\mathrm{d}s\right)^2\right], \quad 0 < t < 1, \\ &u(0) = u'(0) = 0, \qquad D_{0^+}^{5/2,1}u(0) = 0, \\ &D_{0^+}^{1,1}u(1) = \int_0^1 \mathrm{e}^{-1}2t\left[\frac{1}{\sqrt{4+t^2}} + \frac{1}{3(1+t)^2}\sin\bigl(u(t)\bigr) + \frac{1}{3+t}\cos\bigl(v(t)\bigr)\right]\mathrm{d}t, \\ &\varphi_3\bigl(-D_{0^+}^{5/2,1}u(1)\bigr) = \int_0^1 \mathrm{e}^{-(1-t)}\bigl(\varphi_3\bigl(-D_{0^+}^{5/2,1}u(t)\bigr)\bigr)\,\mathrm{d}t, \end{aligned} \tag{27}$$

where $\alpha = 3/2$, $\beta = 5/2$, $\gamma = 1$, $\lambda = 1$, $p = 3$, $q = 3/2$, $\varrho = 1$, $a(t) = 2t$, $H(t,s) = 2ts$, and let

$$f(t,u,v) = \frac{1}{\sqrt{t(1-t)}}\bigl(u^2 + v^2\bigr),$$

$$\psi(t,u,v) = \frac{1}{\sqrt{4+t^2}} + \frac{1}{3(1+t)^2}\sin u + \frac{1}{3+t}\cos v.$$

Clearly, $f \in C((0,1) \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $\psi \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, thus, (H0) and (H6) are satisfied.

Let $L_1 = 1/3$, we have

$$\begin{aligned} \bigl|\psi(t,u,v) - \psi(t,u',v')\bigr| &\leqslant \frac{1}{3(1+t)^2}|\sin u - \sin u'| + \frac{1}{3+t}|\cos v - \cos v'| \\ &\leqslant \frac{1}{3}\bigl(|u-u'| + |v-v'|\bigr), \end{aligned}$$

thus, (H7) is satisfied.

Choose $\vartheta_k(t) = (3/\sqrt{t(1-t)})(1+\sigma)^2k^2$. We can obtain $|f(t,u,v)| \leqslant \vartheta_k(t)$ for all $t \in [0,1]$, $|u|, |v| \leqslant (1+\sigma)k$, thus, (H8) is satisfied. We can also obtain that $\sigma = 1$, $a^* = 1$, $(2/3)(\Gamma(1.5)/\Gamma(2.5)) = 4/9 < 1$. Hence, all the conditions of Theorem 3 are satisfied. By Theorem 3, problem (27) has at least one positive solution.

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