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Global asymptotic synchronization for delayed inertial BAM neural networks without controllers via the fundamental solution matrix method

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Abstract. This paper considers the global asymptotic synchronization (GAS) for master-slave delayed inertial bidirectional associative memory (BAM) neural networks (NNs). Most existing studies on GAS rely on the LMI approach, integral inequality, matrix measure, and maximum-valued methods, usually with additional controllers. In contrast, this work achieves GAS without using the above methods or designing any controllers. Instead, two new sufficient conditions are derived via the fundamental solution matrix method for first-order differential systems. For the first time, the fundamental solution matrix method is introduced to deal with the GAS for the NNs.

Keywords: master-slave inertial BAM neural networks, fundamental solution matrix method, global asymptotic synchronization, controller-free.

1 Introduction

In 1986, Babcock and Westervelt [2] first introduced an inertial term into NNs. Inertial terms can induce rich bifurcation and chaotic dynamics, making inertial neural networks (INNs) more complex and widely studied. Until now, extensive results on equilibrium and periodic solutions have been reported for various INNs [7, 16, 28, 32].

Bidirectional associative memory neural networks (BAMNNs) have attracted considerable attention due to their applications in signal processing, automatic control, image processing, optimization, and pattern recognition. Excellent dynamic results have been achieved for BAMNNs [18, 34, 35].

Recently, synchronization of master-response BAMNNs has been carefully discussed. In [14], the global exponential synchronization (GES) was studied on different time

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domains via time-scale theory and matrix-measure methods with effective feedback controllers. In [2], global robust exponential synchronization was analyzed for interval BAMNNs using direct methods based on system solutions. In [17], global asymptotic synchronization (GAS) was explored via integral inequality techniques. In [10], general decay synchronization (GDS) was investigated using Lyapunov theory and inequality skills. In [31], delayed complex-valued BAMNNs were studied, and a new global asymptotic periodic synchronization condition was derived via the LMI approach.

Until now, the synchronization of non-BAM neural networks has been widely studied. In [21], global exponential synchronization (GES) of two chaotic neural networks was investigated via output or state coupling. Using a Lyapunov function and Young's inequality, criteria were derived to design the coupling matrix. In [20], synchronization of chaotic neural networks with actuator saturation was addressed via a sampled-data controller. A new Lyapunov functional with looped and double-integral terms was constructed, and synchronization criteria were obtained via LMIs. In [9], GES of chaotic neural networks with time-varying delay was studied via impulsive control. Relaxed synchronization conditions were established using the average impulsive interval and delay method. In [37], an aperiodic adaptive event-triggered mechanism (AAETM) was applied to synchronization of neural networks with actuator saturation. Improved criteria were derived using two-side looped functionals and matrix integral inequalities. In [1], projective synchronization of inertial quaternion-valued neural networks with proportional delays was investigated. New criteria were obtained by designing a parameter-tunable Lyapunov functional. In [25], quasi-synchronization of impulsively controlled heterogeneous dynamic neural networks was studied. Sufficient conditions were derived via a comparison system, impulsive delay inequalities, and Lyapunov theory. In [13], synchronization of fuzzy reaction-diffusion neural networks (FRDNNs) was achieved via aperiodic semiintermittent hybrid control, with applications to image encryption. New results were obtained using the LMI approach and appropriate Lyapunov functions. In [29], exponential synchronization of stochastic memristor-based neural networks was investigated via pinning impulsive control. Sufficient conditions were established by designing a hybrid control scheme. In [30], polynomial synchronization (PS) of quaternion-valued fuzzy cellular neural networks was studied via a non-decomposition method and a nonlinear controller. In [8], observer-based adaptive event-triggered quasi-synchronization of Markov jump neural networks was investigated via Lyapunov techniques. In [22], dissipative synchronization of inertial memristor competitive neural networks (IMCNNs) was studied via adaptive sliding mode control. An LMI-based criterion was obtained using delay-dependent reciprocal convex inequalities. In [19], quasi-projective synchronization (QPS) of stochastic quaternion fuzzy cellular neural networks was investigated. Sufficient conditions were derived via a suitable controller, Lyapunov functionals, and stochastic analysis. In [12], general decay synchronization (GDS) of delayed reaction-diffusion neural networks was studied. Robust GDS criteria were obtained using inequality techniques. In [23], synchronization of drive-response stochastic neural networks (SNNs) was investigated via adaptive feedback control (AFC). Criteria were derived based on Lyapunov stability theory and Itô stochastic calculus. In [26], cluster synchronization of stochastic neural networks was achieved via pinning impulsive control. Results were established using stochastic impulsive analysis and Lyapunov stability theory. In [36], GES of Clifford-valued memristive fuzzy neural networks (CLVMFNNs) with delayed impulses was studied. Sufficient conditions were derived via Halanay differential inequality and a linear feedback controller. In [11], PS of delayed inertial quaternion-valued neural networks was investigated. New criteria were presented in component form by constructing a novel Lyapunov functional. In [33], GES of uncertain complex-valued inertial neural networks was studied. A sampling-period-dependent Lyapunov functional and improved complex-valued inequalities were used to derive sufficient conditions. In [24], sampled-data synchronization of inertial neural networks was investigated. Enhanced criteria were obtained via an extended two-sided looped-functional method. In [27], global asymptotic synchronization (GAS) of fuzzy master-slave inertial neural networks was considered. Three criteria were obtained via the maximum-value method and three classes of feedback controllers.

In conclusion, existing synchronization results for BAMNNs and non-BAMNNs are mainly derived via matrix measure method [14], integral inequality method [17], Lyapunov function method [1,8,10,11,15,19,21,25,30], LMI approach [13,20,22], AAETM [9], AID [37], inequality skills [12], Lyapunov stability theory [11, 23, 24, 26, 30, 33], Halanay differential inequality [36], pinning impulsive control [29], and the maximum-valued method [27]. However, very few studies have analyzed the GAS of neural networks using the fundamental solution matrix method for first-order differential systems.

Moreover, almost all related works [1, 3–6, 8–15, 17, 19–27, 29–31, 33, 36, 37] rely on designed controllers to achieve synchronization. Namely, without the controllers designed, the synchronization between master-slave NNs cannot be realized.

Inspired by above discussions, this paper focuses on establishing GAS criteria for a class of master-slave inertial BAMNNs without any controllers or traditional analysis methods. Instead, the fundamental solution matrix method is employed.

Thus, the contributions of this study are as follows: (i) For the first time, the fundamental solution matrix method of first-order differential equation group is introduced to study the GAS for master-slave inertial BAMNNs. So, the study method on the GAS for the BAMNNs is novel. (ii) Without designing controllers, a novel method for the GAS of BAMNNs is proposed. (iii) The techniques for computing the limit are novel. (iv) Two novel criteria for the GAS of the considered master–slave inertial BAMNNs are obtained using the fundamental solution matrix method without designing controllers.

2 Preliminaries

The following delayed inertial BAMNN is considered:

$$\begin{aligned}\frac{\mathrm{d}^2[\sigma_u(t)]}{\mathrm{d}t^2} &= -a_u\frac{\mathrm{d}[\sigma_u(t)]}{\mathrm{d}t} - b_u\sigma_u(t) + \sum_{v=1}^{q} c_{uv} f_v\bigl(\gamma_v(t)\bigr)\\ &\quad + \sum_{v=1}^{q} d_{uv} f_v\bigl(\gamma_v(t-\zeta)\bigr) + I_u, \quad u = 1,2,\dots,p, \qquad (1_1)\end{aligned}$$

$$\frac{\mathrm{d}^2[\gamma_v(t)]}{\mathrm{d}t^2} = -l_v\frac{\mathrm{d}[\gamma_v(t)]}{\mathrm{d}t} - k_v\gamma_v(t) + \sum_{u=1}^{p} e_{vu}g_u\bigl(\sigma_u(t)\bigr) + \sum_{u=1}^{p} r_{vu}g_u\bigl(\sigma_u(t-\eta)\bigr) + J_v, \quad v = 1, 2, \dots, q, \tag{1_2}$$

where $\sigma(t) = (\sigma_1(t), \sigma_2(t), \dots, \sigma_p(t))^{\mathrm{T}} \in \mathbb{R}^p$, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_q)^{\mathrm{T}} \in \mathbb{R}^q$, $\sigma_u(t), \gamma_v(t)$ represent the states of the pth and qth neuron at time t, respectively; $f_v(\cdot)$, $g_u(\cdot) : \mathbb{R} \to \mathbb{R}$ are continuous activation functions; the second derivatives are called inertial terms of system (1); $a_u, b_u, l_v, k_v > 0$, and they denote the rates with which the pth neuron and qth neuron will reset its potential to the resting state in isolation when disconnected from the network and external inputs; c_{uv}, d_{uv}, e_{vu}, r_{vu} are the connection weights. The initial values of system (1) are designed as

$$\begin{aligned}
\sigma_u(\theta) &= \mu_u(\theta), \quad \frac{\mathrm{d}[\sigma_u(\theta)]}{\mathrm{d}t} = \tau_u(\theta), \quad \theta \in [-\eta, 0], \\
\gamma_v(s) &= \tau_v(s), \quad \frac{\mathrm{d}[\gamma_v(s)]}{\mathrm{d}t} = \phi_v(s), \quad s \in [-\zeta, 0],
\end{aligned}$$

where all the above functions are continuous.

Setting $\mathrm{d}[\sigma_u(t)]/\mathrm{d}t + \eta_1\sigma_u(t) = \varTheta_u(t)$, $\mathrm{d}[\gamma_v(t)]/\mathrm{d}t + \eta_2\gamma_v(t) = \varGamma_v(t)$, system (1) becomes the following differential equation group:

$$\begin{aligned}
\sigma_u'(t) &= -\eta_1\sigma_u(t) + \varTheta_u(t), \\
\varTheta_u'(t) &= (\eta_1 - a_u)\varTheta_u(t) - \bigl(\eta_1^2 + b_u\bigr)\sigma_u(t) \\
&\quad + \sum_{v=1}^{q} c_{uv}f_v\bigl(\gamma_v(t)\bigr) + \sum_{v=1}^{q} d_{uv}f_v\bigl(\gamma_v(t-\zeta)\bigr) + I_u, \\
\gamma_v'(t) &= -\eta_2\gamma_v(t) + \varGamma_v(t), \\
\varGamma_v'(t) &= (\eta_2 - l_v)\varGamma_v(t) - \bigl(\eta_2^2 + k_v\bigr)\gamma_v(t) \\
&\quad + \sum_{u=1}^{p} e_{vu}g_u\bigl(\sigma_u(t)\bigr) + \sum_{u=1}^{p} r_{vu}g_u\bigl(\sigma_u(t-\eta)\bigr) + J_v.
\end{aligned} \tag{2}$$

The initial conditions of system (2) are

$$\begin{aligned}
\sigma_u(\theta) &= \mu_u(\theta), \quad \varTheta_u(\theta) = \tau_u(\theta), \quad \theta \in [-\eta, 0], \\
\gamma_v(s) &= \epsilon_v(s), \quad \varGamma_v(s) = \rho_v(s), \quad s \in [-\zeta, 0].
\end{aligned}$$

If system (2) is regarded as the master system, then the slave system is described as

$$\begin{aligned}
\alpha_u'(t) &= -\eta_1\alpha_u(t) + \beta_u(t), \\
\beta_u'(t) &= (\eta_1 - a_u)\beta_u(t) - \bigl(\eta_1^2 + b_u\bigr)\alpha_u(t) \\
&\quad + \sum_{v=1}^{q} c_{uv}f_v\bigl(\omega_v(t)\bigr) + \sum_{v=1}^{q} d_{uv}f_v\bigl(\omega_v(t-\zeta)\bigr) + I_u,
\end{aligned} \tag{3_1}$$

$$\begin{aligned}
\omega_v'(t) &= -\eta_2\omega_v(t) + \rho_v(t),\\
\rho_v'(t) &= (\eta_2 - l_v)\rho_v(t) - \bigl(\eta_2^2 + k_v\bigr)\omega_v(t)\\
&\quad + \sum_{u=1}^{p} e_{vu} g_u\bigl(\alpha_u(t)\bigr) + \sum_{u=1}^{p} r_{vu} g_u\bigl(\alpha_u(t-\eta)\bigr) + J_v,
\end{aligned} \tag{3$_2$}$$

where the parameters are the same as those of system (2); $\alpha_u(t)$, $\beta_u(t)$, $\omega_v(t)$, $\rho_v(t)$ are the status variables. The initial conditions of system (3) are

$$\begin{aligned}
&\alpha_u(\theta) = \mu_{1u}(\theta), \quad \beta_u(\theta) = \tau_{2u}(\theta), \quad \theta \in [-\eta, 0],\\
&\omega_v(s) = \epsilon_{1v}(s), \quad \rho_v(s) = \rho_{2v}(s), \quad \theta \in [-\zeta, 0].
\end{aligned}$$

Let $M_u(t) = \alpha_u(t) - \sigma_u(t)$, $N_u(t) = \beta_u(t) - \Theta_u(t)$, $K_v(t) = \omega_v(t) - \gamma_v(t)$, and $H_v(t) = \rho_v(t) - \Gamma_v(t)$. Then from systems (2) and (3) the corresponding error system is derived:

$$\begin{aligned}
M_u'(t) &= -\eta_1 M_u(t) + N_u(t),\\
N_u'(t) &= -a_u N_u(t) - b_u M_u(t) + \sum_{v=1}^{q} c_{uv}\bigl[f_v\bigl(\omega_v(t)\bigr) - f_v\bigl(\gamma_v(t)\bigr)\bigr]\\
&\quad + \sum_{v=1}^{q} d_{uv}\bigl[f_v\bigl(\omega_v(t-\zeta)\bigr) - f_v\bigl(\gamma_v(t-\zeta)\bigr)\bigr],\\
K_v'(t) &= -\eta_2 K_v(t) + H_v(t),\\
H_v'(t) &= -l_v H_v(t) - k_v K_v(t) + \sum_{u=1}^{p} e_{vu}\bigl[g_u\bigl(\alpha_u(t)\bigr) - g_u\bigl(\sigma_u(t)\bigr)\bigr]\\
&\quad + \sum_{u=1}^{p} r_{vu}\bigl[g_u\bigl(\sigma_u(t-\eta)\bigr) - g_u\bigl(\sigma(t-\eta)\bigr)\bigr].
\end{aligned} \tag{4}$$

Assumption 1. Let there exist two positive constants L_u, Q_v such that for $y_1, y_2 \in \mathbb{R}$, $u = 1, 2, \dots, p$, and $v = 1, 2, \dots, q$,

$$\bigl|f_v(y_1) - f_v(y_2)\bigr| \leqslant Q_v|y_1 - y_2|, \qquad \bigl|g_u(y_1) - g_u(y_2)\bigr| \leqslant L_u|y_1 - y_2|.$$

Definition 1. The drive system (2) and the response system (3) are said to achieve GAS if, for any solutions of systems (2) and (3) denoted by $[\sigma_1(t), \sigma_2(t), \dots, \sigma_p(t);\ \Theta_1(t), \Theta_2(t), \dots, \Theta_p(t);\ \gamma_1(t), \gamma_2(t), \dots, \gamma_q(t);\ \Gamma_1(t), \Gamma_2(t), \dots, \Gamma_q(t)]^{\mathrm{T}}$ and $[\alpha_1(t), \alpha_2(t), \dots, \alpha_p(t);\ \beta_1(t), \beta_2(t), \dots, \beta_p(t);\ \omega_1(t), \omega_2(t), \dots, \omega_q(t);\ \rho_1(t), \rho_2(2), \dots, \rho_q(t)]^{\mathrm{T}}$, the following holds for $u = 1, \dots, p$ and $v = 1, \dots, q$:

$$\begin{aligned}
&\lim_{t\to\infty}\bigl|\sigma_u(t) - \alpha_u(t)\bigr| = 0, \qquad \lim_{t\to\infty}\bigl|\Gamma_u(t) - \beta_u(t)\bigr| = 0,\\
&\lim_{t\to\infty}\bigl|\gamma_v(t) - \omega_v(t)\bigr| = 0, \qquad \lim_{t\to\infty}\bigl|\Gamma_v(t) - \rho_v(t)\bigr| = 0.
\end{aligned}$$

Lemma 1. (*See* [17].) *Consider the following systems of differential equations*:

$$y'(t) = By + g(t), \quad t \geqslant 0, \tag{5}$$

and

$$y'(t) = By, \quad t \geqslant 0, \tag{6}$$

where $B_{n\times n}$ *is a constant matrix, and* $g : \mathbb{R}^+ \to \mathbb{R}^n$ *is a continuous vector-valued function. If* $y(0) = x = [x_1, x_2, \dots, x_n] \in \mathbb{R}^n$*, then the solution of* (5) *is*

$$y(t) = \mathrm{e}^{Bt}x + \int_0^t \mathrm{e}^{(t-s)B} g(s)\,\mathrm{d}s,$$

and the solution of (6) *with the initial value* $y(0) = x$ *is*

$$y(t) = \mathrm{e}^{Bt}x.$$

Lemma 2. (*See* [17].) *If the matrix* B *has* n *linearly independent eigenvectors* $v_1^*, v_2^*, \dots, v_n^*$ *with corresponding eigenvalues* $\lambda_1, \lambda_2, \dots, \lambda_n$*, then the matrix* $\Psi(t) = [\mathrm{e}^{\lambda_1 t}v_1^*, \mathrm{e}^{\lambda_2 t}v_2^*, \dots, \mathrm{e}^{\lambda_n t}v_n^*]$ *is a fundamental solution matrix of system* (6)*.*

Lemma 3. (*See* [17].) *If* $\Psi(t)$ *is a fundamental solution matrix of system* (6)*, then*

$$\Psi(t) = \mathrm{e}^{Bt}\Psi(0).$$

Lemma 4. (*See* [17].) *If* B *has a single eigenvalue* λ *of multiplicity* n*, then*

$$\mathrm{e}^{Bt} = \mathrm{e}^{\lambda t}\sum_{i=0}^{n-1}\frac{t^i}{i!}(B-\lambda I)^i.$$

Some notations are introduced:

$$\xi_1 = \min_{1\leqslant u\leqslant p}\{a_u\} > 0, \qquad \xi_2 = \max_{1\leqslant u\leqslant p}\{b_u\} > 0, \qquad \xi_3 = \max_{1\leqslant v\leqslant q}\left\{\sum_{u=1}^p |c_{uv}|Q_v\right\} > 0,$$

$$\xi_4 = \max_{1\leqslant v\leqslant q}\left\{\sum_{u=1}^p |d_{uv}|Q_v\right\} > 0, \qquad \xi_5 = \min_{1\leqslant v\leqslant q}\{l_v\} > 0, \qquad \xi_6 = \max_{1\leqslant v\leqslant q}\{k_v\} > 0,$$

$$\xi_7 = \max_{1\leqslant u\leqslant p}\left\{\sum_{v=1}^q |e_{vu}|L_u\right\} > 0, \qquad \xi_8 = \max_{1\leqslant u\leqslant p}\left\{\sum_{v=1}^q |r_{vu}|L_u\right\} > 0,$$

$$\xi_9 = m_1 + m_2|\sigma_1^*| + m_3|u_2| + m_4|u_5|, \qquad \xi_{10} = m_1|\sigma_6^*| + m_2|\sigma_7^*| + m_3|\sigma_8^*| + m_4|\sigma_9^*|;$$

$$u_1 = -\frac{\xi_3}{(\xi_1-\eta_2)(\eta_1-\xi_1)}, \qquad u_2 = \frac{\xi_3}{\xi_1-\eta_2}\left[\frac{1}{\eta_1-\xi_1} - \frac{1}{\eta_2-\eta_1}\right],$$

$$u_3 = -\frac{\xi_3(\eta_1-\xi_5)(\xi_1-\xi_5)}{\xi_1-\eta_2}, \qquad u_4 = \xi_3(\eta_1-\xi_5)\left[\frac{\xi_1-\xi_5}{\xi_1-\eta_2} - \frac{1}{\eta_1-\xi_1}\right],$$

$$u_5 = -\xi_3(\eta_1-\xi_5)\left[\frac{\xi_1-\xi_5}{\xi_1-\eta_2} - \frac{1}{\eta_1-\xi_1}\right], \qquad u_6 = \frac{\xi_3}{(\eta_2-\eta_1)(\xi_1-\eta_2)},$$

$$u_7 = (\eta_1-\xi_5)(\xi_1-\xi_5);$$

$$\sigma_1^* = \frac{1}{\xi_1 - \eta_1}, \qquad \sigma_2^* = 1, \qquad \sigma_3^* = 1 + \frac{1}{\xi_1 - \eta_1}, \qquad \sigma_4^* = u_2 + u_1(\eta_1 - \xi_1),$$
$$\sigma_5^* = u_5 + u_4(\eta_1 - \xi_1), \qquad \sigma_6^* = \frac{1}{\eta_2 - \eta_1}, \qquad \sigma_7^* = \frac{1}{\eta_1 - \xi_1} + \frac{1}{(\eta_2 - \eta_1)(\xi_1 - \eta_1)},$$
$$\sigma_8^* = 1 + u_1 + \frac{u_2}{\eta_2 - \eta_1}, \qquad \sigma_9^* = \frac{1}{\eta_2 - \eta_1} + u_4 + \frac{u_3(\xi_1 - \eta_2)}{\xi_3},$$
$$\sigma_{10}^* = \xi_3, \qquad \sigma_{13}^* = \xi_3 u_5 + \xi_3(\eta_1 - \xi_5)u_4 + u_3(\eta_1 - \xi_5)(\xi_1 - \xi_5),$$
$$\sigma_{12}^* = \xi_3 u_2 + \xi_3(\eta_1 - \xi_5)u_1 + \frac{(\eta_1 - \xi_5)\xi_3(\xi_1 - \xi_5)}{\xi_1 - \eta_2}, \qquad \sigma_{11}^* = \frac{\xi_3 + \xi_3(\xi_5 - \eta_1)}{\xi_1 - \eta_1};$$
$$l_1 = x_1\sigma_2^* + x_2\sigma_3^* + x_3\sigma_4^* + x_4\sigma_5^*, \qquad l_3 = x_1\sigma_1^* + x_3\sigma_8^* + x_4\sigma_9^*,$$
$$l_2 = x_1\sigma_2^* + x_2\sigma_3^* + x_3\sigma_4^* + x_4\sigma_5^*, \qquad l_4 = x_1\sigma_{10}^* + x_2\sigma_{11}^* + x_3\sigma_{12}^* + x_4\sigma_{13}^*,$$
$$l_5 = \max\bigl\{\sigma_1^*\xi_2 + u_5\xi_8 e^{\eta_1\zeta} + u_5\xi_7, u_5\xi_6 + \xi_4\sigma_1^* e^{\eta_1\zeta}\bigr\},$$
$$l_6 = \max\bigl\{\xi_6\sigma_5^* + \xi_4 e^{\xi_1\zeta}, \xi_2\sigma_3^* + \xi_8\sigma_5^* e^{\xi_1\eta} + \sigma_5^*\xi_7\bigr\},$$
$$l_7 = \max\bigl\{\xi_2\sigma_7^* + \xi_7\sigma_9^* + \xi_8\sigma_9 e^{\eta_2\eta}, \xi_6\sigma_9^* + \xi_4\sigma_7^*\bigr\},$$
$$l_8 = \max\bigl\{\xi_4\sigma_{11}^* e^{\xi_2\zeta} + \xi_6\sigma_{13}^*, \xi_8\sigma_{13}^* e^{\xi_5\eta} + \sigma_{11}^*\xi_2 + \xi_7\sigma_{13}^*\bigr\};$$
$$y_1(t) = \xi_4 e^{-\eta_1 t}\int_{-\zeta}^{0} e^{\eta_1(s+\zeta)} Z_3(s)\,ds, \qquad y_2(t) = u_5\xi_8 e^{-\eta_1 t}\int_{0}^{t} e^{\eta_1(s+\eta)} Z_1(s)\,ds,$$
$$y_3(t) = \xi_4 e^{-\xi_1 t}\int_{-\zeta}^{0} e^{\xi_1(s+\zeta)} Z_3(s)\,ds, \qquad y_4(t) = \sigma_8^* e^{-\xi_1 t}\int_{-\eta}^{0} e^{\xi_1(s+\eta)} Z_1(s)\,ds,$$
$$y_5(t) = \xi_4 e^{-\eta_2 t}\int_{-\eta_2}^{0} e^{\eta_2(s+\zeta)} Z_3(s)\,ds + \xi_8 e^{-\eta_2 t}\int_{-\zeta}^{0} e^{\eta_2(s+\zeta)} Z_1(s)\,ds,$$
$$y_6(t) = \xi_8 e^{-\eta_2 t}\int_{-\eta}^{0} e^{\eta_2(s+\eta)} Z_1(s)\,ds, \qquad y_7(t) = \xi_4 e^{-\xi_5 t}\int_{-\zeta}^{0} e^{\xi_5(s+\zeta)} Z_3(s)\,ds,$$
$$y_8(t) = \xi_8 e^{-\xi_5 t}\int_{-\eta}^{0} e^{\xi_5(s+\eta)} Z_1(s)\,ds;$$
$$r_1 = \max_{1\leqslant u\leqslant p}\left\{b_u - 2a_u + \sum_{v=1}^{q}\bigl(|c_{uv}| + |d_{uv}|\bigr)Q_v\right\}, \qquad r_2 = \max_{1\leqslant u\leqslant p}\{b_u\},$$
$$r_3 = \max_{1\leqslant v\leqslant q}\bigl\{|c_{uv}|Q_v\bigr\}, \qquad r_4 = \max_{1\leqslant v\leqslant q}\left\{k_v - 2l_v + \sum_{u=1}^{p}\bigl(|e_{uv}| + |r_{uv}|\bigr)L_u\right\},$$
$$r_5 = \max_{1\leqslant v\leqslant q}\{k_v\}, \qquad r_6 = \max_{1\leqslant u\leqslant p}\left\{\sum_{v=1}^{q}|e_{uv}|L_u\right\},$$

$$r_7 = \max_{1\leqslant u\leqslant p}\left\{\sum_{v=1}^{q}|r_{vu}|L_u\right\},\qquad r_8 = \max_{1\leqslant v\leqslant q}\left\{\sum_{u=1}^{p}|d_{uv}|Q_v\right\},$$
$$r_9 = 1-2\eta_1-r_1,\qquad r_{10} = 1-2\eta_2-r_1,\qquad r_{11} = 1-2\eta_2-2r_1+r_4;$$
$$g_5(t) = 1+r_9t+0.5r_9^2t^2,\qquad g_6(t) = t+0.5r_9t^2,\qquad g_7(t) = r_3t+0.5r_3r_{10}t^2,$$
$$g_8(t) = 1+r_{10}t+0.5r_{10}^2t^2,\qquad g_9(t) = 1+0.5(r_4-r_1)^2t^2+(r_4-r_1)t,$$
$$g_{10}(t) = t+0.5r_{11}t^2.$$

3 Main results

In this section, two novel criteria on the GAS for systems (2) and (3) are obtained.

Theorem 1. *Under Assumption* 1*, systems* (2) *and* (3) *achieve GAS if the following conditions are satisfied*:

(h1) $\xi_1 \neq \xi_5 \neq \eta_1 \neq \eta_2$;
(h2) $\eta_1, \eta_2 > 0$.

Proof. Four Lyapunov functions are defined:

$$Z_1(t) = \sum_{u=1}^{p}\bigl|M_u(t)\bigr|,\qquad Z_2(t) = \sum_{u=1}^{p}\bigl|N_u(t)\bigr|,$$
$$Z_3(t) = \sum_{v=1}^{q}\bigl|K_v(t)\bigr|,\qquad Z_4(t) = \sum_{v=1}^{q}\bigl|H_v(t)\bigr|.$$

Computing the derivatives of $Z_i(t)$ ($i = 1,2,3,4$) along the solutions of system (4), we have

$$\begin{aligned}
Z_1'(t) &= \sum_{u=1}^{p}\operatorname{sign}\bigl[M_u(t)\bigr]\bigl(-\eta_1M_u(t)+N_u(t)\bigr) \leqslant \sum_{u=1}^{p}\bigl(\bigl|N_u(t)\bigr|-\eta_1\bigl|M_u(t)\bigr|\bigr)\\
&= -\eta_1Z_1(t)+Z_2(t),
\end{aligned}\tag{7}$$

$$\begin{aligned}
Z_2'(t) &= \sum_{u=1}^{p}\operatorname{sign}\bigl[N_u(t)\bigr]\Biggl\{-a_uN_u(t)-b_uM_u(t)+\sum_{v=1}^{q}c_{uv}\bigl[f_v(\omega_v(t))-f_v\bigl(\gamma_v(t)\bigr)\bigr]\\
&\quad+\sum_{v=1}^{q}d_{uv}\bigl[f_v\bigl(\omega_v(t-\zeta)\bigr)-f_v\bigl(\gamma_v(t-\zeta)\bigr)\bigr]\Biggr\}\\
&\leqslant \sum_{u=1}^{p}\Biggl\{-a_u\bigl|N_u(t)\bigr|+b_u\bigl|M_u(t)\bigr|+\sum_{v=1}^{q}|c_{uv}|Q_v\bigl|K_v(t)\bigr|\\
&\quad+\sum_{v=1}^{q}|d_{uv}|Q_v\bigl|K_v(t-\zeta)\bigr|\Biggr\}\\
&\leqslant -\xi_1Z_2(t)+\xi_2Z_1(t)+\xi_3Z_3(t)+\xi_4Z_3(t-\zeta),
\end{aligned}\tag{8}$$

$$
\begin{aligned}
Z_3'(t) &= \sum_{v=1}^{q} \operatorname{sign}\bigl[K_v(t)\bigr]\bigl\{-\eta_2 K_v(t) + H_v(t)\bigr\} \leqslant \sum_{v=1}^{q} \bigl(\bigl|H_v(t)\bigr| - \eta_2\bigl|K_v(t)\bigr|\bigr) \\
&= -\eta_2 Z_3(t) + Z_4(t),
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
Z_4'(t) &= \sum_{v=1}^{q} \operatorname{sign}\bigl[H_v(t)\bigr]\Biggl\{-l_v H_v(t) - k_v K_v(t) + \sum_{u=1}^{p} e_{vu}\bigl[g_u\bigl(\alpha_u(t)\bigr) - g_u\bigl(\sigma_u(t)\bigr)\bigr] \\
&\quad + \sum_{u=1}^{p} r_{vu}\bigl[g_u\bigl(\alpha_u(t-\eta)\bigr) - g_u\bigl(\sigma(t-\eta)\bigr)\bigr]\Biggr\} \\
&\leqslant \sum_{v=1}^{q}\Biggl\{-l_v\bigl|H_v(t)\bigr| + k_v\bigl|K_v(t)\bigr| + \sum_{u=1}^{p} |e_{vu}| L_u \bigl|M_u(t)\bigr| \\
&\quad + \sum_{v=1}^{q} |r_{vu}| L_u \bigl|M_u(t-\eta)\bigr|\Biggr\} \\
&\leqslant -\xi_5 Z_4(t) + \xi_6 Z_3(t) + \xi_7 Z_1(t) + \xi_8 Z_1(t-\eta).
\end{aligned} \tag{10}
$$

From (7)–(10) it follows that there exist four positive bounded functions $f_i(t)$ ($|f_i(t)| \leqslant m_i$, $i = 1, 2, 3, 4$) such that

$$Z_1'(t) = -\eta_1 Z_1(t) + Z_2(t) - f_1(t), \tag{11}$$

$$Z_2'(t) = -\xi_1 Z_2(t) + \xi_2 Z_1(t) + \xi_3 Z_3(t) + \xi_4 Z_3(t-\zeta) - f_2(t), \tag{12}$$

$$Z_3'(t) = -\eta_2 Z_3(t) + Z_4(t) - f_3(t), \tag{13}$$

and

$$Z_4'(t) = -\xi_5 Z_4(t) + \xi_6 Z_3(t) + \xi_7 Z_1(t) + \xi_8 Z_1(t-\eta) - f_4(t). \tag{14}$$

Let $Z(t) = [Z_1(t), Z_2(t), Z_3(t), Z_4(t)]^{\mathrm{T}}$. Then the differential equations (11)–(14) can be rewritten in the following vector form:

$$Z'(t) = BZ(t) + f(t), \tag{15}$$

where

$$
B = \begin{pmatrix}
-\eta_1 & 1+\mu_1 & 0 & 0 \\
0 & -\xi_1 & \xi_3 & 0 \\
0 & 0 & -\eta_2 & 1+\mu_4 \\
0 & 0 & 0 & -\xi_5
\end{pmatrix}
$$

and

$$
f(t) = \begin{pmatrix}
-f_1(t) \\
\xi_2 Z_1(t) + \xi_4 Z_3(t-\zeta) - f_2(t) \\
-f_3(t) \\
-f_4(t) + \xi_8 Z_1(t-\eta) + \xi_6 Z_3(t) + \xi_7 Z_1(t)
\end{pmatrix}.
$$

The characteristic equation of the matrix B is

$$\begin{vmatrix} \lambda+\eta_1 & -1 & 0 & 0 \\ 0 & \lambda+\xi_1 & -\xi_3 & 0 \\ 0 & 0 & \lambda+\eta_2 & -1 \\ 0 & 0 & 0 & \lambda+\xi_5 \end{vmatrix} = 0.$$

So, the characteristic roots are $\lambda_1 = -\eta_1$, $\lambda_2 = -\xi_1$, $\lambda_3 = -\eta_2$, $\lambda_4 = -\xi_5$. The corresponding eigenvectors are

$$v_1 = (1,0,0,0)^{\mathrm{T}}, \qquad v_3 = \left(\frac{1}{\eta_2-\eta_1},\, 1,\, \frac{\xi_1-\eta_2}{\xi_3},\, 0\right)^{\mathrm{T}},$$
$$v_2 = (1,\, \eta_1-\xi_1,\, 0,\, 0)^{\mathrm{T}}, \qquad v_4 = \bigl(\xi_3,\, \xi_3(\eta_1-\xi_5),\, (\eta_1-\xi_5)(\xi_1-\xi_5),\, 1\bigr)^{\mathrm{T}}.$$

Since $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4$, then above four vectors are linearly independent. By Lemma 2, the matrix

$$\Psi(t) = \bigl(\mathrm{e}^{-\eta_1 t}v_1,\, \mathrm{e}^{-\xi_1 t}v_2,\, \mathrm{e}^{-\eta_2 t}v_3,\, \mathrm{e}^{-\xi_5 t}v_4\bigr)$$

is a fundamental solution matrix of the system $Z'(t) = BZ(t)$. According to Lemma 3, we have

$$\begin{aligned}
\exp(tB) &= \Psi(t)\psi^{-1}(0) \\
&= \bigl(\mathrm{e}^{-\eta_1 t}v_1, \mathrm{e}^{-\xi_1 t}v_2, \mathrm{e}^{-\eta_2 t}v_3, \mathrm{e}^{-\xi_5 t}v_4\bigr)^{\mathrm{T}}(v_1,v_2,v_3,v_4)^{-1} \\
&= \bigl(\mathrm{e}^{-\eta_1 t}v_1, \mathrm{e}^{-\xi_1 t}v_2, \mathrm{e}^{-\eta_2 t}v_3, \mathrm{e}^{-\xi_5 t}v_4\bigr)^{\mathrm{T}} \begin{pmatrix} 1 & \frac{1}{\xi_1-\eta_1} & u_2 & u_5 \\ 0 & \frac{1}{\eta_1-\xi_1} & u_1 & u_4 \\ 0 & 0 & \frac{\xi_3}{\xi_1-\eta_2} & u_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{e}^{-\eta_1 t} & \mathrm{e}^{-\eta_1 t}\sigma_1^* & \mathrm{e}^{-\eta_1 t}u_2 & \mathrm{e}^{-\eta_1 t}u_5 \\ \mathrm{e}^{-\xi_1 t}\sigma_2^* & \mathrm{e}^{-\xi_1 t}\sigma_3^* & \mathrm{e}^{-\xi_1 t}\sigma_4^* & \mathrm{e}^{-\xi_1 t}\,\sigma_5^* \\ \mathrm{e}^{-\eta_2 t}\sigma_6^* & \mathrm{e}^{-\eta_2 t}\sigma_7^* & \mathrm{e}^{-\eta_2 t}\sigma_8^* & \mathrm{e}^{-\eta_2 t}\sigma_9^* \\ \mathrm{e}^{-\xi_5 t}\sigma_{10}^* & \mathrm{e}^{-\xi_5 t}\sigma_{11}^* & \mathrm{e}^{-\xi_5 t}\sigma_{12}^* & \mathrm{e}^{-\xi_5 t}\sigma_{13}^* \end{pmatrix}.
\end{aligned} \tag{16}$$

So, if $Z(0) = x = (x_1,x_2,x_3,x_4)$ for the equation $Z'(t) = BZ(t)$, then, by Lemma 1, the solution of Eq. (15) is

$$\begin{aligned}
Z(t) &= [Z_1(t), Z_2(t), Z_3(t), Z_4(t)]^{\mathrm{T}} \\
&= \begin{pmatrix} \mathrm{e}^{-\eta_1 t} & \mathrm{e}^{-\eta_1 t}\sigma_1^* & \mathrm{e}^{-\eta_1 t}u_2 & \mathrm{e}^{-\eta_1 t}u_5 \\ \mathrm{e}^{-\xi_1 t}\sigma_2^* & \mathrm{e}^{-\xi_1 t}\sigma_3^* & \mathrm{e}^{-\xi_1 t}\sigma_4^* & \mathrm{e}^{-\xi_1 t}\,\sigma_5^* \\ \mathrm{e}^{-\eta_2 t}\sigma_6^* & \mathrm{e}^{-\eta_2 t}\sigma_7^* & \mathrm{e}^{-\eta_2 t}\sigma_8^* & \mathrm{e}^{-\eta_2 t}\sigma_9^* \\ \mathrm{e}^{-\xi_5 t}\sigma_{10}^* & \mathrm{e}^{-\xi_5 t}\sigma_{11}^* & \mathrm{e}^{-\xi_5 t}\sigma_{12}^* & \mathrm{e}^{-\xi_5 t}\sigma_{13}^* \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \\
&\quad + \int_0^t \begin{pmatrix} \mathrm{e}^{-\eta_1 (t-s)} & \mathrm{e}^{-\eta_1 (t-s)}\sigma_1^* & \mathrm{e}^{-\eta_1 (t-s)}u_2 & \mathrm{e}^{-\eta_1 (t-s)}u_5 \\ \mathrm{e}^{-\xi_1 (t-s)}\sigma_2^* & \mathrm{e}^{-\xi_1 (t-s)}\sigma_3^* & \mathrm{e}^{-\xi_1 (t-s)}\sigma_4^* & \mathrm{e}^{-\xi_1 (t-s)}\,\sigma_5^* \\ \mathrm{e}^{-\eta_2 (t-s)}\sigma_6^* & \mathrm{e}^{-\eta_2 (t-s)}\sigma_7^* & \mathrm{e}^{-\eta_2 (t-s)}\sigma_8^* & \mathrm{e}^{-\eta_2 (t-s)}\sigma_9^* \\ \mathrm{e}^{-\xi_5 (t-s)}\sigma_{10}^* & \mathrm{e}^{-\xi_5 (t-s)}\sigma_{11}^* & \mathrm{e}^{-\xi_5 (t-s)}\sigma_{12}^* & \mathrm{e}^{-\xi_5 (t-s)}\sigma_{13}^* \end{pmatrix} f(s)\,\mathrm{d}s.
\end{aligned} \tag{17}$$

From (17) we obtain

$$\begin{aligned} Z_1(t) &= \mathrm{e}^{-\eta_1 t} l_2 \\ &\quad + \int_0^t \mathrm{e}^{-\eta_1(t-s)} \bigl\{ -f_1(s) + \sigma_1^* \bigl[\xi_2 Z_1(s) + \xi_4 Z_3(t-\zeta) - f_2(s) \bigr] \\ &\qquad - u_2 f_3(s) + u_5 \bigl[-f_4(s) + \xi_8 Z_1(s-\eta) + \xi_6 Z_3(s) + \xi_7 Z_1(s) \bigr] \bigr\} \,\mathrm{d}s, \end{aligned} \tag{18}$$

$$\begin{aligned} Z_2(t) &= \mathrm{e}^{-\xi_1 t} l_1 \\ &\quad + \int_0^t \mathrm{e}^{-\xi_1(t-s)} \bigl\{ -\sigma_2^* f_1(s) + \sigma_3^* \bigl[\xi_2 Z_1(s) + \xi_4 Z_3(s-\zeta) - f_2(s) \bigr] \\ &\qquad - \sigma_4^* f_3(s) + \sigma_5^* \bigl[-f_4(s) + \xi_8 Z_1(s-\eta) + \xi_6 Z_3(s) + \xi_7 Z_1(s) \bigr] \bigr\} \,\mathrm{d}s, \end{aligned} \tag{19}$$

$$\begin{aligned} Z_3(t) &= \mathrm{e}^{-\eta_2 t} l_3 \\ &\quad + \int_0^t \mathrm{e}^{-\eta_2(t-s)} \bigl\{ -\sigma_6^* f_1(s) + \sigma_7^* \bigl[\xi_2 Z_1(s) + \xi_4 Z_3(s-\zeta) - f_2(s) \bigr] \\ &\qquad - \sigma_8^* f_3(s) + \sigma_9^* \bigl[-f_4(s) + \xi_8 Z_1(s-\eta) + \xi_6 Z_3(s) + \xi_7 Z_1(s) \bigr] \bigr\} \,\mathrm{d}s, \end{aligned} \tag{20}$$

and

$$\begin{aligned} Z_4(t) &= \mathrm{e}^{-\xi_5 t} l_4 \\ &\quad + \int_0^t \mathrm{e}^{-\xi_5(t-s)} \bigl\{ -\sigma_{10}^* f_1(s) + \sigma_{11}^* \bigl[\xi_2 Z_1(s) + \xi_4 Z_3(s-\zeta) - f_2(s) \bigr] \\ &\qquad - \sigma_{12}^* f_3(s) + \sigma_{13}^* \bigl[-f_4(s) + \xi_8 Z_1(s-\eta) + \xi_6 Z_3(s) + \xi_7 Z_1(s) \bigr] \bigr\} \,\mathrm{d}s. \end{aligned} \tag{21}$$

It is clear that

$$\begin{aligned} &\xi_4 \sigma_1^* \int_0^t \mathrm{e}^{-\eta_1(t-s)} Z_3(s-\zeta) \,\mathrm{d}s + u_5 \xi_8 \int_0^t \mathrm{e}^{-\eta_1(t-s)} Z_1(s-\eta) \,\mathrm{d}s \\ &\quad = \xi_4 \int_{-\zeta}^{t-\zeta} \mathrm{e}^{-\eta_1(t-s-\zeta)} Z_3(s) \,\underset{.}{s} + u_5 \xi_8 \int_{-\eta}^{t-\eta} \mathrm{e}^{-\eta_1(t-s-\eta)} Z_1(s) \,\mathrm{d}s \\ &\quad = \xi_4 \sigma_1^* \int_0^{t-\zeta} \mathrm{e}^{-\eta_1(t-s-\zeta)} Z_3(s) \,\mathrm{d}s + x_1(t) + u_5 \xi_8 \int_0^{t-\eta} \mathrm{e}^{-\eta_1(t-s-\eta)} Z_1(s) \,\mathrm{d}s \\ &\qquad + x_2(t) \\ &\quad \leqslant \xi_4 \sigma_1^* \int_0^t \mathrm{e}^{-\eta_1(t-s-\zeta)} Z_3(s) \,\mathrm{d}s + y_1(t) + u_5 \xi_8 \int_0^t \mathrm{e}^{-\eta_1(t-s-\eta)} Z_1(s) \,\mathrm{d}s \\ &\qquad + y_2(t), \end{aligned} \tag{22}$$

$$\xi_4\sigma_3^*\int_0^t \mathrm{e}^{-\xi_1(t-s)}Z_3(s-\zeta)\,\mathrm{d}s+\xi_8\sigma_5^*\int_0^t \mathrm{e}^{-\xi_1(t-s)}Z_1(s-\eta)\,\mathrm{d}s$$
$$\leqslant \xi_4\sigma_3^*\int_0^t \mathrm{e}^{-\xi_1(t-s-\zeta)}Z_3(s)\,\mathrm{d}s+y_3(t)+\xi_8\sigma_5^*\int_0^t \mathrm{e}^{-\xi_1(t-s-\eta)}Z_1(s)\,\mathrm{d}s$$
$$+\,y_4(t), \tag{23}$$
$$\xi_4\sigma_7^*\int_0^t \mathrm{e}^{-\eta_2(t-s)}Z_3(s-\zeta)\,\mathrm{d}s+\sigma_9^*\xi_8\int_0^t \mathrm{e}^{-\eta_2(t-s)}Z_1(s-\eta)\,\mathrm{d}s$$
$$\leqslant \xi_4\sigma_7^*\int_0^t \mathrm{e}^{-\eta_2(t-s-\zeta)}Z_3(s)\,\mathrm{d}s+y_5(t)+\xi_8\sigma_9^*\int_0^t \mathrm{e}^{-\eta_2(t-s-\eta)}Z_1(s)\,\mathrm{d}s$$
$$+\,y_6(t), \tag{24}$$
and
$$\xi_4\sigma_{11}^*\int_0^t \mathrm{e}^{-\xi_5(t-s)}Z_3(s-\zeta)\,\mathrm{d}s+\xi_8\sigma_{13}^*\int_0^t \mathrm{e}^{-\xi_5(t-s)}Z_1(s-\eta)\,\mathrm{d}s$$
$$\leqslant \xi_4\sigma_{11}^*\int_0^t \mathrm{e}^{-\xi_5(t-s-\zeta)}Z_3(s)\,\mathrm{d}s+y_7(t)+\xi_8\sigma_{13}^*\int_0^t \mathrm{e}^{-\xi_5(t-s-\eta)}Z_1(s)\,\mathrm{d}s$$
$$+\,y_8(t). \tag{25}$$

Substituting (22) into (18), (23) into (19), (24) into (20), (25) into (21), respectively, we have

$$Z_1(t)\leqslant l_2\mathrm{e}^{-\eta_1 t}$$
$$+\int_0^t \mathrm{e}^{-\eta_1(t-s)}\bigl\{-f_1(s)-\sigma_1^*f_2(s)-u_2f_3(s)-u_5f_4(s)\bigr\}\,\mathrm{d}s$$
$$+\int_0^t \mathrm{e}^{-\eta_1(t-s)}\bigl([\sigma_1^*\xi_2+u_5\xi_8\mathrm{e}^{\eta_1\zeta}+u_5\xi_7]Z_1(s)+\bigl[u_5\xi_6+\xi_4\sigma_1^*\mathrm{e}^{\eta_1\zeta}\bigr]Z_3(s)\bigr)\,\mathrm{d}s$$
$$+\,y_1(t)+y_2(t)$$
$$\leqslant l_2\mathrm{e}^{-\eta_1 t}+\int_0^t \mathrm{e}^{-\eta_1(t-s)}\bigl\{-f_1(s)-\sigma_1^*f_2(s)-u_2f_3(s)-u_5f_4(s)+\sigma_1^*p\bigr\}\,\mathrm{d}s$$
$$+\int_0^t \mathrm{e}^{-\eta_1(t-s)}l_5\bigl[Z_1(s)+Z_3(s)\bigr]\,\mathrm{d}s+y_1(t)+y_2(t), \tag{26}$$

$$\begin{aligned}
Z_2(t) \leqslant{}& l_1\mathrm{e}^{-\xi_1 t} \\
&+ \int\limits_0^t \mathrm{e}^{-\xi_1(t-s)}\bigl[-\sigma_2^* f_1(s) - \sigma_3^* f_2(s) + p\mu_2\mathrm{e}^{\mu_3 s} - \sigma_4^* f_3(s) + \sigma_5^* f_4(s)\bigr]\,\mathrm{d}s \\
&+ \int\limits_0^t \mathrm{e}^{-\xi_1(t-s)}\bigl\{\bigl[\sigma_3^*\xi_2 + \xi_8\sigma_5^*\mathrm{e}^{\xi_1\eta} + \sigma_5^*\xi_7\bigr]Z_1(s) + \bigl[\xi_6\sigma_5^* + \xi_4\mathrm{e}^{\xi_1\zeta}\bigr]Z_3(s)\bigr\}\,\mathrm{d}s \\
\leqslant{}& l_1\mathrm{e}^{-\xi_1 t} + \int\limits_0^t \mathrm{e}^{-\xi_1(t-s)}\bigl[-\sigma_2^* f_1(s) - \sigma_3^* f_2(s) - \sigma_4^* f_3(s) + \sigma_5^* f_4(s)\bigr]\,\mathrm{d}s \\
&+ \int\limits_0^t \mathrm{e}^{-\xi_1(t-s)} l_6\bigl[Z_1(s) + Z_3(s)\bigr]\,\mathrm{d}s + y_3(t) + y_4(t),
\end{aligned} \tag{27}$$

$$\begin{aligned}
Z_3(t) \leqslant{}& l_3\mathrm{e}^{-\eta_2 t} \\
&+ \int\limits_0^t \mathrm{e}^{-\eta_2(t-s)}\bigl\{\bigl[\xi_2\sigma_7^* + \xi_7\sigma_9^* + \xi_8\sigma_9^*\mathrm{e}^{\eta_2\eta}\bigr]Z_1(s) + \bigl[\xi_6\sigma_9^* + \xi_4\sigma_7^*\bigr]Z_3(s)\bigr\}\,\mathrm{d}s \\
&+ y_5(t) + y_6(t) + \int\limits_0^t \mathrm{e}^{-\eta_2(t-s)}\bigl[-\sigma_6^* f_1(s) - \sigma_7^* f_2(s) - \sigma_8^* f_3(s) - \sigma_9^* f_4(s)\bigr]\,\mathrm{d}s \\
\leqslant{}& l_3\mathrm{e}^{-\eta_2 t} + l_7\int\limits_0^t \mathrm{e}^{-\eta_2(t-s)}\bigl[Z_1(s) + Z_3(s)\bigr]\,\mathrm{d}s + y_5(t) + y_6(t) \\
&+ \int\limits_0^t \mathrm{e}^{-\eta_2(t-s)}\bigl[-\sigma_6^* f_1(s) - \sigma_7^* f_2(s) - \sigma_8^* f_3(s) - \sigma_9^* f_4(s)\bigr]\,\mathrm{d}s,
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
Z_4(t) \leqslant{}& l_4\mathrm{e}^{-\xi_5 t} + \int\limits_0^t \mathrm{e}^{-\xi_5(t-s)}\bigl\{-\sigma_{10}^* f_1(s) - \sigma_{11}^* f_2(s) - \sigma_{12}^* f_3(s) - \sigma_{13}^* f_4(s)\bigr\}\,\mathrm{d}s \\
&+ \int\limits_0^t \mathrm{e}^{-\xi_5(t-s)}\bigl\{\bigl[\xi_4\sigma_{11}^*\mathrm{e}^{\xi_5\zeta} + \xi_6\sigma_{13}^*\bigr]Z_3(s) \\
&\quad + \bigl[\xi_8\sigma_{13}^*\mathrm{e}^{\xi_5\eta} + \sigma_{11}^*\xi_2 + \xi_7\sigma_{13}^*\bigr]Z_1(s)\bigr\}\,\mathrm{d}s \\
\leqslant{}& l_4\mathrm{e}^{-\xi_5 t} + \int\limits_0^t \mathrm{e}^{-\xi_5(t-s)}\bigl\{-\sigma_{10}^* f_1(s) - \sigma_{11}^* f_2(s) - \sigma_{12}^* f_3(s) - \sigma_{13}^* f_4(s)\bigr\}\,\mathrm{d}s \\
&+ \int\limits_0^t \mathrm{e}^{-\xi_5(t-s)} l_8\bigl[Z_3(s) + Z_1(s)\bigr]\,\mathrm{d}s + y_7(t) + y_8(t).
\end{aligned} \tag{29}$$

Summing (26) and (28) yields

$$\begin{aligned}0 &\leqslant Z_1(t)+Z_3(t)\\ &\leqslant \int_0^t \left[l_5\mathrm{e}^{-\eta_1(t-s)}+l_7\mathrm{e}^{-\eta_2(t-s)}\right]\left[Z_1(s)+Z_3(s)\right]\mathrm{d}s\\ &\quad + l_2\mathrm{e}^{-\eta_1 t}+l_3\mathrm{e}^{-\eta_2 t}+y_1(t)+y_2(t)+y_5(t)+y_6(t)\\ &\quad + \int_0^t \mathrm{e}^{-\eta_1(t-s)}\left[-f_1(s)-\sigma_1^* f_2(s)-u_2f_3(s)-u_5f_4(s)+\sigma_1^*\right]\mathrm{d}s\\ &\quad + \int_0^t \mathrm{e}^{-\eta_2(t-s}\left[-\sigma_6^* f_1(s)-\sigma_7^* f_2(s)-\sigma_8^* f_3(s)-\sigma_9^* f_4(s)\right]\mathrm{d}s. \end{aligned}\tag{30}$$

By the mean-value theorem for integrals, it follows from (30) that there exists $\theta_1\in(0,1)$ such that

$$\begin{aligned}Z_1(t)+Z_3(t) &\leqslant \left[l_5\mathrm{e}^{-\eta_1(1-\theta_1)t}+l_7\mathrm{e}^{-\eta_2(1-\theta_1)t}\right]\int_0^t \left[Z_1(s)+Z_3(s)\right]\mathrm{d}s\\ &\quad + l_2\mathrm{e}^{-\eta_1 t}+l_3\mathrm{e}^{-\eta_2 t}+y_1(t)+y_2(t)+y_5(t)+y_6(t)\\ &\quad + \int_0^t \mathrm{e}^{-\eta_1(t-s)}\left[-f_1(s)-\sigma_1^* f_2(s)-u_2f_3(s)-u_5f_4(s)\right]\mathrm{d}s\\ &\quad + \int_0^t \mathrm{e}^{-\eta_2(t-s}\left[-\sigma_6^* f_1(s)-\sigma_7^* f_2(s)-\sigma_8^* f_3(s)-\sigma_9^* f_4(s)\right]\mathrm{d}s. \end{aligned}\tag{31}$$

Multiplying (31) by $\mathrm{e}^{-\int_0^t[l_5\mathrm{e}^{-\eta_1(1-\theta_1)s}+l_7\mathrm{e}^{-\eta_2(1-\theta_1)s}]\,\mathrm{d}s}$ yields

$$\begin{aligned}&\frac{\mathrm{d}[\mathrm{e}^{-\int_0^t[l_5\mathrm{e}^{-\eta_1(1-\theta_1)s}+l_7\mathrm{e}^{-\eta_2(1-\theta_1)s}]\,\mathrm{d}s}\int_0^t[Z_1(s)+Z_3(s)]\,\mathrm{d}s]}{\mathrm{d}t}\\ &\quad\leqslant \mathrm{e}^{-\int_0^t[l_5\mathrm{e}^{-\eta_1(1-\theta_1)s}+l_7\mathrm{e}^{-\eta_2(1-\theta_1)s}]\,\mathrm{d}s}\\ &\qquad\times\Bigg\{l_2\mathrm{e}^{-\eta_1 t}+l_3\mathrm{e}^{-\eta_2 t}+y_1(t)+y_2(t)+y_5(t)+y_6(t)\\ &\qquad + \int_0^t \mathrm{e}^{-\eta_1(t-s)}\left[-f_1(s)-\sigma_1^* f_2(s)-u_2f_3(s)-u_5f_4(s)\right]\mathrm{d}s\\ &\qquad + \int_0^t \mathrm{e}^{-\eta_2(t-s}\left[-\sigma_6^* f_1(s)-\sigma_7^* f_2(s)-\sigma_8^* f_3(s)-\sigma_9^* f_4(s)\right]\mathrm{d}s\Bigg\}. \end{aligned}\tag{32}$$

By integrating (32) over $[0, t]$, we obtain

$$\begin{aligned}
&\mathrm{e}^{\frac{l_5}{\eta_1(\theta_1-1)}[1-\mathrm{e}^{\eta_1(\theta_1-1)t}]+\frac{l_7}{\eta_2(\theta_1-1)}[1-\mathrm{e}^{\eta_2(\theta_1-1)t}]} \int_0^t \big[Z_1(s)+Z_3(s)\big]\,\mathrm{d}s \\
&\quad\leqslant \int_0^t \mathrm{e}^{\frac{l_5}{\eta_1(\theta_1-1)}[1-\mathrm{e}^{\eta_1(\theta_1-1)z}]+\frac{l_7}{\eta_2(\theta_1-1)}[1-\mathrm{e}^{\eta_2(\theta_1-1)z}]} \\
&\qquad\times \Bigg\{ l_2\mathrm{e}^{-\eta_1 z}+l_3\mathrm{e}^{-\eta_2 z}+y_1(z)+y_2(z)+y_5(z)+y_6(z) \\
&\qquad+\int_0^z \mathrm{e}^{-\eta_1(z-s)}\big[m_1+|\sigma_1^*|m_2+|u_2|m_3+|u_5|m_4\big]\,\mathrm{d}s \\
&\qquad+\int_0^z \mathrm{e}^{-\eta_2(z-s)}\big[|\sigma_6^*|m_1+|\sigma_7^*|m_2+|\sigma_8^*|m_3+|\sigma_9^*|m_4\big]\,\mathrm{d}s \Bigg\}\,\mathrm{d}z. \qquad (33)
\end{aligned}$$

By using mean-value theorem for integrals, it follows from (33) that there exist $\theta_2, \theta_3 \in (0, 1)$ such that

$$\begin{aligned}
&\mathrm{e}^{\frac{l_5}{\eta_1(\theta_1-1)}[1-\mathrm{e}^{\eta_1(\theta_1-1)t}]+\frac{l_7}{\eta_2(\theta_1-1)}[1-\mathrm{e}^{\eta_2(\theta_1-1)t}]} \int_0^t \big[Z_1(s)+Z_3(s)\big]\,\mathrm{d}s \\
&\quad\leqslant \int_0^t \mathrm{e}^{\frac{l_5}{\eta_1(\theta_1-1)}[1-\mathrm{e}^{\eta_1(\theta_1-1)z}]+\frac{l_7}{\eta_2(\theta_1-1)}[1-\mathrm{e}^{\eta_2(\theta_1-1)z}]} \\
&\qquad\times \big\{ l_2\mathrm{e}^{-\eta_1 z}+l_3\mathrm{e}^{-\eta_2 z}+y_1(z)+y_2(z) \\
&\qquad+y_5(z)+y_6(z)+z\mathrm{e}^{-\eta_1(z-\theta_2 z)}\xi_9+z\mathrm{e}^{-\eta_2(z-\theta_2 z)}\xi_{10}\big\}\,\mathrm{d}z \\
&\quad= t\mathrm{e}^{\frac{l_5}{\eta_1(\theta_1-1)}[1-\mathrm{e}^{\eta_1(\theta_1-1)\theta_3 t}]+\frac{l_7}{\eta_2(\theta_1-1)}[1-\mathrm{e}^{\eta_2(\theta_1-1)\theta_3 t}]} \\
&\qquad\times \big\{ l_2\mathrm{e}^{-\eta_1\theta_3 t}+l_3\mathrm{e}^{-\eta_2\theta_3 t}+y_1(\theta_3 t)+y_2(\theta_3 t) \\
&\qquad+y_5(\theta_3 t)+y_6(\theta_3 t)+\theta_3 t\mathrm{e}^{-\eta_1(1-\theta_2)\theta_3 t}\xi_9+\theta_3 t\mathrm{e}^{-\eta_2(1-\theta_2)\theta_3 t}\xi_{10}\big\}. \qquad (34)
\end{aligned}$$

From (34) we have

$$\begin{aligned}
&\lim_{t\to\infty} \mathrm{e}^{\frac{l_5}{\eta_1(\theta_1-1)}[1-\mathrm{e}^{\eta_1(\theta_1-1)t}]+\frac{l_7}{\eta_2(\theta_1-1)}[1-\mathrm{e}^{\eta_2(\theta_1-1)t}]} \lim_{t\to\infty}\int_0^t \big[Z_1(s)+Z_3(s)\big]\,\mathrm{d}s \\
&\quad= \lim_{t\to\infty} \mathrm{e}^{\frac{l_5}{\eta_1(\theta_1-1)}[1-\mathrm{e}^{\eta_1(\theta_1-1)\theta_3 t}]+\frac{l_7}{\eta_2(\theta_1-1)}[1-\mathrm{e}^{\eta_2(\theta_1-1)\theta_3 t}]} \\
&\qquad\times \lim_{t\to\infty} t\big\{ l_2\mathrm{e}^{-\eta_1\theta_3 t}+l_3\mathrm{e}^{-\eta_2\theta_3 t}+y_1(\theta_3 t)+y_2(\theta_3 t)+y_5(\theta_3 t)+y_6(\theta_3 t) \\
&\qquad+\theta_3 t\mathrm{e}^{-\eta_1(1-\theta_2)\theta_3 t}\xi_9+\theta_3 t\mathrm{e}^{-\eta_2(1-\theta_2)\theta_3 t}\xi_{10}\big\}. \qquad (35)
\end{aligned}$$

Since $\eta_1, \eta_2, \xi_1 > 0$, then

$$\lim_{t\to\infty} \mathrm{e}^{\frac{l_5}{\eta_1(\theta_1-1)}[1-\mathrm{e}^{\eta_1(\theta_1-1)t}]+\frac{l_7}{\eta_2(\theta_1-1)}[1-\mathrm{e}^{\eta_2(\theta_1-1)t}]} = \mathrm{e}^{\frac{l_5}{\eta_1(\theta_1-1)}+\frac{l_7}{\eta_2(\theta_1-1)}}, \tag{36}$$

$$\lim_{t\to\infty} \mathrm{e}^{\frac{l_5}{\eta_1(\theta_1-1)}[1-\mathrm{e}^{\eta_1(\theta_1-1)\theta_3 t}]+\frac{l_7}{\eta_2(\theta_1-1)}[1-\mathrm{e}^{\eta_2(\theta_1-1)\theta_3 t}]} = \mathrm{e}^{\frac{l_5}{\eta_1(\theta_1-1)}+\frac{l_7}{\eta_2(\theta_1-1)}}, \tag{37}$$

$$\lim_{t\to\infty} ty_1(\theta_3 t) = \xi_4 \frac{t}{\mathrm{e}^{\eta_1 t}\int_{-\zeta}^{0} \mathrm{e}^{\eta_1(s+\zeta)} Z_3(s)\,\mathrm{d}s}\left(\frac{\infty}{\infty}\right) = 0, \tag{38}$$

$$\lim_{t\to\infty} ty_2(\theta_3 t) = \lim_{t\to\infty} ty_5(\theta_3 t) = \lim_{t\to\infty} ty_6(\theta_3 t) = 0, \tag{39}$$

$$\lim_{t\to\infty} ty_6(\theta_3 t) = \lim_{t\to\infty} l_2 t\mathrm{e}^{-\eta_1\theta_3 t} = \lim_{t\to\infty} l_3 t\mathrm{e}^{-\eta_2\theta_3 t} = 0, \tag{40}$$

$$\begin{aligned}\lim_{t\to\infty} \theta_3 t^2 \mathrm{e}^{-\eta_1(1-\theta_2)\theta_3 t} &= \lim_{t\to\infty} \frac{t^2}{\mathrm{e}^{\eta_1(1-\theta_2)\theta_3 t}} = 2\lim_{t\to\infty} \frac{t}{\eta_1(1-\theta_2)\theta_3 \mathrm{e}^{\eta_1(1-\theta_2)\theta_3 t}} \\ &= 2\lim_{t\to\infty} \frac{1}{[\eta_1(1-\theta_2)\theta_3]^2 \mathrm{e}^{\eta_1(1-\theta_2)\theta_3 t}} = 0,\end{aligned} \tag{41}$$

and

$$\lim_{t\to\infty} \theta_3 t^2 \mathrm{e}^{-\eta_2(1-\theta_2)\theta_3 t} = 0. \tag{42}$$

Substituting (36)–(42) into (35) gives

$$\lim_{t\to\infty} \big[Z_1(t) + Z_3(t)\big] = 0,$$

which implies

$$\lim_{t\to\infty} Z_1(t) = \lim_{t\to\infty} Z_3(t) = 0. \tag{43}$$

By using mean-valued theorems for integrals, from (27) and (29) it follows that there exist $\theta_4, \theta_5, \theta_6, \theta_7 \in (0, 1)$ such that

$$\begin{aligned}&\lim_{t\to\infty} Z_2(t) \\ &\quad\leqslant \lim_{t\to\infty} l_1 \mathrm{e}^{-\xi_1 t} + \lim_{t\to\infty} t\mathrm{e}^{-\xi_1(1-\theta_4)t}\big[-\sigma_2^* f_1(\theta_4 t) + \sigma_3^* f_2(s) + \sigma_4^* f_3(s) + \sigma_3^* f_4(s)\big] \\ &\qquad + l_6 \lim_{t\to\infty} \mathrm{e}^{-\xi_1(t-\theta_5 t)} \lim_{t\to\infty} \int_0^t \big[Z_1(s) + Z_3(s)\big]\,\mathrm{d}s + \lim_{t\to\infty} \big[y_3(t) + y_4(t)\big] \\ &\quad\leqslant \lim_{t\to\infty} l_1 \mathrm{e}^{-\xi_1 t} + \lim_{t\to\infty} t\mathrm{e}^{-\xi_1(1-\theta_4)t}\big[|\sigma_2^*|m_1 + |\sigma_3^*|m_2 + |\sigma_4^*|m_3 + |\sigma_3^*|m_4\big] \\ &\qquad + l_6 \lim_{t\to\infty} \mathrm{e}^{-\xi_1(t-\theta_5 t)} \lim_{t\to\infty} \int_0^t \big[Z_1(s) + Z_3(s)\big]\,\mathrm{d}s + \lim_{t\to\infty} \big[y_3(t) + y_4(t)\big] \\ &\quad= 0 + 0 + 0 \cdot 0 + 0 = 0\end{aligned} \tag{44}$$

and

$$\begin{aligned}\lim_{t\to\infty} Z_4(t) &\leqslant \lim_{t\to\infty}\left[l_4\mathrm{e}^{-\xi_3 t}+y_7(t)+y_8(t)\right]\\&\quad+\lim_{t\to\infty}\frac{t}{\mathrm{e}^{\xi_5(t-\theta_6)t}}\left[|\sigma_{10}^* m_1+|\sigma_{11}^*|m_2+|\sigma_{12}|m_3|\sigma_{13}^*|m_4\right]\\&\quad+\lim_{t\to\infty} l_8\mathrm{e}^{-\xi_5(t-\theta_7 t)}\lim_{t\to\infty}\int_0^t\left[Z_1(s)+Z_3(s)\right]\mathrm{d}s\\&=0+0+0=0.\end{aligned}\tag{45}$$

From (43)–(45) the proof of Theorem 1 is completed. □

Theorem 2. *Under Assumption* 1*, systems* (2) *and* (3) *achieve GAS if the following conditions are satisfied*:

- (h3) $1-2\eta_1=r_1=1-2\eta_2=r_4$;
- (h4) $r_1<0, r_4<0, 2\eta_1>1, 2\eta_2>1$.

Proof. Four Lyapunov functions are defined as follows:

$$\begin{aligned}Z_5(t)&=\sum_{u=1}^{p}M_u^2(t), & Z_6(t)&=\sum_{u=1}^{p}N_u^2(t),\\Z_7(t)&=\sum_{v=1}^{q}K_v^2(t), & Z_8(t)&=\sum_{v=1}^{q}H_v^2(t).\end{aligned}$$

Computing the derivatives of $Z_i(t)$ $(i=5,6,7,8)$ along the solutions of system (4), we have

$$\begin{aligned}Z_5'(t)&=2\sum_{u=1}^{p}M_u(t)\left[-\eta_1 M_u(t)+N_u(t)\right]\leqslant\sum_{u=1}^{p}\left[-2\eta_1 M_u^2(t)+M_u^2(t)+N_u^2(t)\right]\\&=(1-2\eta_1)Z_5(t)+Z_6(t),\end{aligned}\tag{46}$$

$$\begin{aligned}Z_6'(t)&=2\sum_{u=1}^{p}N_u(t)\big\{-a_u N_u(t)-b_u M_u(t)\\&\quad+\sum_{v=1}^{q}c_{uv}[f_v(\omega_v(t))-f_v(\gamma_v(t))]+\sum_{v=1}^{q}d_{uv}\left[f_v(\omega_v(t-\zeta))-f_v(\gamma_v(t-\zeta))\right]\big\}\\&=\sum_{u=1}^{p}\big\{-2a_u N_u^2(t)+b_u\left[M_u^2(t)+N_u^2(t)\right]+\sum_{v=1}^{q}|c_{uv}|Q_v\left[N_u^2(t)+K_v^2(t)\right]\\&\quad+\sum_{v=1}^{q}|d_{uv}|Q_v\left[N_u^2(t)+K_v^2(t-\zeta)\right]\big\}\end{aligned}$$

$$\begin{aligned}
&= \sum_{u=1}^{p}\left[b_u - 2a_u + \sum_{v=1}^{q}\big(|c_{uv}| + |d_{uv}|\big)Q_v\right]N_u^2(t) + \sum_{u=1}^{p} b_u M_u^2(t)\\
&\quad + \sum_{v=1}^{q}\left(\sum_{u=1}^{p} c_{uv}\right)Q_v K_v^2(t) + \sum_{v=1}^{q}\left(\sum_{u=1}^{p} d_{uv}\right)Q_v K_v^2(t-\zeta)\\
&\leqslant r_1 Z_6(t) + r_2 Z_5(t) + r_3 Z_7(t) + r_8 Z_7(t-\zeta), \qquad (47)
\end{aligned}$$

$$\begin{aligned}
Z_7'(t) &= 2\sum_{v=1}^{q} K_v(t)\big\{-\eta_2 K_v(t) + H_v(t)\big\} \leqslant \sum_{v=1}^{q}\big[-2\eta_2 K_v^2(t) + K_v^2(t) + H_v^2(t)\big]\\
&\leqslant (1-\eta_2)Z_7(t) + Z_8(t), \qquad (48)
\end{aligned}$$

and

$$\begin{aligned}
Z_8'(t) &= 2\sum_{v=1}^{q} H_v(t)\Bigg\{-l_v H_v(t) - k_v K_v(t) + \sum_{u=1}^{p} e_{vu}\big[g_u(\alpha_u(t)) - g_u\big(\sigma_u(t)\big)\big]\\
&\quad + \sum_{u=1}^{p} r_{vu}\big[g_u\big(\alpha_u(t-\eta)\big) - g_u\big(\sigma(t-\eta)\big)\big]\Bigg\}\\
&\leqslant \sum_{v=1}^{q}\Bigg\{-2l_v H_v^2(t) + k_v\big[H_v^2(t) + K_v^2(t)\big] + \sum_{u=1}^{p}|e_{uv}|L_u\big[H_v^2(t) + M_u^2(t)\big]\\
&\quad + \sum_{u=1}^{p}|r_{vu}|L_u\big[H_v^2(t) + M_u^2(t-\eta)\big]\Bigg\}\\
&= \sum_{v=1}^{q}\left[k_v - 2l_v + \sum_{u=1}^{p}\big(|e_{uv}| + |r_{uv}|\big)\right]H_v^2(t) + \sum_{v=1}^{q} k_v K_v^2(t)\\
&\quad + \sum_{u=1}^{p}\left(\sum_{v=1}^{q}|e_{uv}|L_u\right)M_u^2(t) + \sum_{u=1}^{p}\left(\sum_{v=1}^{q}|r_{vu}|L_u\right)M_u^2(t-\eta)\\
&\leqslant r_4 Z_8(t) + r_5 Z_7(t) + r_6 Z_6(t) + r_7 Z_6(t-\eta). \qquad (49)
\end{aligned}$$

From (46)–(49) it follows that there exist four positive and bounded functions $g_i(t)$ ($i = 1, 2, 3, 4$) such that

$$Z_5'(t) = (1-2\eta_1)Z_5(t) + Z_6(t) - g_1(t), \qquad (50)$$

$$Z_6'(t) = r_1 Z_6(t) + r_2 Z_5(t) + r_3 Z_7(t) + r_8 Z_7(t-\zeta) - g_2(t), \qquad (51)$$

$$Z_7'(t) = (1-\eta_2)Z_7(t) + Z_8(t) - g_3(t), \qquad (52)$$

and

$$Z_8'(t) = r_4 Z_8(t) + r_5 Z_7(t) + r_6 Z_5(t) + r_7 Z_5(t-\eta) - g_4(t). \qquad (53)$$

Let $X(t) = [Z_5(t), Z_6(t), Z_7(t), Z_8(t)]^{\mathrm{T}}$. Then the system of differential equations (50)–(53) can be expressed as

$$X'(t) = CX(t) + g(t), \qquad (54)$$

$$C=\begin{pmatrix}1-2\eta_1 & 1 & 0 & 0\\ 0 & r_1 & r_3 & 0\\ 0 & 0 & 1-2\eta_2 & 1\\ 0 & 0 & 0 & r_4\end{pmatrix},\qquad g(t)=\begin{pmatrix}-g_1(t)\\ r_2Z_5(t)+r_8Z_7(t-\zeta)-g_2(t)\\ -g_3(t)\\ r_5Z_7(t)+r_6Z_5(t)+r_7Z_5(t-\eta)-g_4(t)\end{pmatrix}.$$

The characteristic equation of the matrix C is

$$\begin{vmatrix}1-2\eta_1 & 1 & 0 & 0\\ 0 & r_1 & r_3 & 0\\ 0 & 0 & 1-2\eta_2 & 1\\ 0 & 0 & 0 & r_4\end{vmatrix}=0.$$

By condition (h3), the matrix C has a single characteristic root $\lambda=r_1=r_4=1-2\eta_1=1-2\eta_2$ of multiplicity 3. By Lemma 4, we have

$$\begin{aligned}
&\exp(tC)\\
&\quad=\mathrm{e}^{r_1t}\sum_{i=0}^{2}\frac{t^i}{i!}(C-r_1E)^i=\mathrm{e}^{r_1t}\biggl[E+t(C-r_1E)+\frac{t^2}{2}(C-r_1E)^2\biggr]\\
&\quad=\mathrm{e}^{r_1t}\begin{pmatrix}1+r_9t+0.5r_9^2t^2 & t+0.5r_9t^2 & 0.5r_3t^2 & 0\\ 0 & 1 & r_3t+0.5r_3r_{10}t^2 & 0.5r_3t^2\\ 0 & 0 & 1+r_{10}t+0.5r_{10}^2t^2 & t+0.5r_{11}t^2\\ 0 & 0 & 0 & \begin{matrix}1+(r_4-r_1)t\\ +0.5(r_4-r_1)^2t^2\end{matrix}\end{pmatrix}.
\end{aligned}\tag{55}$$

So, if $X(0)=x=(x_1,x_2,x_3,x_4)$ for the equation $X'(t)=CX(t)$, then, by Lemma 1 and (55), the solution of Eq. (54) is

$$\begin{aligned}
X(t)&=\bigl[Z_5(t),Z_6(t),Z_7(t),Z_8(t)\bigr]^{\mathrm{T}}\\
&=\mathrm{e}^{r_1t}\begin{pmatrix}g_5(t) & g_6(t) & 0.5r_3t^2 & 0\\ 0 & 1 & g_7(t) & 0.5r_3t^2\\ 0 & 0 & g_8(t) & g_{10}(t)\\ 0 & 0 & 0 & g_9(t)\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}\\
&\quad+\int\limits_0^t\mathrm{e}^{r_1(t-s)}\begin{pmatrix}g_5(t-s) & g_6(t-s) & 0.5r_3(t-s)^2 & 0\\ 0 & 1 & g_7(t-s) & 0.5r_3(t-s)^2\\ 0 & 0 & g_8(t-s) & g_{10}(t-s)\\ 0 & 0 & 0 & g_9(t-s)\end{pmatrix}g(s)\,\mathrm{d}s.
\end{aligned}\tag{56}$$

According to (56),

$$\begin{aligned}
Z_5(t)&=\mathrm{e}^{r_1t}\bigl[x_1g_5(t)+x_2g_6(t)+0.5r_3x_3t^2\bigr]\\
&\quad+\int\limits_0^t\mathrm{e}^{r_1(t-s)}\bigl\{-g_1(s)g_5(t-s)+g_6(t-s)\bigl[r_2Z_5(s)+r_8Z_7(s-\zeta)-g_2(s)\bigr]\\
&\qquad-0.5r_3(t-s)^2g_3(s)\bigr\}\,\mathrm{d}s,
\end{aligned}\tag{57}$$

$$Z_6(t) = \mathrm{e}^{r_1 t}\left[x_2 + x_3 g_7(t) + 0.5 r_3 x_4 t^2\right] + \int_0^t \mathrm{e}^{r_1(t-s)}\Big\{r_2 Z_5(s) + r_8 Z_7(s-\zeta) - g_2(s) - g_7(t-s) g_3(s) + 0.5 r_3 (t-s)^2\left[r_5 Z_7(s) + r_6 Z_5(s) + r_7 Z_5(s-\eta)\right]\Big\}\,\mathrm{d}s, \tag{58}$$

$$Z_7(t) = \mathrm{e}^{r_1 t}\left[x_3 g_8(t) + x_4 g_{10}(t)\right] + \int_0^t \mathrm{e}^{r_1(t-s)}\Big\{-g_8(t-s) g_3(s) + g_{10}(t-s)\big[r_5 Z_7(s) + r_6 Z_5(s) + r_7 Z_5(s-\eta)\big]\Big\}\,\mathrm{d}s, \tag{59}$$

and

$$Z_8(t) = x_4 g_9(t)\mathrm{e}^{r_1 t} + \int_0^t \mathrm{e}^{r_1(t-s)} g_9(t-s)\left[r_5 Z_7(s) + r_6 Z_5(s) + r_7 Z_5(s-\eta)\right]\mathrm{d}s. \tag{60}$$

Since the remaining proofs are the same as the corresponding part of Theorem 1, they are omitted. Thus, from (57)–(60) the proof is completed. □

Remark 1. So far, various methods, including integral inequalities [17], Lyapunov functions [1, 8, 10, 11, 15, 19, 21, 25, 30], LMIs [13, 20, 22], AAETM [9], AID [37], inequality techniques [12], Lyapunov stability theory [11, 23, 24, 26, 30, 33], Halanay differential inequalities [36], pinning impulsive control [29], and the maximum-valued approach [27] have been applied to investigate global asymptotic synchronization (GAS) in master-slave neural networks.

Remark 2. In almost papers which discussed the synchronization (control) [1, 3–6, 8–15, 17, 19–27, 29–31, 33, 36, 37], the controllers have been utilized to assure the synchronization for the master-slave NNs. However in our study, without applying the controllers, we also can establish the synchronization criteria for the considered master-slave inertial BAMNNs.

4 Examples

In this section, two specific examples are presented to verify the validity of our results.

Example 1. We consider systems (2) and (3) for $u, v = 1, 2$, where $a_1 = 3$, $a_2 = 4$, $b_1 = 3$, $b_2 = 2$, $l_1 = 4$, $l_2 = 5$, $k_1 = 1$, $k_2 = 2$, $c_{11} = c_{12} = c_{21} = c_{22} = 1$, $d_{11} = d_{22} = 1$, $d_{12} = -1$, $d_{21} = 2$, $e_{11} = e_{21} = 1$, $e_{12} = 2$, $e_{22} = -1$, $r_{11} = r_{22} = 2$, $r_{12} = 1$, $r_{21} = -1$, $I_1 = -1$, $I_2 = 1$, $J_1 = -1$, $J_2 = 1$, $f_1(x) = |x|$, $f_2(x) = 2x - 1$, $g_1(x) = 2|x|$, $g_2(x) = |x - 1|$, $\zeta = 3$, $\eta = 4$. Thus, in Theorem 1, $\xi_1 = 3$, $\xi_5 = 4$, $\eta_1 = 1$, $\eta_2 = 2$. It is easy to confirm that all the conditions in Theorem 1 are satisfied. The drive system (1)

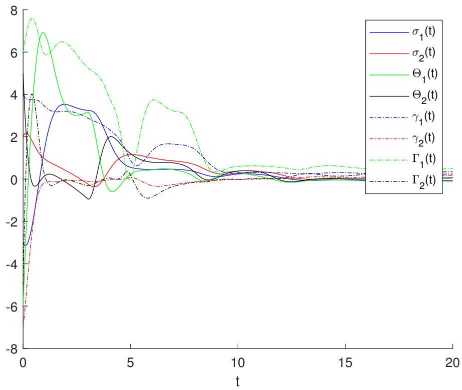

Figure 1. Curves of the master system for Example 1.

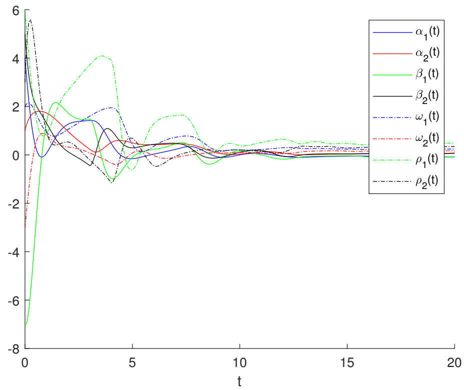

Figure 2. Curves of the slave system for Example 1.

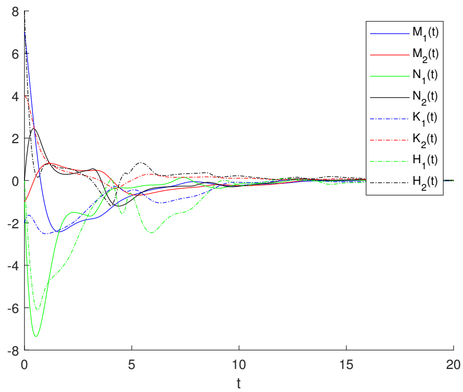

Figure 3. Curves of the error system for Example 1.

and the response system (2) can attain GAS without utilizing controllers. Unlike the methods in [1, 8–15, 17, 19–27, 29–31, 33, 36, 37], the method employed in this paper exhibits significant differences; therefore, those methods cannot be used to verify our results.

The curves of master system with variables $\sigma_u(t)$, $\varTheta_u(t)$, $\gamma_v(t)$, $\varGamma_v(t)$ are shown in the following Fig. 1. The curves of slave system with variables $\alpha_u(t)$, $\beta_u(t)$, $\omega_v(t)$, $\rho_v(t)$ are shown in the following Fig. 2. The error curves with variables $M_u(t)$, $N_u(t)$, $K_v(t)$, $H_v(t)$ are shown in the following Fig. 3.

Example 2. We consider systems (2) and (3) for $u, v = 1, 2$, where $a_1 = 3$, $a_2 = 4$, $b_1 = b_2 = 1$, $k_1 = k_2 = 1$, $l_1 = 5$, $l_2 = 4$, $c_{11} = c_{12} = c_{21} = c_{22} = 1$, $d_{11} = d_{21} = d_{22} = 1$, $d_{12} = -1$, $e_{11} = e_{21} = e_{22} = 1$, $e_{12} = -1$, $r_{11} = r_{21} = r_{22} = 1$, $r_{12} = -1$, $I_1 = -2$, $I_2 = -2$, $J_1 = -2$, $J_2 = 1$, $f_1(x) = \tanh x$, $f_2(x) = \tanh(x-1)$, $g_1(x) = 2|x|$, $g_2(x) = \tanh(x+2)$, $\zeta = 2$, $\eta = 1$. Thus, in Theorem 2, $\eta_1 = \eta_2 = 1$, $r_1 = r_4 = -1$. The conditions of Theorem 2 are obviously satisfied. Thus, the drive system (2) and the response system (3) achieve GAS without controller design. The method adopted in this paper differs significantly

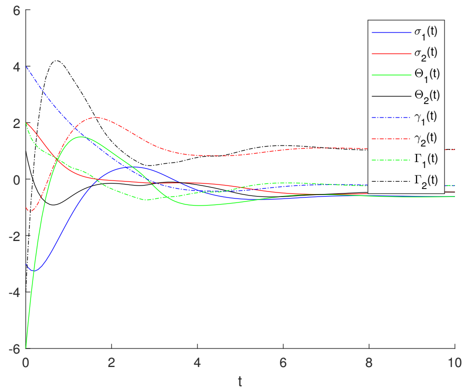

Figure 4. Curves of the master system for Example 2.

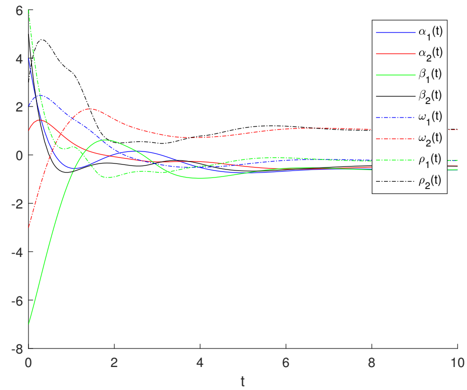

Figure 5. Curves of the slave system for Example 2.

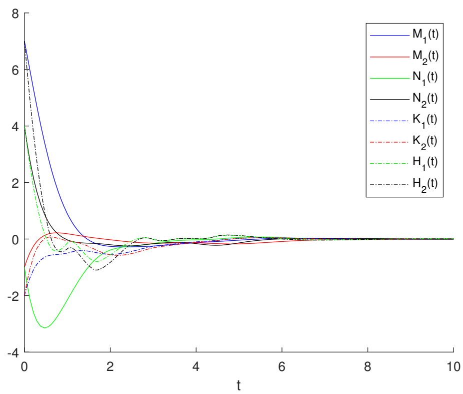

Figure 6. Curves of the error system for Example 2.

from those in [1, 8–15, 17, 19–27, 29–31, 33, 36, 37]; therefore, those methods cannot be used to verify our results.

Figures 4–6 depict the curves of the variables $\sigma_u(t)$, $\varTheta_u(t)$, $\gamma_v(t)$, $\varGamma_v(t)$ (master system), $\alpha_u(t)$, $\beta_u(t)$, $\omega_v(t)$, $\rho_v(t)$ (slave system), and $M_u(t)$, $N_u(t)$, $K_v(t)$, $H_v(t)$ (error system).

5 Conclusion

In this article, the GAS of a class of master–slave BAMNNs is investigated. By employing the fundamental solution matrix method for a first-order system of differential equations, two sufficient criteria are established. To the best of our knowledge, this study is the first to introduce the *fundamental solution matrix method* for analyzing the GAS of such BAMNNs. In future work, we will investigate finite-time synchronization for neural networks using this method and further study the GAS and finite-time synchronization of discrete-time neural networks.

Conflicts of interest. The authors declare no conflicts of interest.

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