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Remetrization techniques for mappings contracting perimeters of triangles*

Marija Cvetković

Faculty of Sciences and Mathematics, University of Niš,
Višegradska 33, 18000 Niš, Serbia
marija.cvetkovic@pmf.edu.rs

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Abstract. There exists a metric in which a mapping contracting perimeters of triangles becomes a graphic contraction, while the completeness of the underlying space is preserved. The reverse question is also considered. We discuss in which cases those two metrics are equivalent and obtain some stability results of the fixed point problem with respect to the original metric and the induced one. The theory is substantiated with numerous examples and a discussion on the advantages regarding the scope of applications.

Keywords: graphic contraction, mapping contracting perimeters of triangles, Ulam–Hyers stability, weak well-posedness.

1 Introduction

The notion of a mapping contracting perimeters of triangles was introduced by Petrov [10]. It presents a class of continuous mappings that possess fixed point or a periodic point of a prime period two on a complete metric space. Banach contraction is an example of a mapping contracting perimeters of triangles, while these two concepts coincide on a set of accumulation points of the underlying metric space. However, there exist mappings contracting perimeters of triangles that have more than one fixed point and hence that are not Banach contractions. This class of mappings was further studied and generalized by several authors; see, for example, [2, 9, 11, 12, 20]. Mappings contracting perimeters of triangles are weakly Picard operators due to the lack of uniqueness of a fixed point and the convergence of a sequence of Picard iterations.

Graphic contraction, as another example of weakly Picard operator, originated in the work of Rheinboldt [19] as a more general concept. The contractive condition of a graphic contraction is restricted to the graph of a mapping f, $\Gamma(f) = \{(x, fx),\, x \in X\}$. In 1972, Hicks and Rhoades [6] obtained the existence result for the graphic contraction by

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loosening the continuity requirement of Subrahmanyam [26]. Rus provided some stability results for an orbitally continuous graphic contraction on a complete metric space in [21].

We collect basic definitions and main results related to these three classes of mappings.

Weakly Picard operator (WPO) is defined in [22].

Definition 1. (See [22].) If (X, d) is a metric space and $f : X \mapsto X$ is a mapping such that, for any $x \in X$, the sequence $(f^n x)$ converges to the fixed point of a mapping f, then f is a weakly Picard operator.

Picard operators are weakly Picard operators.

Definition 2. A mapping $f : X \mapsto X$ is a Picard operator (PO) if, for any $x \in X$, the sequence $(f^n x)$ converges to the unique fixed point of a mapping f.

However, not every weakly Picard operator is a Picard operator.

Example 1. Let $X = [0, 1]$ be equipped by the Euclidean metric $d : X \times X \mapsto [0, \infty)$ defined by

$$d(x, y) = |x - y| \tag{1}$$

for all $x, y \in X$. Evidently, (X, d) is a complete metric space.

Observe the identity mapping $f : X \mapsto X$ defined by $fx = x$ for all $x \in X$, then it is a weakly Picard operator that is not a Picard operator since the mapping has infinitely many fixed points.

A graphic contraction was presented through a more general concept in the work of Rheinboldt [19] and further researched as a concept of graphic contraction and related extensions by multiple authors [5, 6, 13–16, 22, 23, 25].

Definition 3. A mapping $f : X \mapsto X$ is a graphic contraction on a metric space (X, d) if the inequality

$$d\bigl(fx, f^2x\bigr) \leqslant qd(x, fx)$$

holds for some $q \in [0, 1)$ and all $x \in X$.

The following result of Rheinboldt, known as the Basic majorant theorem, acquires the existence of a fixed point for a graphic contraction by taking

$$\varphi\bigl(d(fx, x), d(fx, x_0), d(x, x_0)\bigr) = qd(x, fx)$$

for any $x, x_0 \in X$.

Theorem 1. (*See* [19].) *Let $f : A \subseteq X \mapsto X$ be a mapping on a complete metric space (X, d) such that there exists a nondecreasing function φ defined on a hypercube $I_1 \times I_2 \times I_3$, where I_j is an interval of the form $\{[0, a), [0, a], [0, \infty)\}$ for some $a > 0$, $j = 1, 2, 3$, and a point $x_0 \in A$ such that, on some set $A_0 \subseteq A$,*

$$d\bigl(f^2x, fx\bigr) \leqslant \varphi\bigl(d(fx, x), d(fx, x_0), d(x, x_0)\bigr)$$

whenever x and fx belong to A_0. Assume that, for $t_0 = 0$ and $t_1 = d(fx_0, x)$, the solution of the difference equation

$$t_{n+1} - t_n = \varphi(t_n - t_{n-1}, t_n, t_{n-1}), \quad n \in \mathbb{N},$$

exists. If the sequence (x_n) defined by $x_n = f^n x_0$ for $n \in \mathbb{N}$, is contained in A_0, then (t_n) majorizes (x_n); and if $\lim_{n\to\infty} t_n = t^* < +\infty$, then also $\lim_{n\to\infty} x_n = x^*$ exists, and the error estimate

$$d(x_n, x^*) \leqslant t^* - t_n$$

holds for $n \in \mathbb{N}$. If $x^* \in A$ and f is continuous at x^*, then x^* is a fixed point of the mapping f.

The existence of a fixed point of a graphic contraction on a complete metric space involves the assumption of orbital continuity.

Definition 4. Let $x \in X$, and let $f : X \mapsto X$ be a mapping. An orbit of a mapping $f : X \mapsto X$ at the point $x \in X$ is a set $O_f(x) \equiv O(x) = \{f^n x,\, n \in \mathbb{N}_0\}$.

Orbital continuity [3] presents a restriction of a continuity request on $O(x)$ for any point $x \in X$.

Definition 5. (See [3].) A mapping $f : X \mapsto X$ is orbitally continuous on X if, for any $x \in X$, the existence of a limit $\lim_{n\to\infty} f^{m_n} x$ for some $(m_n) \subseteq \mathbb{N}$ implies that $f(\lim_{n\to\infty} f^{m_n} x) = \lim_{n\to\infty} f(f^{m_n} x)$.

Weakly Picard operators are orbitally continuous, but there exist orbitally continuous mappings that are not weakly Picard operators.

Example 2. Let $X = \mathbb{N}$, and let $d : X \times X \mapsto [0, \infty)$ be described as in (1) of Example 1 for all $x, y \in X$. Let $f : X \mapsto X$ be a mapping defined by $fn = n + 1$ for all $n \in X$. Then it is an orbitally continuous mapping that is not a weakly Picard operator.

If $f : X \mapsto X$ is a weakly Picard operator, let $f^\infty : X \mapsto X$ be defined as

$$f^\infty x = \lim_{n\to\infty} f^n x$$

for all $x \in X$.

For a fixed point x^* of a mapping f, let $X_{x^*} = \{x \in X \mid \lim_{n\to\infty} f^n x = x^*\}$.

The following fixed point theorem is the main result of [6] and presents an improvement of the existence result in [26].

Theorem 2. (*See* [6].) *Let (X, d) be a complete metric space and $q \in [0, 1)$. Suppose there exists $x \in X$ such that*

$$d\big(fy, f^2 y\big) \leqslant q d(y, fy)$$

holds for all $y \in O(x)$. Then:

(i) $\lim_{n\to\infty} f^n x = x^*$ *exists.*
(ii) $d(f^n x, x^*) \leqslant q/(1 - q) d(x, fx)$.

(iii) *x^* is a fixed point of f if and only if the functional $g : X \to \mathbb{R}_+$ given by*

$$gx = d(x, fx)$$

is f-orbitally lower semicontinuous at x^*, i.e., if $(x_n)_{n\in\mathbb{N}}$ is a sequence in $O(x)$ and $\lim_{n\to\infty} x_n = x^*$, then

$$gx^* \leqslant \liminf_{n\to\infty} gx_n.$$

The saturated principle of a graphic contraction is the main result of [25] in which the stability properties of the fixed point problem are discussed for an orbitally continuous graphic contraction.

Theorem 3. (*See* [25].) *Let (X, d) be a complete metric space, and let $f : X \mapsto X$ be a graphic contraction for some $q \in [0, 1)$. Then, for all $x \in X$, $(f^n x)$ converges, and $\sum_{n=0}^{\infty} d(f^n x, f^{n+1} x) < +\infty$. In addition, if f is an orbitally continuous mapping, then*:

(i) *$F_f = F_{f^n} \neq \emptyset$ for all $n \in \mathbb{N}$.*
(ii) *f is a WPO.*
(iii) *$d(x, f^\infty(x)) \leqslant d(x, f(x))/(1-q)$ for all $x \in X$.*
(iv) *The fixed point problem $fx = x$ is weakly well posed in the sense of Reich and Zaslavski.*
(v) *If $q < 1/3$, then $d(f(x), f^\infty(x)) \leqslant d(x, f^\infty(x))/(1-2q)$ for all $x \in X$.*
(vi) *If $q < 1/3$, then the operator f has the Ostrowski property.*

Recall also the definition of a c-weakly Picard operator.

Definition 6. If (X, d) is a metric space and $f : X \mapsto X$ is a mapping such that, for any $x \in X$, the sequence $(f^n x)$ converges to the fixed point of a mapping f and there exists some $c > 0$ such that

$$d\big(x, f^\infty x\big) \leqslant cd(x, fx)$$

holds for all $x \in X$, then f is a c-weakly Picard operator.

Evidently, due to the saturated principle of graphic contraction, any orbitally continuous graphic contraction on a complete metric space is a c-weakly Picard operator for $c = 1/(1-q)$.

Mappings contracting perimeters of triangles [10] present a modification of a Banach contractive condition for three mutually distinct points.

Definition 7. (See [10].) A mapping $f : X \mapsto X$ on a metric space (X, d) is a mapping contracting perimeters of triangles if there exists some $q \in [0, 1)$ such that, for all mutually distinct points $x, y, z \in X$, we have

$$d(fx, fy) + d(fy, fz) + d(fz, fx) \leqslant q\big(d(x, y) + d(y, z) + d(z, x)\big). \qquad (2)$$

The assumption that a mapping has no periodic points of a prime period two, i.e., that $f^2 x = x$ implies $fx = x$ for $x \in X$, is necessary to assert the existence of a fixed point of a mapping contracting perimeters of triangles on a complete metric space.

Theorem 4. (*See* [10].) *Let (X, d) be a complete metric space with at least three points, and let a mapping $f : X \mapsto X$ be a mapping contracting perimeters of triangles that has no periodic points of a prime period two. Then f has a fixed point, and the number of fixed points is at most two.*

From the proof of Theorem 4 given in [10] it may by deduced that a mapping contracting the perimeters of triangles with no periodic points of a prime period two is a weakly Picard operator. However, it does not need to be a Picard operator. The weakly Picard operator is not necessarily a mapping contracting perimeters of triangles; see Example 1.

We present a remetrization under which a mapping contracting perimeters of triangles without periodic points of prime period two transforms into a graphic contraction, while continuity of the mapping and completeness of the underlying space are preserved. Additionally, we will comment on a reverse question of the existence of a metric under which graphic contraction becomes a mapping contracting perimeters of triangles, and we will provide examples when the answer is negative. Stability results for a fixed point problem of a mapping contracting perimeters of triangles without periodic points of a prime period two will be acquired through the presented remetrization approach. In addition, we will present another remetrization approach and compare the benefits.

2 Relation of mappings contracting perimeters of triangles and graphic contractions

The class of mappings contracting perimeters of triangles represents a class of weakly Picard operators as well as graphic contraction. However, there exist mappings contracting perimeters of triangles that are not graphic contractions.

Example 3. Let $X = \{x_n,\ n \in \mathbb{N}_0\}$, where $n \neq m$ implies $x_n \neq x_m$, be equipped with the metric $d : X \times X \mapsto [0, \infty)$ such that

$$d(x_n, x_m) = \begin{cases} 0, & n = m, \\ \frac{1}{2^{[(n-1)/2]}}, & m = n + 1,\ n > 0, \\ \frac{1}{2^{[(m-1)/2]}}, & n = m + 1,\ m > 0, \\ \sum_{i=n}^{m-1} d(x_i, x_{i+1}), & m > n + 1, \\ \sum_{i=m}^{n-1} d(x_i, x_{i+1}), & n > m + 1, \\ 4 - \sum_{i=1}^{n-1} d(x_i, x_{i+1}), & m = 0,\ n > 0, \\ 4 - \sum_{i=1}^{m-1} d(x_i, x_{i+1}), & n = 0,\ m > 0. \end{cases}$$

Observe a mapping $f : X \mapsto X$ such that

$$fx_n = \begin{cases} x_0, & n = 0, \\ x_{n+1}, & n > 0. \end{cases}$$

In [10], it is proved that f is a mapping contracting perimeters of triangles with $q = 7/8$. Nevertheless, it is not a graphic contraction on (X, d); this can be verified by taking $x = x_1$.

It is obvious that there exists a graphic contraction that is not a mapping contracting perimeters of triangles, for example, identity mapping.

Since these two concepts are independent on the observed metric space, we are discussing on the existence of a metric in which a mapping contracting perimeters of triangles is a graphic contraction.

Theorem 5. *Suppose (X, d) is a metric space and $f : X \mapsto X$ is a mapping contracting perimeters of triangles for some $q \in [0, 1)$ without periodic points of a prime period two. In that case, there exists a metric $\rho : X \times X \mapsto [0, \infty)$ such that f is a graphic contraction on (X, ρ) with respect to q.*

Proof. Define a mapping $\rho : X \times X \mapsto [0, \infty)$ by

$$\rho(x, y) = \begin{cases} d(x, fx) + d(y, fy) + d(x, fy) + d(y, fx), & x \neq y, \\ 0, & x = y. \end{cases} \tag{3}$$

Evidently, ρ is a well-defined mapping taking into account the fact that d is a metric on X. Moreover, $x = y$ implies $\rho(x, y) = 0$, while $\rho(x, y) = 0$ for $x \neq y$ leads to the contradiction due to (3) and the observation that $d(x, y) \leqslant \rho(x, y)/2$ holds for all $x, y \in X$. Hence, $\rho(x, y) = 0$ if and only if $x = y$. A mapping ρ is symmetric since d is symmetric. To prove the triangle inequality, let $x, y, z \in X$ be arbitrary. Then

$$\begin{aligned} \rho(x, y) &\leqslant d(x, fx) + d(y, fy) + d(x, fy) + d(y, fx) \\ &\leqslant d(x, fx) + d(fy, y) + d(x, fz) + d(fz, z) \\ &\quad + d(z, fy) + d(y, fz) + d(fz, z) + d(z, fx) \\ &= \rho(x, z) + \rho(z, y), \end{aligned}$$

assuming that $z \notin \{x, y\}$. Evidently, if $z \in \{x, y\}$, the triangle inequality also holds.

Consequently, (X, ρ) is a metric space.

For arbitrary $x \in X$, observe

$$\begin{aligned} \rho(fx, f^2x) &= d\big(fx, f^2x\big) + d\big(f^2x, f^3x\big) + d\big(fx, f^3x\big) \\ &\leqslant q\big(d(x, fx) + d\big(fx, f^2x\big) + d\big(f^2x, x\big)\big) \\ &= q\rho(x, fx), \end{aligned}$$

while the estimations are based on the fact that f is a mapping contracting perimeters of triangles for some $q \in [0, 1)$ and under an additional assumption that x, fx, and f^2x are three mutually distinct points.

If $fx = x$, then $f^2x = fx$ and $\rho(fx, f^2x) = 0$ as well as for case $f^2x = fx$. However, it is impossible to have $f^2x = x$ without $fx = x$ since f has no periodic points of a prime period two.

Therefore f is a graphic contraction on a metric space (X, ρ). $\square$

Example 4. Observe a metric space (X,d) and a mapping $f : X \mapsto X$ defined as in Example 3.

In the light of Theorem 5,

$$\rho(x_n,x_m) = \begin{cases} 2\sum_{i=n}^{m} \frac{1}{2^{[(i-1)/2]}}, & n,m \in \mathbb{N},\ m > n,\\ 2\sum_{i=m}^{n} \frac{1}{2^{[(i-1)/2]}}, & n,m \in \mathbb{N},\ m > n,\\ 2(4 - \sum_{i=1}^{m-1} \frac{1}{2^{[(i-1)/2]}}), & n = 0,\ m \in \mathbb{N},\\ 2(4 - \sum_{i=1}^{n-1} \frac{1}{2^{[(i-1)/2]}}), & m = 0,\ n \in \mathbb{N},\\ 0, & n = m. \end{cases}$$

However,

$$\rho(fx_n, f^2x_n) = \begin{cases} 2(\frac{1}{2^{[n/2]}} + \frac{1}{2^{[(n+1)/2]}}), & n \in \mathbb{N},\\ 0, & n = 0, \end{cases}$$

and

$$\rho(x_n, fx_n) = \begin{cases} 2(\frac{1}{2^{[(n-1)/2]}} + \frac{1}{2^{n/2]}}), & n \in \mathbb{N},\\ 0, & n = 0, \end{cases}$$

so $\rho(fx_n, f^2x_n) \leqslant (3/4)\rho(x_n, fx_n)$ holds for all $n \in \mathbb{N}_0$.

Note that the values of contractive constants do not coincide in Example 4, and there may exist a lower bound for $\sup_{x \neq fx} \rho(fx, f^2x)/\rho(x, fx)$ that is smaller than the contractive constant of a mapping contracting perimeters of triangles.

To obtain Theorem 4 as a consequence of Theorem 3, it is necessary to comment on a continuity of a mapping and completeness of the induced metric space.

As noted in [10], a mapping contracting the perimeters of triangles is continuous.

Theorem 6. *Suppose (X,d) is a metric space and $f : X \mapsto X$ is a mapping contracting perimeters of triangles for some $q \in [0,1)$ without periodic points of a prime period two. In that case, there exists a metric $\rho : X \times X \mapsto [0,\infty)$ defined by* (3) *such that f is a continuous mapping on (X,ρ).*

Proof. Sequential continuity will be discussed. Let $(x_n) \subseteq X$ be a sequence converging to x in (X,ρ), where obviously $fx = x$ must hold since $d(x,fx) \leqslant \rho(x_n,x)$ for $n \in \mathbb{N}$. If the sequence is stationary, meaning that $x_n = x$ starting from some $n \geqslant n_0$, then $\lim_{n\to\infty} fx_n = x$ with respect to the metric ρ since $fx_n = fx = x$ for $n \geqslant n_0$ and hence $\rho(fx_n,x) = 0$ for $n \geqslant n_0$. Similarly, if $fx_n = x$ for infinitely many $n \in \mathbb{N}$, then $\rho(fx_n,x) = 0$ for those indices. Alternatively, if $fx_n = x_n$, then (x_n) has a stationary subsequence, and the conclusion easily follows. Otherwise,

$$\begin{aligned} \rho(fx_n, fx) = \rho(fx_n, x) &= d\bigl(fx_n, f^2x_n\bigr) + d(fx_n, x) + d\bigl(f^2x_n, x\bigr)\\ &\leqslant q\bigl(d(x_n, fx_n) + d(x_n, x) + d(fx_n, x)\bigr) = q\rho(x_n, x), \end{aligned}$$

and by letting $n \to \infty$, we conclude that f is continuous mapping on a metric space (X,ρ). $\square$

Since Banach contraction represents a subclass of mappings contracting perimeters of triangles, we may claim that completeness (or adequate orbital completeness) of a metric space is a necessary condition in order to obtain a fixed point. Completeness is preserved under presented remetrization (3) if (X, d) is a complete metric space.

Theorem 7. *If (X, d) is a complete metric space and $f : X \mapsto X$ is a mapping contracting perimeters of triangles for some $q \in [0, 1)$ without periodic points of a prime period two and a metric $\rho : X \times X \mapsto [0, \infty)$ defined by* (3)*, then (X, ρ) is a complete metric space.*

Proof. Under imposed assumptions, (X, ρ) is a metric space due to Theorem 5. Assume that (x_n) is a Cauchy sequence in (X, ρ). Then

$$\lim_{n\to\infty} d(x_n, fx_n) = \lim_{n,m\to\infty} d(x_n, fx_m) = 0.$$

Further,

$$d(x_n, x_m) \leqslant d(x_n, fx_n) + d(fx_n, x_m) \leqslant \rho(x_n, x_m)$$

holds for all $n, m \in \mathbb{N}$, and (x_n) is also a Cauchy sequence in a metric space (X, d). Hence, it is convergent, implying the existence of some $x \in X$ such that the limit $\lim_{n\to\infty} d(x_n, x) = 0$. Due to the continuity of a mapping f on (X, d), we get that $\lim_{n\to\infty} d(fx_n, fx) = 0$. Since

$$d(fx, x) \leqslant d(fx, fx_n) + d(fx_n, x_n) + d(x_n, x)$$

holds for $n \in \mathbb{N}$, x is a fixed point of a mapping f.

Moreover,

$$\rho(x_n, x) = d(x_n, fx_n) + d(x, fx) + d(x_n, fx) + d(fx_n, x)$$

indicates that x is a limit of the sequence (x_n) in a metric space (X, ρ). Hence, the induced metric space (X, ρ) is also complete. □

Theorems 5–7 jointly imply that the existence of a fixed point of a mapping contracting perimeters of triangles without periodic points of a prime period two on a complete metric space is a direct corollary of the saturated principle of graphic contraction.

Corollary 1. *If $f : X \mapsto X$ is a mapping contracting perimeters of triangles without periodic points of a prime period two on a complete metric space (X, d), then it possesses a fixed point in X.*

Proof. Theorem 5 asserts that (X, ρ) is a metric space, where the metric $\rho : X \times X \mapsto [0, \infty)$ is defined by (3), and that a mapping contracting the perimeters of triangles is a graphic contraction with respect to ρ. By Theorem 7, the observed metric space is complete, while Theorem 6 claims the continuity of such a mapping in (X, ρ). According to Theorem 3, the mapping possesses a fixed point in X. □

Example 5. Let $X = \{a, b, c\}$ be a set equipped with the discrete metric $d : X \times X \mapsto [0, \infty)$ such that

$$d(x, y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y, \end{cases}$$

for $x, y \in X$ and a mapping $f : X \mapsto X$ defined by

$$f : \begin{pmatrix} a & b & c \\ b & a & a \end{pmatrix}.$$

The equalities

$$d(fa.fb) + d(fb, fc) + d(fc, fa) = 2d(a, b) = 2,$$
$$d(a, b) + d(b, c) + d(c, a) = 3$$

imply that (2) is true for $q = 2/3$ and all pairwise distinct $x, y, z \in X$.

Moreover, suppose that $\rho : X \times X \mapsto [0, \infty)$ is defined by (3) and

$$\rho(x, fx) = \begin{cases} 3, & x = c, \\ 2, & x \in \{a, b\}, \end{cases} \qquad \rho\big(fx, f^2x\big) = 2, \quad x \in X,$$

so $\rho(fx, f^2x)$ in general cannot be majorized with $\alpha\rho(x, fx)$ for some $\alpha \in [0, 1)$.

The reason for it lies in the fact that f has periodic points of a prime period two and hence no fixed points since they cannot coexist for a mapping contracting perimeters of triangles.

It is important to discuss the topological and strong equivalence of the presented remetrization and the underlying metric.

Definition 8. Let d_1 and d_2 be metrics on a set X. The metrics d_1 and d_2 are said to be equivalent if they generate the same topology on X.

In the case of equivalence of metrics, the topological spaces (X, τ_{d_1}) and (X, τ_{d_2}), where τ_{d_1} and τ_{d_2} are the topologies induced by the open balls in (X, d_1) and (X, d_2), respectively, are equal, $\tau_{d_1} = \tau_{d_2}$.

Definition 9. Let d_1 and d_2 be metrics on a set X. The metrics d_1 and d_2 are said to be strongly equivalent (bi-Lipschitz equivalent or uniformly equivalent) if there exist constants $a, b > 0$ such that, for all $x, y \in X$,

$$ad_2(x, y) \leqslant d_1(x, y) \leqslant bd_2(x, y).$$

Strong equivalence of metrics implies equivalence, while reverse does not hold.

In the presented case, what can easily be deduced is that

$$2d(x, y) \leqslant \rho(x, y)$$

holds for all $x, y \in X$. However, the reverse inequality does not hold in general for any positive constant since, in (X, ρ), all nonstationary convergent sequences will have a fixed point as a limit, which need not be the case in the metric space (X, d).

Theorem 8. *Let metrics d and ρ on a nonempty set X be strongly equivalent, and let ρ be defined by* (3)*, induced by a mapping $f : X \mapsto X$ that contracts perimeters of triangles and has no periodic points of prime period two. Then f is a Lipschitz mapping on (X, d).*

Proof. Assume that the metrics are strongly equivalent, then there exists some $b > 0$ such that

$$\rho(x, y) \leqslant bd(x, y)$$

holds for all $x, y \in X$. Then

$$\begin{aligned} d(fx, fy) &\leqslant \frac{1}{2}\big(d(fx, x) + d(x, fy) + d(fx, y) + d(y, fy)\big) \\ &= \frac{1}{2}\rho(x, y) \leqslant \frac{b}{2}d(x, y) \end{aligned}$$

holds for all $x, y \in X$. □

Example 6. The metrics d and ρ, defined in Examples 3 and 4, are strongly equivalent for $a = b = 2$. Indeed, f is a nonexpansive mapping on (X, d).

3 Other remetrization approaches

Question of the converse of the Banach fixed point theorem was investigated by several authors, among whom we will reference Mayers [8] and Bessaga [1]. The statements and approaches are slightly different but lead to the same conclusion: for every Picard operator, there exists a complete metric under which it becomes a Banach contraction.

Theorem 9. (*See* [8].) *Let X be a nonempty set, and let $f : X \mapsto X$ be a mapping. Suppose f satisfies the following conditions*:

- (i) *f is a Picard operator on (X, d).*
- (ii) *There exists an open neighborhood U of a fixed point x^* such that, for any neighborhood V of x^*, there exists $n \in \mathbb{N}$ such that $f^n(U) \subseteq V$ ($f^n(U) \to \{x^*\}$).*

Then, for each $q \in (0, 1)$, there exists a complete metric on X for which f is a Banach contraction with a contraction constant q.

Condition (ii) is modified in the following result.

Theorem 10. (*See* [1].) *Let X be a nonempty set, and let $f : X \mapsto X$ be a mapping such that*

$$\mathrm{Fix}(f) = \mathrm{Fix}\big(f^n\big) = \{x^*\}$$

holds for all $n \in \mathbb{N}$. For any $q \in (0, 1)$, there exists a metric d on X such that

- (i) *(X, d) is a complete metric space*;
- (ii) *f is a contraction on (X, d).*

However, proof techniques in both cases rely on the sequence of metrics and are not similar to the approach presented in the previous section. Moreover, they also cannot be applied to mappings contracting perimeters of triangles since, in general, they are not Picard operators.

Rus extended the results of [1] for a class of weakly Picard operators.

Theorem 11. (*See* [22].) *Let X be a nonempty set, and let $f : X \to X$ be a mapping. The following statements are equivalent*:

(i) *There exists a complete metric d on X such that $f : (X, d) \to (X, d)$ is a weakly Picard mapping.*

(ii) $\mathrm{Fix}(f) \neq \emptyset$ *and* $\mathrm{Fix}(f) = \mathrm{Fix}(f^n)$ *for all* $n \in \mathbb{N}$.

(iii) *There exists a partition of X, $X = \bigcup_{i \in I} X_i$, such that:*
   (a) $X_i \in I(f)$ *for each* $i \in I$;
   (b) $f|_{X_i}$ *is a Bessaga mapping for each* $i \in I$.

(iv) *For each $q \in (0, 1)$, there exists a complete metric d on X and a partition of X, $X = \bigcup_{i \in I} X_i$, such that*:
   (a) $d(f^2x, fx) \leqslant qd(x, fx)$ *for all* $x \in X$;
   (b) $f(X_i) \subseteq X_i$ *and* $\mathrm{card}(\mathrm{Fix}(f) \cap X_i) = 1$;
   (c) $f|_{X_i} : (X_i, d) \to (X_i, d)$ *is continuous.*

(v) *For each $q \in (0, 1)$, there exists a complete metric d on X, such that:*
   (a) $d(f^2x, fx) \leqslant qd(x, fx)$ *for all* $x \in X$;
   (b) *the mapping $\varphi : (X, d) \to [0, \infty)$ defined by $\varphi(x) = d(x, fx)$ is f-orbitally lower semicontinuous.*

(vi) *There exists a complete metric d on X such that $f|_{O(x)}$ is a Picard mapping for all $x \in X$.*

Although we can claim that Theorem 4 is a direct corollary of the Banach theorem solely based on Theorem 11, the approach is different, and the proof is based on creating a sequence of Bessaga mappings for which we are already creating a family of metrics. Hence, the presented remetrization in Section 2 is specifically chosen for mappings contracting perimeters of triangles and given in the explicit form. Moreover, the stability results obtained in Section 5 cannot be provided directly in this way.

4 Question of a reverse remetrization approach

It is reasonable to discuss on the existence of a metric, which would potentially transform an arbitrary graphic contraction into a mapping contracting perimeters of triangles. Note that the graphic contraction cannot possess periodic points of a prime period two. However, the answer to this question is obviously negative in general, while some specific cases may be considered.

Theorem 12. *If (X, d) is a metric space and $f : X \mapsto X$ is a graphic contraction for some $q \in [0, 1)$ with more than two fixed points, then there does not exist a metric $\lambda : X \times X \mapsto [0, \infty)$ such that f is a mapping contracting perimeters of triangles on (X, λ).*

Proof. Mappings contracting perimeters of triangles have at most two fixed points, so there cannot exist a described metric. □

It is evident that the mapping defined in Example 1 on a set with at least three points is the example of a graphic contraction for which does not exist a metric such that f is a mapping contracting perimeters of triangles on a newly obtained metric space.

Further discussion on this topic in the case where a graphic contraction has exactly two fixed points will depend on the topological properties.

Theorem 13. *If (X, d) is a metric space and $f : X \mapsto X$ is a graphic contraction for some $q \in [0, 1)$ with two fixed points, then*:

(i) *If any of the fixed point is an accumulation point of the orbit of some point x of the space (X, d) endowed with the metric topology, then there does not exist a metric $\lambda : X \times X \mapsto [0, \infty)$ such that f is a mapping contracting perimeters of triangles on (X, λ).*

(ii) *If both fixed points are isolated points of the space (X, d) endowed with the metric topology, then there exists a metric $\lambda : X \times X \mapsto [0, \infty)$ such that f is a mapping contracting perimeters of triangles on (X, λ).*

Proof. (i) Assume that $\mathrm{Fix}(f) = \{x^*, y^*\}$ and, without loss of generality, that x^* is an accumulation point. Hence, there exists some nonstationary sequence $(x_n) \subseteq O(x)$ for some $x \in X$ such that $\lim_{n\to\infty} x_n = x^*$ in the original metric topology. Additionally, $x_n \neq y^*$ for $n \in \mathbb{N}$.

Assume, to the contrary, that there exists such a metric λ on X. Then

$$\lambda(x^*, y^*) + \lambda(x^*, fx_n) + \lambda(fx_n, y^*) \leqslant q\big(\lambda(x^*, y^*) + \lambda(x^*, x_n) + \lambda(x_n, y^*)\big)$$

as $n \to \infty$. We obtain a contradiction since a mapping contracting perimeters of triangles is continuous and a weakly Picard operator.

(ii) Recall that a graphic contraction is a weakly Picard operator by Theorem 2, so $(f^n x)$ converges to a fixed point of a mapping f for any $x \in X$. If $\mathrm{Fix}(f) = \{x^*, y^*\}$ and both fixed points are isolated points, then, since $(f^n x)$ converges to the point that is not an accumulation point of X, it must be stationary at the limit. So, for any $x \in X$, there exists some $n(x) \in \mathbb{N}$ such that $f^{n(x)}x = x^*$ or $f^{n(x)}x = y^*$. Hence, we can define a partition of the metric space $X = X^* \cup Y^*$ such that

$$X^* = X_{x^*} = \big\{x \in X \bigm| \exists n(x) \in \mathbb{N}\colon\ f^{n(x)}x = x^*\big\},$$
$$Y^* = X_{y^*} = \big\{x \in X \bigm| \exists n(x) \in \mathbb{N}\colon\ f^{n(x)}x = y^*\big\}.$$

Evidently, $X^* \cap Y^* = \emptyset$ since f is a weakly Picard operator, and X^* and Y^* are nonempty sets. Define mappings $d_{X^*} : X^* \times X^* \mapsto [0, \infty)$ and $d_{Y^*} : Y^* \times Y^* \mapsto [0, \infty)$ such that

$$d_{X^*}(x, y) = \begin{cases} d(x, fx) + d(y, fy), & x \neq y, \\ 0, & x = y, \end{cases}$$

for all $x, y \in X^*$ and, similarly,

$$d_{Y^*}(x, y) = \begin{cases} d(x, fx) + d(y, fy), & x \neq y, \\ 0, & x = y, \end{cases}$$

for all $x, y \in Y^*$.

To prove that d_{X^*} is a metric on X^* and, analogously, that d_{Y^*} is a metric on Y^*, note that d_{X^*} is a well-defined mapping and that $d_{X^*}(x, y) = 0$ if and only if $x = y$ since f has a unique fixed point in X^*. It is also symmetric since d is a metric on X, and the triangle inequality holds for all $x, y, z \in X^*$ (analogously for Y^*) due to

$$\begin{aligned} d_{X^*}(x, y) &\leqslant d(x, fx) + d(y, fy) \\ &\leqslant d(x, fx) + d(z, fz) + d(z, fz) + d(y, fy) \\ &= d_{X^*}(x, z) + d_{X^*}(z, y), \end{aligned} \tag{4}$$

assuming that $z \notin \{x, y\}$; otherwise, the inequality is trivial.

Consequently, (X^*, d_{X^*}) and (Y^*, d_{Y^*}) are metric spaces.

If (x_n) is a Cauchy sequence in (X^*, d_{X^*}), then $\lim_{n\to\infty} d(x_n, fx_n) = 0$. Moreover,

$$d_{X^*}(x_n, x^*) = d(x_n, fx_n),$$

and x^* is the limit of a sequence (x_n) in (X^*, d_{X^*}). Hence, (X^*, d_{X^*}) and, analogously, (Y^*, d_{Y^*}) are complete metric spaces.

Define a mapping $\lambda : X \times X \mapsto [0, \infty)$ such that

$$\lambda(x, y) = \begin{cases} d_{X^*}(x, y), & x, y \in X^*, \\ d_{Y^*}(x, y), & x, y \in Y^*, \\ 1 + d(x, fx) + d(y, fy) & \text{otherwise}. \end{cases}$$

Evidently, λ is well defined, symmetric, and since $x^* \neq y^*$ are the only fixed points of a mapping f, also $\lambda(x, y) = 0$ if and only if $x = y$ holds. The last conclusion is derived from the fact that the fixed point problem (5) is weakly well posed and that, by assumption, both fixed points are isolated.

The discussion on the triangle inequality will be divided into three cases.

If x, y, and z belong to the same set of the partition, the triangle inequality holds since d_{X^*} and d_{Y^*} are metrics.

If $x, y \in X^*$ and $z \in Y^*$ (or $x, y \in Y^*$ and $z \in X^*$), then

$$\begin{aligned} \lambda(x, y) &= d_{X^*}(x, y) \\ &\leqslant d(x, fx) + d(y, fy) \\ &\leqslant 1 + d(x, fx) + d(z, fz) + 1 + d(z, fz) + d(y, fy) \\ &= \lambda(x, z) + \lambda(z, y). \end{aligned}$$

If $x, z \in X^*$ and $y \in Y^*$ (or $x \in X^*$ and $y, z \in Y^*$), then

$$\begin{aligned}\lambda(x,y) &= 1 + d(x,fx) + d(y,fy) \\ &\leqslant d(x,fx) + d(z,fz) + 1 + d(z,fz) + d(y,fy) \\ &= \lambda(x,z) + \lambda(z,y).\end{aligned}$$

Accordingly, (X,λ) is a metric space.

If $(x_n) \subseteq X$ is a Cauchy sequence in (X,λ), then, starting from some $n_0 \in \mathbb{N}$, all elements of the sequence must belong to the same set of the partition. Without loss of generality, assume that $(x_n) \subseteq X^*$. The metric space (X, d_{X^*}) is complete, so the sequence (x_n) converges to x^* with respect to the metric d_{X^*} and, consequently, with respect to λ. Hence, (X,λ) is a complete metric space.

Let $x, y, z \in X$ be three pairwise distinct points. If all three belong to the same subset of the partition, note that $f(X^*) \subseteq X^*$ and $f(Y^*) \subseteq Y^*$, so

$$\begin{aligned}&\lambda(fx,fy) + \lambda(fy,fz) + \lambda(fz,fx) \\ &\quad\leqslant 2\big(d\big(fx,f^2x\big) + d\big(fy,f^2y\big) + d\big(fz,f^2z\big)\big) \\ &\quad\leqslant 2q\big(d(x,fx) + d(y,fy) + d(z,fz)\big) \\ &\quad= q\big(\lambda(x,y) + \lambda(y,z) + \lambda(z,x)\big),\end{aligned}$$

and (2) holds.

If $x, z \in X^*$ and $y \in Y^*$ (or $x \in X^*$ and $y, z \in Y^*$), then

$$\begin{aligned}&\lambda(fx,fy) + \lambda(fy,fz) + \lambda(fz,fx) \\ &\quad\leqslant 2\big(1 + d\big(fx,f^2x\big) + d\big(fy,f^2y\big) + d\big(fz,f^2z\big)\big) \\ &\quad\leqslant 2 + 2q\big(d(x,fx) + d(y,fy) + d(z,fz)\big) \\ &\quad\leqslant \frac{1+qa}{1+a}\big(\lambda(x,y) + \lambda(y,z) + \lambda(z,x)\big),\end{aligned}$$

where

$$a = \min\Big\{\inf_{\substack{x,y\in X^* \\ x\neq y}} d(x,fx) + d(y,fy),\ \inf_{\substack{x,y\in Y^* \\ x\neq y}} d(x,fx) + d(y,fy)\Big\}.$$

Since both X^* and Y^* have a unique fixed point of a mapping f and these fixed points are isolated, it follows that $a > 0$. To elaborate that conclusion, assume that, for some sequence $(x_n) \subset X^*$ (analogously, $(x_n) \subset Y^*$), we have $\lim_{n\to\infty} d(x_n, fx_n) = 0$. Let $\varepsilon = \inf_{x\in X^*\setminus\{x^*\}} d(x,x^*)$ and notice that $\varepsilon > 0$ since x^* is an isolated point. Let $n_0 \in \mathbb{N}$ be such that $d(x_n, fx_n) < (1-q)\varepsilon$ for all $n \geqslant n_0$. Then

$$\begin{aligned}d(x^*, x_{n_0}) = d\big(f^{n(x_{n_0})}x_{n_0}, x_{n_0}\big) &\leqslant \sum_{i=0}^{n(x_{n_0})-1} q^i d(x_{n_0}, fx_{n_0}) \\ &< \frac{1}{1-q} d(x_{n_0}, fx_{n_0}) < \varepsilon\end{aligned}$$

leads to the contradiction. Implicitly, inequality (2) holds for all pairwise distinct points for a contractive constant $(1 + qa)/(1 + a) \in [0, 1)$. □

If a graphic contraction has a unique fixed point, we obtain a stronger result.

Theorem 14. *If* (X, d) *is a* (*complete*) *metric space and* $f : X \mapsto X$ *is a graphic contraction for some* $q \in [0, 1)$ *with a unique fixed point, then there exists a metric* $\lambda : X \times X \mapsto [0, \infty)$ *such that* f *is a Banach contraction on a* (*complete*) *metric space* (X, λ).

Proof. Define a mapping $\lambda : X \times X \mapsto [0, \infty)$ such that

$$\lambda(x, y) = \begin{cases} d(x, fx) + d(y, fy), & x \neq y, \\ 0, & x = y. \end{cases}$$

Then (X, λ) is a metric space since λ is a well-defined symmetric function and f has a unique fixed point $x^* \in X$, so $\lambda(x, y) = 0$ is equivalent to $x = y$, while the triangle inequality evidently holds as in (4). Moreover, the metric space (X, λ) is complete since, for a Cauchy sequence (x_n) with respect to λ, we have $\lim_{n\to\infty} d(x_n, fx_n) = 0$ and $\lambda(x_n, x^*) \leqslant d(x_n, fx_n)$, so

$$\lim_{n\to\infty} \lambda(x_n, x^*) = 0.$$

Moreover, for arbitrary distinct $x, y \in X$,

$$\begin{aligned} \lambda(fx, fy) &\leqslant d(fx, f^2x) + d\big(fy, f^2y\big) \leqslant q\big(d(x, fx) + d(y, fy)\big) \\ &= q\lambda(x, y), \end{aligned}$$

and f is a Banach contraction on a (complete) metric space (X, λ) with respect to the contractive constant q. □

Corollary 2. *If* (X, d) *is a* (*complete*) *metric space and* $f : X \mapsto X$ *is a graphic contraction for some* $q \in [0, 1)$ *with a unique fixed point, then there exists a metric* $\lambda : X \times X \mapsto [0, \infty)$ *such that* f *is a mapping contracting perimeters of triangles on a* (*complete*) *metric space* (X, λ).

5 Stability results

The results obtained thanks to presented remetrization may be used in obtaining stability results of the fixed point problem $fx = x$ under the assumption that f is a mapping contracting perimeters of triangles without periodic points of a prime period two on a complete metric space.

We will recall some definitions and well-known results on Ulam–Hyers stability and well-posedness in the sense of Reich and Zaslavski of a fixed point problem $fx = x$ on a complete metric space.

The question of problem of the stability of functional equations was posed by Ulam [27, 28] with the significant progress made by Hyers [7] and Rassias [17]. Nowadays, the question of stability is investigated in fixed point theory for a fixed point problem $fx = x$ as well as for coincidence problems, fixed point inclusions, and related formulations. In 2009, Rus [24] studied the relationship between Picard operators and the stability of fixed point problems.

Definition 10. (See [24].) Let (X, d) be a metric space, and let $f : X \mapsto X$ be an operator. The fixed point equation

$$fx = x \tag{5}$$

is Ulam–Hyers stable if there exists a real number $c_f > 0$ such that, for each $\varepsilon > 0$ and each x satisfying

$$d(x, fx) \leqslant \varepsilon,$$

there exists a fixed point x^* of the mapping f such that

$$d(x, x^*) \leqslant c_f \varepsilon.$$

Remark 1. [24] If f is a c-weakly Picard operator, then the fixed point equation (5) is Ulam–Hyers stable.

Theorem 15. *If (X, d) is a complete metric space and $f : X \mapsto X$ is a mapping contracting perimeters of triangles without periodic points of a prime period two, then the fixed point problem* (5) *is Ulam–Hyers stable.*

Proof. Graphic contraction is, as mentioned, a c-weakly Picard operator, so the fixed point problem (5) is Ulam–Hyers stable on a metric space (X, ρ) because of Theorems 5–7 since f is a continuous graphic contraction on a complete metric space (X, ρ). Hence, whenever $\rho(x, fx) < \varepsilon$ for some $\varepsilon > 0$, then there exists some fixed point x^* of a mapping f such that $\rho(x, x^*) \leqslant c_\rho \varepsilon$ for some $c_\rho > 0$.

Let $d(x, fx) < \varepsilon$ for some $\varepsilon > 0$. If $fx = x$, then $x^* = x$, and the constant can be an arbitrary positive number, while for $f^2x = fx$, the lower bound for c_f may be 1. Otherwise,

$$\begin{aligned}
2d\big(fx, f^2x\big) &\leqslant d\big(fx, f^2x\big) + d\big(fx, f^3x\big) + d\big(f^3x, fx\big) \\
&\leqslant q\big(d(x, fx) + d\big(fx, f^2x\big) + d\big(f^2x, x\big)\big) \\
&\leqslant 2q\big(d(x, fx) + d\big(fx, f^2x\big)\big),
\end{aligned}$$

so

$$d\big(fx, f^2x\big) \leqslant \frac{q}{1-q} d(x, fx) < \frac{q}{1-q}\varepsilon$$

and

$$\begin{aligned}
\rho(x, fx) &= d(x, fx) + d\big(fx, f^2x\big) + d\big(f^2x, x\big) \\
&\leqslant 2\big(d(x, fx) + d\big(fx, f^2x\big)\big) < \frac{2}{1-q}\varepsilon.
\end{aligned}$$

Due to remarks concerning Ulam–Hyers stability of (5) on (X, ρ), we obtain

$$\rho(x, x^*) \leqslant \frac{2c_\rho}{1-q}\varepsilon.$$

Since $d(x, x^*) \leqslant \rho(x, x^*)/2$, the fixed point problem (5) is Ulam–Hyers stable with respect to the metric d for a constant $c = c_\rho/(1-q)$. $\square$

The well-posedness acquires a unique fixed point, so it is impossible to discuss on well-posedness of a fixed point problem (5) for a mapping contracting perimeters of triangles. However, we can prove that the fixed point problem for a mapping contracting perimeters of triangles on a complete metric space is weakly well posed, which was established as a stability property by De Blassi and Myjak [4]. Also, the fixed point problem is weakly well posed in the sense of Reich and Zaslavski [18].

Definition 11. (See [4].) The fixed point problem (5) is weakly well posed if the set of fixed points of f is nonempty and compact and if every sequence $(x_n) \subseteq X$ such that $\lim_{n\to\infty} d(x_n, fx_n) = 0$ is relatively compact.

Another way to discuss on well-posedness of a fixed point problem is to use the approach of Rus [25] since it is the approach of Reich and Zaslavski modified for the weakly Picard operators. In the original work of Reich and Zaslavski [18], it is requested that the mapping has a unique fixed point, but in several references this approach is also called a well-posedness in the sense of Reich and Zaslavski, so we will use a term of weak well-posedness in the sense of Reich and Zaslavski for the case that there exist multiple fixed points.

Definition 12. (See [18].) The fixed point problem (5) is well posed in the sense of Reich and Zaslavski if there exists a unique fixed point $x^* \in X$ of the mapping f and the sequence (x_n) converges to x^* whenever $\lim_{n\to\infty} d(x_n, fx_n) = 0$.

Rus modification is the observation of well-posedness in the sense of Reich and Zaslavski on the set X_{x^*} associated with any fixed point x^* of a mapping f.

Definition 13. (See [25].) The fixed point problem (5) is weakly well posed in the sense of Reich and Zaslavski if, for every fixed point $x^* \in X$ of the mapping f and any sequence $(x_n) \subseteq X$ such that Picard sequence $(f^m x_n)$ converges to x^* ($(x_n) \subseteq X_{x^*}$), the sequence (x_n) converges to x^* whenever $\lim_{n\to\infty} d(x_n, fx_n) = 0$.

Theorem 16. *If (X, d) is a complete metric space and $f : X \mapsto X$ is a mapping contracting perimeters of triangles without periodic points of a prime period two, then the fixed point problem* (5) *is weakly well posed in the sense of Reich and Zaslavski on* (X, d).

Proof. Assume that $\lim_{n\to\infty} d(x_n, fx_n) = 0$ for some sequence $(x_n) \subseteq X$ such that $\lim_{m\to\infty} f^m x_n = x^*$ with respect to the metric d for all $n \in \mathbb{N}$ and some fixed point x^* of a mapping f. Since a mapping contracting perimeters of triangles is continuous, it also implies that $\lim_{n\to\infty} d(fx_n, f^2x_n) = 0$. Thus,

$$\begin{aligned}\rho(x_n, fx_n) &= d(x_n, fx_n) + d\bigl(fx_n, f^2x_n\bigr) + d\bigl(x_n, f^2x_n\bigr)\\ &\leqslant 2\bigl(d(x_n, fx_n) + d\bigl(fx_n, f^2x_n\bigr)\bigr)\end{aligned}$$

tends to zero as $n \to \infty$. Knowing, as stated in Theorem 3(iv), that the fixed point problem $fx = x$ is weakly well posed in the sense of Reich and Zaslavski assuming that f is an orbitally continuous graphic contraction on a complete metric space, Theorems 5–7 imply that $fx = x$ is weakly well posed in the sense of Reich and Zaslavski in (X, ρ). Accordingly, $x^* \in X$ must be the limit of a sequence (x_n) with respect to the metric ρ. Moreover,

$$d(x_n, x^*) \leqslant \rho(x_n, x^*),$$

and the fixed point problem (5) is weakly well posed in the sense of Reich and Zaslavski on a metric space (X, d). □

Note that in the original statement of Theorem 3 in [25] it is written that the fixed point problem is well posed. However, from the previous theorem it is evident that it is in the sense of Definition 13. To avoid using a double defined term due to the results of [4], we modified that part by using a notion of weak well-posedness in the sense of Reich and Zaslavski.

Corollary 3. *If* (X, d) *is a complete metric space and* $f : X \mapsto X$ *is a mapping contracting perimeters of triangles without periodic points of a prime period two, then the fixed point problem* (5) *is well posed in the sense of Reich and Zaslavski on* X_{x^*} *for any fixed point* $x^* \in X$ *of a mapping* f.

Weak well-posedness can be obtained in a similar manner.

Theorem 17. *If* (X, d) *is a complete metric space and* $f : X \mapsto X$ *is a mapping contracting perimeters of triangles without periodic points of a prime period two, then the fixed point problem* (5) *is weakly well posed on* (X, d).

Proof. As previously mentioned, the set of fixed points of mappings contracting perimeters has at most two points, so it is compact as a finite set.

Moreover, note that, in the proof of Theorem 16, the assumption that Picard sequences of all initial points x_n converge to the fixed point was not used until the conclusion that point must coincide with the limit of the sequence (x_n) with respect to ρ. However, not knowing this fact about Picard sequences does not affect the rest of the proof since $\lim_{n\to\infty} \rho(x_n, x^*) = 0$ for some fixed point x^* of the mapping f implies that $\lim_{n\to\infty} d(x_n, x^*) = 0$ and the set $\{x_n,\ n \in \mathbb{N}\}$ is relatively compact. □

6 Conclusion

A mapping contracting perimeters of triangles and graphic contractions are independent classes with nonempty intersection. However, it is possible to define a complete metric that will preserve continuity of a mapping contracting perimeters of triangles without periodic points of prime period two and in which a mapping will fulfill a graphic contraction contractive condition. Hence, all fixed point existence results can be derived from the theorems claiming the existence of a fixed point of a graphic contraction stated in [6, 21, 26]. However, these two metrics are not necessarily strongly equivalent, hence, stability results cannot be transferred directly. Nevertheless, the Ulam–Hyers stability and weak well-posedness, along with well-posedness in the sense of Reich and Zaslavski on the sets of partitions or weak well-posedness, follow directly from the saturated principle of graphic contraction of [25]. It is the author's belief that the same approach may be applied to other modifications of the class of mappings contracting the perimeters of triangles, including those studied in [2, 9, 11, 12], among many others.

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References

1. C. Bessaga, On the converse of the Banach fixed point principle, *Colloq. Math.*, **7**:41–43, 1959, `https://doi.org/10.4064/cm-7-1-41-43`.
2. R.K. Bisht, E. Petrov, Three point analogue of ćirić-reich-rus type mappings with non-unique fixed points, *J. Anal.*, **32**:2609–2627, 2024, `https://doi.org/10.1007/s41478-024-00743-2`.
3. Lj.B. Ćirić, On contraction type mappings, *Math. Balk.*, **1**:52–57, 1971.
4. F.S. De Blasi, J. Myjak, Sur la porosité des contractions sans point fixe, *C. R. Acad. Sci. Paris, Sér. I*, **308**(2):51–54, 1989.
5. A.-D. Filip, Metric conditions, graphic contractions and weakly Picard operators, *Stud. Univ. Babeş-Bolyai, Math.*, **70**(1):161–174, 2025, `https://doi.org/10.24193/subbmath.2025.1.11`.
6. T.L. Hicks, B.E. Rhoades, Banach type fixed point theorem, *Math. Japon.*, **24**(3):327–330, 1979.
7. D.H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. USA*, **27**(4): 222–224, 1941, `https://doi.org/10.1073/pnas.27.4.222`.
8. P.R. Meyers, A converse to Banach's contraction theorem, *J. Res. Natl. Bur. Stand., Sect. B*, **71**(2–3):73–76, 1967, `https://doi.org/10.6028/jres.071B.014`.
9. C.M. Pacurar, O. Popescu, Fixed point theorem for generalized Chatterjea type mappings, *Acta Math. Hung.*, **173**(2):500–509, 2024, `https://doi.org/10.1007/s10474-024-01455-6`.
10. E. Petrov, Fixed point theorem for mappings contracting perimeters of triangles, *J. Fixed Point Theory Appl.*, **25**(3):74, 2023, `https://doi.org/10.1007/s11784-023-01078-4`.
11. E. Petrov, R.K. Bisht, Fixed point theorem for generalized Kannan type mappings, *Rend. Circ. Mat. Palermo (2)*, **73**(8):2895–2912, 2024, `https://doi.org/10.1007/s12215-024-01079-3`.

12. E. Petrov, R. Salimov, Fixed point theorem for mappings contracting perimeters of triangles and its generalizations, in *Algebraic and Geometric Methods of Analysis, Odesa, Ukraine, May 29 – June 1, 2023*, year, Volume 25, 2023, `https://imath.kiev.ua/~topology/conf/agma2023/contents/agma2023-theses.pdf`.
13. A. Petruşel, G. Petruşel, Graphical contractions and common fixed points in b-metric spaces, *Arab. J. Math.*, **12**:423–430, 2023, `https://doi.org/10.1007/s40065-022-00396-8`.
14. A. Petruşel, G. Petruşel, J.-C. Yao, Fixed points and coupled fixed points for graphic contractions in complete b-metric spaces, *J. Nonlinear Convex Anal.*, **19**(11):1869–1882, 2018, `https://doi.org/10.24193/fpt-ro.2018.2.39`.
15. A. Petruşel, I.A. Rus, Graphic contraction principle and applications, in T.M. Rassias, P.M. Pardalos (Eds.), *Mathematical Analysis and Applications*, Springer Optim. Appl., Springer, Cham, 2019, pp. 411–432, `https://doi.org/10.1007/978-3-030-31339-5_15`.
16. A. Petruşel, I.A. Rus, M.-A. Şerban, Nonexpansive operators as graphic contractions, *J. Nonlinear Convex Anal.*, **17**(7):1409–1415, 2016, `http://www.yokohamapublishers.jp/online2/opjnca/vol17/p1409.html`.
17. Th.M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Am. Math. Soc.*, **72**(2):297–300, 1978, `https://doi.org/10.2307/2042795`.
18. S. Reich, A.J. Zaslavski, Well-posedness of fixed point problems, *Far East J. Math. Sci.*, **Spec. Vol., Pt. III**:393–401, 2001.
19. W.C. Rheinboldt, A unified convergence theory for a class of iterative processes, *SIAM J. Numer. Anal.*, **5**:42–63, 1968, `https://doi.org/10.1137/0705003`.
20. K. Roy, Mappings contracting axes of ellipse, *J. Anal.*, **32**:3557–3563, 2024, `https://doi.org/10.1007/s41478-024-00819-z`.
21. I.A. Rus, On the method of successive approximations, *Rev. Roum. Math. Pures Appl.*, **17**:1433–1437, 1972.
22. I.A. Rus, Weakly Picard mappings, *Commentat. Math. Univ. Carol.*, **34**(4):769–773, 1993, `https://eudml.org/doc/21986`.
23. I.A. Rus, Weakly Picard operators and applications, *Semin. Fixed Point Theory Cluj-Napoca*, **2**:41–57, 2001, `https://scispace.com/pdf/weakly-picard-operators-and-applications-55cmmwgbrh.pdf`.
24. I.A. Rus, Remarks on Ulam stability of the operatorial equations, *Fixed Point Theory*, **10**(2):305–320, 2009, `https://www.math.ubbcluj.ro/~nodeacj/download.php?f=092rus-final.pdf`.
25. I.A. Rus, Relevant classes of weakly Picard operators, *An. Univ. Vest Timiş., Ser. Mat.-Inform.*, **54**(2):131–147, 2016, `https://doi.org/10.1515/awutm-2016-0019`.
26. P.V. Subrahmanyam, Remarks on some fixed point theorems related to Banach's contraction principle, *J. Math. Phys. Sci.*, **8**:445–457, 1974.
27. S.M. Ulam, *A Collection of Mathematical Problems*, Intersci. Tracts Pure Appl. Math., No. 8, Interscience, New York, London, 1960.
28. S.M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1960.