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Positive solutions to a system of ψ-Riemann–Liouville fractional differential equations with coupled boundary conditions

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Abstract. We investigate the existence of positive solutions for a system of ψ-Riemann–Liouville fractional differential equations with positive parameters and sign-changing nonlinearities subject to nonlocal coupled boundary conditions containing Riemann–Stieltjes integrals and fractional derivatives of various orders. To prove our main result, we use the nonlinear alternative of Leray–Schauder type.

Keywords: ψ-Riemann–Liouville fractional differential equations, coupled boundary conditions, sign-changing functions, positive solutions.

1 Introduction

In this paper, we consider the system of ψ-Riemann–Liouville fractional differential equations

$$\begin{aligned} &D_{a+}^{\alpha,\psi}u(t) + \lambda f\bigl(t,u(t),v(t)\bigr) = 0, \quad t \in (a,b),\\ &D_{a+}^{\beta,\psi}v(t) + \mu g\bigl(t,u(t),v(t)\bigr) = 0, \quad t \in (a,b), \end{aligned} \tag{1}$$

supplemented with the coupled boundary conditions

$$\begin{aligned} &u^{(i)}(a) = 0, \quad i = 0,1,\dots,n-2,\\ &D_{a+}^{\varsigma,\psi}u(b) = \sum_{i=1}^{c}\int_a^b D_{a+}^{\varrho_i,\psi}u(s)\,\mathrm{d}\mathcal{H}_i(s) + \sum_{i=1}^{d}\int_a^b D_{a+}^{\sigma_i,\psi}v(s)\,\mathrm{d}\mathcal{K}_i(s), \end{aligned} \tag{2_1}$$

$$
\begin{aligned}
&v^{(i)}(a) = 0, \quad i = 0, 1, \dots, m-2, \\
&D_{a+}^{\vartheta,\psi} v(b) = \sum_{i=1}^{p} \int_a^b D_{a+}^{\eta_i,\psi} u(s)\,\mathrm{d}\mathcal{P}_i(s) + \sum_{i=1}^{q} \int_a^b D_{a+}^{\theta_i,\psi} v(s)\,\mathrm{d}\mathcal{Q}_i(s).
\end{aligned} \tag{2_2}
$$

Here $a, b \in \mathbb{R}$, $0 \leqslant a < b$, $\alpha \in (n-1, n)$, $\beta \in (m-1, m)$, $n, m \in N$, $n, m \geqslant 3$, $\psi \in C^{\rho}[a,b]$, $\rho = \max\{n, m\}$ with $\psi'(t) > 0$ for all $t \in [a,b]$, $c, d, p, q \in N$, $1 \leqslant \varsigma < \alpha - 1$, $0 \leqslant \varrho_i \leqslant \varsigma$, $i = 1 \dots, c$, $0 \leqslant \eta_j \leqslant \varsigma$, $j = 1, \dots, p$, $1 \leqslant \vartheta < \beta - 1$, $0 \leqslant \sigma_k \leqslant \vartheta$, $k = 1, \dots, d$, $0 \leqslant \theta_\iota \leqslant \vartheta$, $\iota = 1, \dots, q$, λ and μ are positive parameters. $D_{a+}^{\kappa,\psi} u$ and $D_{a+}^{\widetilde{\kappa},\psi} v$ denote the ψ-Riemann–Liouville fractional derivatives of functions u and v, respectively, of order κ and $\widetilde{\kappa}$ for $\kappa \in \{\alpha, \varsigma, \varrho_i, \eta_j,\ i = 1, \dots, c,\ j = 1, \dots, p\}$, $\widetilde{\kappa} \in \{\beta, \vartheta, \sigma_k, \theta_\iota,\ k = 1, \dots, d,\ \iota = 1, \dots, q\}$. The integrals from the boundary conditions (2) are Riemann–Stieltjes integrals with $\mathcal{H}_i$, $\mathcal{P}_j$, $\mathcal{K}_k$, $\mathcal{Q}_\iota$, $i = 1, \dots, c$, $j = 1, \dots, p$, $k = 1, \dots, d$, $\iota = 1, \dots, q$, bounded variation functions, and the functions f and g are sign-changing continuous functions.

In this paper, under some assumptions on the nonlinearities f and g, we present intervals for the parameters λ and μ such that problem (1)–(2) has at least one positive solution. So we have here a system of semipositone boundary value problems. To prove our main result, we use the nonlinear alternative of Leray–Schauder type. The ψ-Riemann–Liouville fractional derivative generalizes the Riemann–Liouville derivative (for $\psi(t) = t$) and the Hadamard derivative (for $\psi(t) = \ln t$). Next, we present several papers that study various fractional differential equations and systems connecting to our problem. In [21], the authors investigated the ψ-Riemann–Liouville fractional differential equation

$$
D_{a+}^{\alpha,\psi} u(t) + \lambda f\bigl(t, u(t)\bigr) = 0, \quad t \in (a, b), \tag{3}
$$

with the nonlocal boundary conditions

$$
\begin{aligned}
&u^{(i)}(a) = 0, \quad i = 0, 1, \dots, n-2, \\
&D_{a+}^{\varsigma,\psi} u(b) = \sum_{i=1}^{m} \int_a^b D_{a+}^{\varrho_i,\psi} u(s)\,\mathrm{d}H_i(s).
\end{aligned} \tag{4}
$$

Here $0 \leqslant a < b$, $\alpha > 0$, $n \in N$, $n \geqslant 3$, $\alpha \in (n-1, n]$, $\psi \in C^n[a,b]$ with $\psi'(t) > 0$ for all $t \in [a,b]$, $m \in N$, λ is a positive parameter, $1 \leqslant \varsigma < \alpha - 1$, $0 \leqslant \varrho_i \leqslant \varsigma$, $i = 1, \dots, m$, the function f may change sign and may be singular at the points $t = a$ and/or $t = b$, and $H_i : [a,b] \to \mathbb{R}$, $i = 1, \dots, m$, are bounded variation functions. Based on the Guo–Krasnosel'skii fixed point theorem, they proved the existence of positive solutions to problem (3)–(4) for various intervals of λ. In [14], the authors explored the system of nonlinear fractional differential equations

$$
\begin{aligned}
&D_{0+}^{\alpha} u(t) + \lambda f\bigl(t, u(t), v(t)\bigr) = 0, \quad t \in (0, 1), \\
&D_{0+}^{\beta} v(t) + \mu g\bigl(t, u(t), v(t)\bigr) = 0, \quad t \in (0, 1),
\end{aligned} \tag{5}
$$

subject to the coupled integral boundary conditions

$$
\begin{aligned}
&u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0+}^{p} u(1) = \int_0^1 v(s)\,\mathrm{d}H(s), \\
&v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \qquad D_{0+}^{q} v(1) = \int_0^1 u(s)\,\mathrm{d}K(s).
\end{aligned} \tag{6}
$$

Here $n, m \in N$, $n, m \geqslant 3$, $\alpha, \beta \in \mathbb{R}$, $\alpha \in (n-1, n]$, $\beta \in (m-1, m]$, $p, q \in N$, $p \in [1, n-2]$, $q \in [1, m-2]$, λ and μ are positive parameters, D_{0+}^{κ} denotes the Riemann–Liouville fractional derivative of order κ for $\kappa \in \{\alpha, \beta, p, q\}$, the functions f and g are sign-changing continuous functions, which may be singular at $t = 0$ and/or $t = 1$, and H and K are bounded variation functions. They presented intervals for the parameters λ and μ such that problem (5)–(6) has at least one positive solution. We also draw attention to the works [1, 4, 5, 11, 17, 18, 22–26, 28] in which the authors examined the existence of positive solutions for a variety of semipositone or singular Riemann–Liouville fractional differential equations subject to different types of boundary conditions. Other systems of Riemann–Liouville fractional differential equations subject to various nonlocal boundary conditions have been studied in [6, 8, 9, 12, 16]. For comprehensive treatments of Riemann–Liouville, Caputo, and Hadamard fractional differential equations and systems with various applications and related nonlinear dynamics, the reader is referred to the monographs [2, 3, 13, 29] and the papers [7, 10, 20, 27]. For definitions and various properties of the ψ-Riemann–Liouville fractional integrals and derivatives, we refer the reader to the monographs [15, 19] and the paper [21].

The novelty of the present paper lies in the use of ψ-Riemann–Liouville fractional derivatives both in the equations of system (1) and in the associated boundary conditions (2). Moreover, the nonlinearities f and g in (1) are not assumed to be nonnegative; instead, they may change sign. Finally, the boundary conditions (2), formulated in terms of Riemann–Stieltjes integrals, represent a unifying framework that encompasses multipoint boundary conditions, classical integral conditions, as well as combinations thereof.

The remainder of the paper is structured as follows. In Section 2, we analyze the linear version of problem (1)–(2), including the construction of the corresponding Green functions and a discussion of their properties. Section 3 is dedicated to establishing the main existence result for positive solutions of problem (1)–(2). In Section 4, an example is provided to highlight the applicability of the theoretical results. Finally, Section 5 contains the concluding remarks.

2 Auxiliary results

In this section, we study the linear problem associated with our problem (1)–(2), including its solution, the corresponding Green functions, and their properties.

We consider the system of fractional differential equations

$$\begin{aligned} &D_{a+}^{\alpha,\psi}u(t)+h(t)=0, \quad t\in(a,b),\\ &D_{a+}^{\beta,\psi}v(t)+k(t)=0, \quad t\in(a,b), \end{aligned} \tag{7}$$

subject to the boundary conditions (2), under the assumptions presented in Section 1 after (2), with $h,k\in C(a,b)\cap L^1(a,b)$.

We mention that the conditions $u^{(i)}(a)=0$, $i=0,1,\dots,n-2$, are equivalent to the conditions $\mathcal{D}^i u(a)=0$, $i=0,1,\dots,n-2$, where $\mathcal{D}=(1/\psi'(t))\,\mathrm{d}/\mathrm{d}t$. In the same way, $v^{(i)}(a)=0$, $i=0,1,\dots,m-2$, are equivalent to the conditions $\mathcal{D}^i v(a)=0$, $i=0,1,\dots,m-2$.

We denote by

$$\begin{aligned} \Lambda_1 &= \frac{\Gamma(\alpha)}{\Gamma(\alpha-\varsigma)}\big(\psi(b)-\psi(a)\big)^{\alpha-\varsigma-1}\\ &\quad -\sum_{i=1}^{c}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\varrho_i)}\int_a^b\big(\psi(s)-\psi(a)\big)^{\alpha-\varrho_i-1}\,\mathrm{d}\mathcal{H}_i(s),\\ \Lambda_2 &= \sum_{i=1}^{d}\frac{\Gamma(\beta)}{\Gamma(\beta-\sigma_i)}\int_a^b\big(\psi(s)-\psi(a)\big)^{\beta-\sigma_i-1}\,\mathrm{d}\mathcal{K}_i(s),\\ \Lambda_3 &= \sum_{i=1}^{p}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\eta_i)}\int_a^b\big(\psi(s)-\psi(a)\big)^{\alpha-\eta_i-1}\,\mathrm{d}\mathcal{P}_i(s),\\ \Lambda_4 &= \frac{\Gamma(\beta)}{\Gamma(\beta-\vartheta)}\big(\psi(b)-\psi(a)\big)^{\beta-\vartheta-1}\\ &\quad -\sum_{i=1}^{q}\frac{\Gamma(\beta)}{\Gamma(\beta-\theta_i)}\int_a^b\big(\psi(s)-\psi(a)\big)^{\beta-\theta_i-1}\,\mathrm{d}\mathcal{Q}_i(s),\\ \Delta &= \Lambda_1\Lambda_4-\Lambda_2\Lambda_3. \end{aligned} \tag{8}$$

Lemma 1. *If $\Delta\neq 0$ (that is, $\Lambda_1\Lambda_4\neq\Lambda_2\Lambda_3$), then the solution of problem* (7), (2) *is given by*

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(s)\big(\psi(t)-\psi(s)\big)^{\alpha-1}h(s)\,\mathrm{d}s+\frac{\Lambda_4U+\Lambda_2V}{\Delta}\big(\psi(t)-\psi(a)\big)^{\alpha-1},\\ v(t) &= -\frac{1}{\Gamma(\beta)}\int_a^t\psi'(s)\big(\psi(t)-\psi(s)\big)^{\beta-1}k(s)\,\mathrm{d}s+\frac{\Lambda_3U+\Lambda_1V}{\Delta}\big(\psi(t)-\psi(a)\big)^{\beta-1} \end{aligned} \tag{9}$$

for all $t \in [a,b]$, *where*

$$\begin{aligned}
U &= \frac{1}{\Gamma(\alpha-\varsigma)} \int_a^b \psi'(s)\big(\psi(b)-\psi(s)\big)^{\alpha-\varsigma-1} h(s)\,\mathrm{d}s \\
&\quad - \sum_{i=1}^{c} \frac{1}{\Gamma(\alpha-\varrho_i)} \int_a^b \left(\int_a^s \psi'(\tau)\big(\psi(s)-\psi(\tau)\big)^{\alpha-\varrho_i-1} h(\tau)\,\mathrm{d}\tau \right) \mathrm{d}\mathcal{H}_i(s) \\
&\quad - \sum_{i=1}^{d} \frac{1}{\Gamma(\beta-\sigma_i)} \int_a^b \left(\int_a^s \psi'(\tau)\big(\psi(s)-\psi(\tau)\big)^{\beta-\sigma_i-1} k(\tau)\,\mathrm{d}\tau \right) \mathrm{d}\mathcal{K}_i(s), \\
V &= \frac{1}{\Gamma(\beta-\vartheta)} \int_a^b \psi'(s)\big(\psi(b)-\psi(s)\big)^{\beta-\vartheta-1} k(s)\,\mathrm{d}s \\
&\quad - \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha-\eta_i)} \int_a^b \left(\int_a^s \psi'(\tau)\big(\psi(s)-\psi(\tau)\big)^{\alpha-\eta_i-1} h(\tau)\,\mathrm{d}\tau \right) \mathrm{d}\mathcal{P}_i(s) \\
&\quad - \sum_{i=1}^{q} \frac{1}{\Gamma(\beta-\theta_i)} \int_a^b \left(\int_a^s \psi'(\tau)\big(\psi(s)-\psi(\tau)\big)^{\beta-\theta_i-1} k(\tau)\,\mathrm{d}\tau \right) \mathrm{d}\mathcal{Q}_i(s).
\end{aligned} \tag{10}$$

Proof. By Lemma 5 from [21], the solution of system (7) can be written as

$$\begin{aligned}
u(t) &= -I_{a+}^{\alpha,\psi} h(t) + \sum_{j=1}^{n} a_j \big(\psi(t)-\psi(a)\big)^{\alpha-j} \\
&= -\frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)\big(\psi(t)-\psi(s)\big)^{\alpha-1} h(s)\,\mathrm{d}s + \sum_{j=1}^{n} a_j \big(\psi(t)-\psi(a)\big)^{\alpha-j}, \\
v(t) &= -I_{a+}^{\beta,\psi} k(t) + \sum_{j=1}^{m} b_j \big(\psi(t)-\psi(a)\big)^{\beta-j} \\
&= -\frac{1}{\Gamma(\beta)} \int_a^t \psi'(s)\big(\psi(t)-\psi(s)\big)^{\beta-1} k(s)\,\mathrm{d}s + \sum_{j=1}^{m} b_j \big(\psi(t)-\psi(a)\big)^{\beta-j}
\end{aligned} \tag{11}$$

for all $t \in [a,b]$, where $a_i, b_j \in \mathbb{R}$, $i = 1,\dots,n$, $j = 1,\dots,m$.

Using (11) in the conditions $u(a) = u'(a) = \cdots = u^{(n-2)}(a) = 0$ and $v(a) = v'(a) = \cdots = v^{(m-2)}(a) = 0$, we find that $a_n = a_{n-1} = \cdots = a_2 = 0$ and $b_m = b_{m-1} = \cdots = b_2 = 0$. We will show next that the conditions $u(a) = u'(a) = \cdots = u^{(n-2)}(a) = 0$ indeed imply $a_n = a_{n-1} = \cdots = a_2 = 0$. The last term of u (from (11)) is $a_n(\psi(t)-\psi(a))^{\alpha-n}$, which has negative power. If we suppose that $a_n \neq 0$, then u is defined only on $(a,b]$, and u becomes unbounded near $t=a$; so it is impossible to impose a condition for $u(a)$. Then $a_n=0$, and therefore we obtain

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\big(\psi(t)-\psi(s)\big)^{\alpha-1}h(s)\,\mathrm{d}s + \sum_{j=1}^{n-1} a_j\big(\psi(t)-\psi(a)\big)^{\alpha-j}.$$

The first derivative of u is

$$\begin{aligned} u'(t) = &-\frac{1}{\Gamma(\alpha-1)}\int_a^t \psi'(s)\big(\psi(t)-\psi(a)\big)^{\alpha-2}\psi'(t)h(s)\,\mathrm{d}s \\ &+ \sum_{j=1}^{n-1} a_j(\alpha-j)\big(\psi(t)-\psi(a)\big)^{\alpha-j-1}\psi'(t). \end{aligned}$$

The last term in u' is $a_{n-1}(\alpha-n+1)\psi'(t)(\psi(t)-\psi(a))^{\alpha-n}$, which also has a negative power. Hence the condition $u'(a)=0$ implies, from the same reason as above, that $a_{n-1}=0$. Continuing in this way, we conclude that all coefficients vanish until the condition $u^{(n-2)}(a)=0$, which implies $a_2=0$.

In consequence, system (11) takes the following form:

$$\begin{aligned} u(t) &= -I_{a+}^{\alpha,\psi}h(t) + a_1\big(\psi(t)-\psi(a)\big)^{\alpha-1} \\ &= -\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\big(\psi(t)-\psi(s)\big)^{\alpha-1}h(s)\,\mathrm{d}s + a_1\big(\psi(t)-\psi(a)\big)^{\alpha-1}, \\ v(t) &= -I_{a+}^{\beta,\psi}k(t) + b_1\big(\psi(t)-\psi(a)\big)^{\beta-1} \\ &= -\frac{1}{\Gamma(\beta)}\int_a^t \psi'(s)\big(\psi(t)-\psi(s)\big)^{\beta-1}k(s)\,\mathrm{d}s + b_1\big(\psi(t)-\psi(a)\big)^{\beta-1}, \end{aligned} \tag{12}$$

where $t\in[a,b]$.

From (12) we have

$$\begin{aligned} D_{a+}^{\kappa,\psi}u(t) &= -D_{a+}^{\kappa,\psi}I_{a+}^{\alpha,\psi}h(t) + a_1 D_{a+}^{\kappa,\psi}\big(\psi(t)-\psi(a)\big)^{\alpha-1} \\ &= -I_{a+}^{\alpha-\kappa,\psi}h(t) + a_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\kappa)}\big(\psi(t)-\psi(a)\big)^{\alpha-\kappa-1} \end{aligned} \tag{13}$$

for $t\in(a,b)$, $\kappa=\varsigma,\varrho_i,\eta_j$, $i=1,\dots,c$, $j=1,\dots,p$, and

$$\begin{aligned} D_{a+}^{\widetilde{\kappa},\psi}v(t) &= -D_{a+}^{\widetilde{\kappa},\psi}I_{a+}^{\beta,\psi}k(t) + b_1 D_{a+}^{\widetilde{\kappa},\psi}\big(\psi(t)-\psi(a)\big)^{\beta-1} \\ &= -I_{a+}^{\beta-\widetilde{\kappa},\psi}k(t) + b_1\frac{\Gamma(\beta)}{\Gamma(\beta-\widetilde{\kappa})}\big(\psi(t)-\psi(a)\big)^{\beta-\widetilde{\kappa}-1} \end{aligned} \tag{14}$$

for $t\in(a,b)$, $\widetilde{\kappa}=\vartheta,\sigma_i,\theta_j$, $i=1,\dots,d$, $j=1,\dots,q$.

Inserting (13) and (14) in the integral boundary conditions of (2), we deduce

$$\begin{aligned} &a_1\left[\frac{\Gamma(\alpha)}{\Gamma(\alpha-\varsigma)}\big(\psi(b)-\psi(a)\big)^{\alpha-\varsigma-1}-\sum_{i=1}^{c}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\varrho_i)}\int_a^b\big(\psi(s)-\psi(a)\big)^{\alpha-\varrho_i-1}\,\mathrm{d}\mathcal{H}_i(s)\right]\\ &\quad-b_1\sum_{i=1}^{d}\frac{\Gamma(\beta)}{\Gamma(\beta-\sigma_i)}\int_a^b\big(\psi(s)-\psi(a)\big)^{\beta-\sigma_i-1}\,\mathrm{d}\mathcal{K}_i(s)=U, \qquad (15)\\ &-a_1\sum_{i=1}^{p}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\eta_i)}\int_a^b\big(\psi(s)-\psi(a)\big)^{\alpha-\eta_i-1}\,\mathrm{d}\mathcal{P}_i(s)+b_1\left[\frac{\Gamma(\beta)}{\Gamma(\beta-\vartheta)}\big(\psi(b)-\psi(a)\big)^{\beta-\vartheta-1}\right.\\ &\quad\left.-\sum_{i=1}^{q}\frac{\Gamma(\beta)}{\Gamma(\beta-\theta_i)}\int_a^b\big(\psi(s)-\psi(a)\big)^{\beta-\theta_i-1}\,\mathrm{d}\mathcal{Q}_i(s)\right]=V, \qquad (16)\end{aligned}$$

where U and V are given by (10).

Using notations (8) in (15) and (16), we obtain

$$\Lambda_1 a_1-\Lambda_2 b_1=U,\qquad -\Lambda_3 a_1+\Lambda_4 b_1=V. \qquad (17)$$

The determinant of system (17) in the unknowns a_1 and b_1 is $\Delta=\Lambda_1\Lambda_4-\Lambda_2\Lambda_3\neq 0$. Then the solution of system (17) is unique, namely

$$a_1=\frac{1}{\Delta}(\Lambda_4 U+\Lambda_2 V),\qquad b_1=\frac{1}{\Delta}(\Lambda_3 U+\Lambda_1 V). \qquad (18)$$

By replacing the formulas for a_1 and b_1 (given by (18)) in (12), we obtain the solution $(u(t),v(t))$, $t\in[a,b]$, of problem (7), (2) presented in (9).

Conversely, the functions u and v given by (9) satisfy system (7). Indeed, it follows from (9) that for $t\in(a,b)$,

$$\begin{aligned} D_{a+}^{\alpha,\psi}u(t)&=D_{a+}^{\alpha,\psi}\left(-I_{a+}^{\alpha,\psi}h(t)+\frac{\Lambda_4 U+\Lambda_2 V}{\Delta}\big(\psi(t)-\psi(a)\big)^{\alpha-1}\right)\\ &=-h(t),\\ D_{a+}^{\beta,\psi}v(t)&=D_{a+}^{\beta,\psi}\left(-I_{a+}^{\beta,\psi}k(t)+\frac{\Lambda_3 U+\Lambda_1 V}{\Delta}\big(\psi(t)-\psi(a)\big)^{\beta-1}\right)\\ &=-k(t).\end{aligned}$$

In addition, u and v satisfy the boundary conditions (2). By (9) we obtain $u(a)=0$. The first derivative of u is

$$u'(t)=-Z_{11}(t)I_{a+}^{\alpha-1,\psi}h(t)+\frac{\Lambda_4 U+\Lambda_2 V}{\Delta}\widetilde{Z}_{11}(t)\big(\psi(t)-\psi(a)\big)^{\alpha-2}$$

with $Z_{11}(t) = \psi'(t)$ and $\widetilde{Z}_{11}(t) = (\alpha - 1)\psi'(t)$. So $u'(a) = 0$. The second derivative of u is

$$\begin{aligned} u''(t) &= -Z_{21}(t) I_{a+}^{\alpha-1,\psi} h(t) - Z_{22}(t) I_{a+}^{\alpha-2,\psi} h(t) \\ &\quad + \frac{\varLambda_4 U + \varLambda_2 V}{\varDelta} \bigl[\widetilde{Z}_{21}(t)\bigl(\psi(t) - \psi(a)\bigr)^{\alpha-2} + \widetilde{Z}_{22}(t)\bigl(\psi(t) - \psi(a)\bigr)^{\alpha-3}\bigr], \end{aligned}$$

where $Z_{21}(t) = \psi''(t)$, $Z_{22}(t) = (\psi'(t))^2$, $\widetilde{Z}_{21}(t) = (\alpha - 1)\psi''(t)$, $\widetilde{Z}_{22}(t) = (\alpha - 1) \times (\alpha - 2)(\psi'(t))^2$. Then $u''(a) = 0$. For the third derivative of u, we find

$$\begin{aligned} u'''(t) &= -Z_{31}(t) I_{a+}^{\alpha-1,\psi} h(t) - Z_{32}(t) I_{a+}^{\alpha-2,\psi} h(t) - Z_{33}(t) I_{a+}^{\alpha-3,\psi} h(t) \\ &\quad + \frac{\varLambda_4 U + \varLambda_2 V}{\varDelta} \bigl[\widetilde{Z}_{31}(t)\bigl(\psi(t) - \psi(a)\bigr)^{\alpha-2} + \widetilde{Z}_{32}(t)\bigl(\psi(t) - \psi(a)\bigr)^{\alpha-3} \\ &\quad + \widetilde{Z}_{33}(t)\bigl(\psi(t) - \psi(a)\bigr)^{\alpha-4}\bigr], \end{aligned}$$

where $Z_{31}(t) = \psi'''(t)$, $Z_{32}(t) = 3\psi'(t)\psi''(t)$, $Z_{33}(t) = (\psi'(t))^3$, $\widetilde{Z}_{31}(t) = (\alpha - 1) \times \psi'''(t)$, $\widetilde{Z}_{32}(t) = 3(\alpha-1)(\alpha-2)\psi'(t)\psi''(t)$, and $\widetilde{Z}_{33}(t) = (\alpha-1)(\alpha-2)(\alpha-3)(\psi'(t))^3$. Hence $u'''(a) = 0$. We continue in this way until the derivative

$$\begin{aligned} u^{(n-2)}(t) &= -\sum_{i=1}^{n-2} Z_{n-2,i}(t) I_{a+}^{\alpha-i,\psi} h(t) \\ &\quad + \frac{\varLambda_4 U + \varLambda_2 V}{\varDelta} \sum_{i=1}^{n-2} \widetilde{Z}_{n-2,i}(t)\bigl(\psi(t) - \psi(a)\bigr)^{\alpha-i-1}, \end{aligned}$$

where $Z_{n-2,i}$ and $\widetilde{Z}_{n-2,i}$, $i = 1, \dots, n-2$, are functions, which depend on the ψ', ψ'', $\dots$, $\psi^{(n-2)}$. We obtain $u^{(n-2)}(a) = 0$.

The condition in the point b for the function u, namely

$$D_{a+}^{\varsigma,\psi} u(b) = \sum_{i=1}^{c} \int_a^b D_{a+}^{\varrho_i,\psi} u(s)\,\mathrm{d}\mathcal{H}_i(s) + \sum_{i=1}^{d} \int_a^b D_{a+}^{\sigma_i,\psi} v(s)\,\mathrm{d}\mathcal{K}_i(s),$$

is equivalent to

$$\begin{aligned} &-I_{a+}^{\alpha-\varsigma,\psi} h(b) + \frac{\varLambda_4 U + \varLambda_2 V}{\varDelta} \frac{\Gamma(\alpha)}{\Gamma(\alpha - \varsigma)} \bigl(\psi(b) - \psi(a)\bigr)^{\alpha-\varsigma-1} \\ &\quad = \sum_{i=1}^{c} \int_a^b \biggl[-I_{a+}^{\alpha-\varrho_i,\psi} h(s) + \frac{\varLambda_4 U + \varLambda_2 V}{\varDelta} \frac{\Gamma(\alpha)}{\Gamma(\alpha - \varrho_i)} \bigl(\psi(s) - \psi(a)\bigr)^{\alpha-\varrho_i-1}\biggr] \mathrm{d}\mathcal{H}_i(s) \\ &\quad + \sum_{i=1}^{d} \int_a^b \biggl[-I_{a+}^{\beta-\sigma_i,\psi} k(s) + \frac{\varLambda_3 U + \varLambda_1 V}{\varDelta} \frac{\Gamma(\beta)}{\Gamma(\beta - \sigma_i)} \bigl(\psi(s) - \psi(a)\bigr)^{\beta-\sigma_i-1}\biggr] \mathrm{d}\mathcal{K}_i(s). \end{aligned}$$

This last relation is also verified (see relation (15) and the definition of the constants a_1 and b_1 (18)).

In a similar manner, we obtain $v(a) = v'(a) = \cdots = v^{(m-2)}(a) = 0$, and

$$D_{a+}^{\vartheta,\psi} v(b) = \sum_{i=1}^{p} \int_a^b D_{a+}^{\eta_i,\psi} u(s)\,\mathrm{d}\mathcal{P}_i(s) + \sum_{i=1}^{q} \int_a^b D_{a+}^{\theta_i,\psi} v(s)\,\mathrm{d}\mathcal{Q}_i(s). \qquad \square$$

After quite long computations, we deduce the next result.

Lemma 2. *If $\varDelta \neq 0$, then the solution of problem* (7), (2) (*given by* (9)) *can be expressed as*

$$\begin{aligned} u(t) &= \int_a^b G_1(t,s)h(s)\,\mathrm{d}s + \int_a^b G_2(t,s)k(s)\,\mathrm{d}s, \quad t \in [a,b], \\ v(t) &= \int_a^b G_3(t,s)h(s)\,\mathrm{d}s + \int_a^b G_4(t,s)k(s)\,\mathrm{d}s, \quad t \in [a,b], \end{aligned} \tag{19}$$

where the Green functions G_i, $i = 1, \dots, 4$, are given by

$$\begin{aligned} G_1(t,s) &= g_1(t,s) + \frac{(\psi(t)-\psi(a))^{\alpha-1}}{\varDelta} \\ &\quad \times \left[\varLambda_4 \sum_{i=1}^{c} \int_a^b g_{1i}(\tau,s)\,\mathrm{d}\mathcal{H}_i(\tau) + \varLambda_2 \sum_{i=1}^{p} \int_a^b g_{2i}(\tau,s)\,\mathrm{d}\mathcal{P}_i(\tau) \right], \\ G_2(t,s) &= \frac{(\psi(t)-\psi(a))^{\alpha-1}}{\varDelta} \\ &\quad \times \left[\varLambda_4 \sum_{i=1}^{d} \int_a^b g_{3i}(\tau,s)\,\mathrm{d}\mathcal{K}_i(\tau) + \varLambda_2 \sum_{i=1}^{q} \int_a^b g_{4i}(\tau,s)\,\mathrm{d}\mathcal{Q}_i(\tau) \right], \\ G_3(t,s) &= \frac{(\psi(t)-\psi(a))^{\beta-1}}{\varDelta} \\ &\quad \times \left[\varLambda_3 \sum_{i=1}^{c} \int_a^b g_{1i}(\tau,s)\,\mathrm{d}\mathcal{H}_i(\tau) + \varLambda_1 \sum_{i=1}^{p} \int_a^b g_{2i}(\tau,s)\,\mathrm{d}\mathcal{P}_i(\tau) \right], \\ G_4(t,s) &= g_2(t,s) + \frac{(\psi(t)-\psi(a))^{\beta-1}}{\varDelta} \\ &\quad \times \left[\varLambda_3 \sum_{i=1}^{d} \int_a^b g_{3i}(\tau,s)\,\mathrm{d}\mathcal{K}_i(\tau) + \varLambda_1 \sum_{i=1}^{q} \int_a^b g_{4i}(\tau,s)\,\mathrm{d}\mathcal{Q}_i(\tau) \right] \end{aligned}$$

for all $t, s \in [a, b]$ *with*

$$g_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \psi'(s)[(\psi(t)-\psi(a))^{\alpha-1}(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)})^{\alpha-\varsigma-1} \\ \quad -(\psi(t)-\psi(s))^{\alpha-1}], & a \leqslant s \leqslant t \leqslant b, \\ \psi'(s)(\psi(t)-\psi(a))^{\alpha-1}(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)})^{\alpha-\varsigma-1}, & a \leqslant t \leqslant s \leqslant b, \end{cases}$$

$$g_2(t,s) = \frac{1}{\Gamma(\beta)} \begin{cases} \psi'(s)[(\psi(t)-\psi(a))^{\beta-1}(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)})^{\beta-\vartheta-1} \\ \quad -(\psi(t)-\psi(s))^{\beta-1}], & a \leqslant s \leqslant t \leqslant b, \\ \psi'(s)(\psi(t)-\psi(a))^{\beta-1}(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)})^{\beta-\vartheta-1}, & a \leqslant t \leqslant s \leqslant b, \end{cases}$$

$$g_{1i}(\tau,s) = \frac{1}{\Gamma(\alpha-\varrho_i)} \begin{cases} \psi'(s)[(\psi(\tau)-\psi(a))^{\alpha-\varrho_i-1}(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)})^{\alpha-\varsigma-1} \\ \quad -(\psi(\tau)-\psi(s))^{\alpha-\varrho_i-1}], & a \leqslant s \leqslant \tau \leqslant b, \\ \psi'(s)(\psi(\tau)-\psi(a))^{\alpha-\varrho_i-1}(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)})^{\alpha-\varsigma-1}, & a \leqslant \tau \leqslant s \leqslant b, \end{cases}$$

$$g_{2j}(\tau,s) = \frac{1}{\Gamma(\alpha-\eta_j)} \begin{cases} \psi'(s)[(\psi(\tau)-\psi(a))^{\alpha-\eta_j-1}(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)})^{\alpha-\varsigma-1} \\ \quad -(\psi(\tau)-\psi(s))^{\alpha-\eta_j-1}], & a \leqslant s \leqslant \tau \leqslant b, \\ \psi'(s)(\psi(\tau)-\psi(a))^{\alpha-\eta_j-1}(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)})^{\alpha-\varsigma-1}, & a \leqslant \tau \leqslant s \leqslant b, \end{cases}$$

$$g_{3k}(\tau,s) = \frac{1}{\Gamma(\beta-\sigma_k)} \begin{cases} \psi'(s)[(\psi(\tau)-\psi(a))^{\beta-\sigma_k-1}(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)})^{\beta-\vartheta-1} \\ \quad -(\psi(\tau)-\psi(s))^{\beta-\sigma_k-1}], & a \leqslant s \leqslant \tau \leqslant b, \\ \psi'(s)(\psi(\tau)-\psi(a))^{\beta-\sigma_k-1}(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)})^{\beta-\vartheta-1}, & a \leqslant \tau \leqslant s \leqslant b, \end{cases}$$

$$g_{4\iota}(\tau,s) = \frac{1}{\Gamma(\beta-\theta_\iota)} \begin{cases} \psi'(s)[(\psi(\tau)-\psi(a))^{\beta-\theta_\iota-1}(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)})^{\beta-\vartheta-1} \\ \quad -(\psi(\tau)-\psi(s))^{\beta-\theta_\iota-1}], & a \leqslant s \leqslant \tau \leqslant b, \\ \psi'(s)(\psi(\tau)-\psi(a))^{\beta-\theta_\iota-1}(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)})^{\beta-\vartheta-1}, & a \leqslant \tau \leqslant s \leqslant b \end{cases}$$

for $i = 1, \dots, c$, $j = 1, \dots, p$, $k = 1, \dots, d$, $\iota = 1, \dots, q$.

Based on the definitions of functions G_i, $i = 1, \dots, 4$, and the properties of the functions g_1, g_2, g_{1i}, g_{2j}, g_{3k}, $g_{4\iota}$, $i = 1, \dots, c$, $j = 1, \dots, p$, $k = 1, \dots, d$, $\iota = 1, \dots, q$ (deduced from [21, Lemma 8]), we obtain the next lemma.

Lemma 3. *Assume that* $\Lambda_1 \geqslant 0$, $\Lambda_4 \geqslant 0$, $\Delta > 0$, *and* $\mathcal{H}_i$, $\mathcal{K}_j$, $\mathcal{P}_k$, $\mathcal{Q}_\iota$, $i = 1, \dots, c$, $j = 1, \dots, d$, $k = 1, \dots, p$, $\iota = 1, \dots, q$, *are nondecreasing functions. Then the functions* G_κ, $\kappa = 1, \dots, 4$, *are continuous on* $[a, b] \times [a, b]$ *and, for* $t, s \in [a, b]$, *satisfy the following inequalities*:

(i) $G_1(t, s) \leqslant J_1(s)$, *where*

$$J_1(s) = h_1(s) + \frac{(\psi(b)-\psi(a))^{\alpha-1}}{\Delta} \times \left[\Lambda_4 \sum_{i=1}^{c} \int_a^b g_{1i}(\tau,s)\,\mathrm{d}\mathcal{H}_i(\tau) + \Lambda_2 \sum_{i=1}^{p} \int_a^b g_{2i}(\tau,s)\,\mathrm{d}\mathcal{P}_i(\tau) \right]$$

with

$$h_1(s)=\frac{1}{\Gamma(\alpha)}\psi'(s)\bigl(\psi(b)-\psi(s)\bigr)^{\alpha-\varsigma-1}\bigl[\bigl(\psi(b)-\psi(a)\bigr)^{\varsigma}-\bigl(\psi(b)-\psi(s)\bigr)^{\varsigma}\bigr];$$

(ii) $G_1(t,s)\geqslant\gamma^{\alpha-1}(t)J_1(s)$, *where* $\gamma(t)=(\psi(t)-\psi(a))/(\psi(b)-\psi(a))$;

(iii) $G_2(t,s)\leqslant J_2(s)$, *where*

$$J_2(s)=\frac{(\psi(b)-\psi(a))^{\alpha-1}}{\varDelta}\times\Biggl[\varLambda_4\sum_{i=1}^{d}\int_a^b g_{3i}(\tau,s)\,\mathrm{d}\mathcal{K}_i(\tau)+\varLambda_2\sum_{i=1}^{q}\int_a^b g_{4i}(\tau,s)\,\mathrm{d}\mathcal{Q}_i(\tau)\Biggr];$$

(iv) $G_2(t,s)=\gamma^{\alpha-1}(t)J_2(s)$;

(v) $G_3(t,s)\leqslant J_3(s)$, *where*

$$J_3(s)=\frac{(\psi(b)-\psi(a))^{\beta-1}}{\varDelta}\times\Biggl[\varLambda_3\sum_{i=1}^{c}\int_a^b g_{1i}(\tau,s)\,\mathrm{d}\mathcal{H}_i(\tau)+\varLambda_1\sum_{i=1}^{p}\int_a^b g_{2i}(\tau,s)\,\mathrm{d}\mathcal{P}_i(\tau)\Biggr];$$

(vi) $G_3(t,s)=\gamma^{\beta-1}(t)J_3(s)$;

(vii) $G_4(t,s)\leqslant J_4(s)$, *where*

$$J_4(s)=h_2(s)+\frac{(\psi(b)-\psi(a))^{\beta-1}}{\varDelta}\times\Biggl[\varLambda_3\sum_{i=1}^{d}\int_a^b g_{3i}(\tau,s)\,\mathrm{d}\mathcal{K}_i(\tau)+\varLambda_1\sum_{i=1}^{p}\int_a^b g_{4i}(\tau,s)\,\mathrm{d}\mathcal{Q}_i(\tau)\Biggr]$$

with

$$h_2(s)=\frac{1}{\Gamma(\beta)}\psi'(s)\bigl(\psi(b)-\psi(s)\bigr)^{\beta-\vartheta-1}\bigl[\bigl(\psi(b)-\psi(a)\bigr)^{\vartheta}-(\psi(b)-\psi(s))^{\vartheta}\bigr];$$

(viii) $G_4(t,s)\geqslant\gamma^{\beta-1}(t)J_4(s)$.

Remark 1. Under the assumptions of Lemma 3, we deduce that $\varLambda_1>0$, $\varLambda_2\geqslant0$, $\varLambda_3\geqslant0$, and $\varLambda_4>0$.

Lemma 4. *We assume that the assumptions of Lemma* 3 *are satisfied. If* $h(t)\geqslant0$ *and* $k(t)\geqslant0$ *for all* $t\in[a,b]$, *then the solution* $(u(t),v(t))$, $t\in[a,b]$, *of problem* (7), (2) (*given by* (19)) *satisfies the inequalities* $u(t)\geqslant0$, $v(t)\geqslant0$ *for all* $t\in[a,b]$ *and* $u(t)\geqslant\gamma^{\alpha-1}(t)u(\tau)$, $v(t)\geqslant\gamma^{\beta-1}(t)v(\tau)$ *for all* $t,\tau\in[a,b]$.

Proof. By using Lemma 3 we obtain the following inequalities for all $t, \tau \in [a,b]$:

$$
\begin{aligned}
u(t) &= \int_a^b G_1(t,s)h(s)\,\mathrm{d}s + \int_a^b G_2(t,s)k(s)\,\mathrm{d}s \leqslant \int_a^b J_1(s)h(s)\,\mathrm{d}s + \int_a^b J_2(s)k(s)\,\mathrm{d}s,\\
u(t) &\geqslant \int_a^b \gamma^{\alpha-1}(t)J_1(s)h(s)\,\mathrm{d}s + \int_a^b \gamma^{\alpha-1}(t)J_2(s)k(s)\,\mathrm{d}s\\
&= \gamma^{\alpha-1}(t)\left[\int_a^b J_1(s)h(s)\,\mathrm{d}s + \int_a^b J_2(s)k(s)\,\mathrm{d}s\right] \geqslant \gamma^{\alpha-1}(t)u(\tau),\\
v(t) &= \int_a^b G_3(t,s)h(s)\,\mathrm{d}s + \int_a^b G_4(t,s)k(s)\,\mathrm{d}s \leqslant \int_a^b J_3(s)h(s)\,\mathrm{d}s + \int_a^b J_4(s)k(s)\,\mathrm{d}s,\\
v(t) &\geqslant \int_a^b \gamma^{\beta-1}(t)J_3(s)h(s)\,\mathrm{d}s + \int_a^b \gamma^{\beta-1}(t)J_4(s)k(s)\,\mathrm{d}s\\
&= \gamma^{\beta-1}(t)\left[\int_a^b J_3(s)h(s)\,\mathrm{d}s + \int_a^b J_4(s)k(s)\,\mathrm{d}s\right] \geqslant \gamma^{\beta-1}(t)v(\tau). \qquad \square
\end{aligned}
$$

3 Existence of positive solutions

In this section, we present our main result related to the existence of positive solutions for problem (1)–(2). We first give the assumptions that we will use in our main theorem.

(A1) $a, b \in \mathbb{R}$, $0 \leqslant a < b$; $\alpha \in (n-1,\, n)$, $\beta \in (m-1,\, m)$; $n, m \in N$; $n, m \geqslant 3$; $\psi \in C^\rho[a,b]$, $\rho = \max\{n, m\}$ with $\psi'(t) > 0$ for all $t \in [a,b]$; $c, d, p, q \in N$; $\varsigma \in [1,\, \alpha-1)$; $\varrho_i \in [0, \varsigma]$, $i = 1, \dots, c$; $\eta_j \in [0, \varsigma]$, $j = 1, \dots, p$; $\vartheta \in [1, \beta-1)$; $\sigma_k \in [0, \vartheta]$, $k = 1, \dots, d$; $\theta_\iota \in [0, \vartheta]$, $\iota = 1, \dots, q$; $\lambda, \mu > 0$; $\mathcal{H}_i, \mathcal{K}_j, \mathcal{P}_k, Q_\iota : [a,b] \to \mathbb{R}$ are nondecreasing functions for all $i = 1, \dots, c$, $j = 1, \dots, d$, $k = 1, \dots, p$, $\iota = 1, \dots, q$. In addition, $\Lambda_1 \geqslant 0$, $\Lambda_4 \geqslant 0$, and $\Delta > 0$.

(A2) The functions $f, g \in C([a,b] \times \mathbb{R}_+ \times \mathbb{R}_+)$, and there exist the functions $z_i \in C([a,b], \mathbb{R}_+)$, $i = 1, 2$, such that $f(t,x,y) \geqslant -z_1(t)$, $g(t,x,y) \geqslant -z_2(t)$ for $t \in [a,b]$, $x, y \in \mathbb{R}_+$. Here $\mathbb{R}_+ = [0, \infty)$.

(A3) $f(t,0,0) > 0$, $g(t,0,0) > 0$ for all $t \in [a,b]$.

Now we consider the system of nonlinear ψ-fractional differential equations

$$
\begin{aligned}
D_{a+}^{\alpha,\psi} x(t) + \lambda\bigl[f\bigl(t, \bigl[x(t) - \delta_1(t)\bigr]^*, \bigl[y(t) - \delta_2(t)\bigr]^*\bigr) + z_1(t)\bigr] = 0, \quad t \in (a,b),\\
D_{a+}^{\beta,\psi} y(t) + \mu\bigl[g\bigl(t, \bigl[x(t) - \delta_1(t)\bigr]^*, \bigl[y(t) - \delta_2(t)\bigr]^*\bigr) + z_2(t)\bigr] = 0, \quad t \in (a,b),
\end{aligned} \tag{20}
$$

with the boundary conditions

$$\begin{aligned}
&x^{(i)}(a) = 0, \quad i = 0,1,\dots,n-2, \qquad y^{(i)}(a) = 0, \quad i = 0,1,\dots,m-2,\\
&D_{a+}^{\varsigma,\psi}x(b) = \sum_{i=1}^{c}\int_a^b D_{a+}^{\varrho_i,\psi}x(s)\,\mathrm{d}\mathcal{H}_i(s) + \sum_{i=1}^{d}\int_a^b D_{a+}^{\sigma_i,\psi}y(s)\,\mathrm{d}\mathcal{K}_i(s),\\
&D_{a+}^{\vartheta,\psi}y(b) = \sum_{i=1}^{p}\int_a^b D_{a+}^{\eta_i,\psi}x(s)\,\mathrm{d}\mathcal{P}_i(s) + \sum_{i=1}^{q}\int_a^b D_{a+}^{\theta_i,\psi}y(s)\,\mathrm{d}\mathcal{Q}_i(s),
\end{aligned} \tag{21}$$

where $\chi^*(t) = \chi(t)$ if $\chi(t) \geqslant 0$, and $\chi^*(t) = 0$ if $\chi(t) < 0$.

Here $(\delta_1(t), \delta_2(t))$, $t \in [a,b]$, given by

$$\delta_1(t) = \lambda\int_a^b G_1(t,s)z_1(s)\,\mathrm{d}s + \mu\int_a^b G_2(t,s)z_2(s)\,\mathrm{d}s, \quad t \in [a,b],$$

$$\delta_2(t) = \lambda\int_a^b G_3(t,s)z_1(s)\,\mathrm{d}s + \mu\int_a^b G_4(t,s)z_2(s)\,\mathrm{d}s, \quad t \in [a,b],$$

is the solution of the system of ψ-fractional differential equations

$$D_{a+}^{\alpha,\psi}\delta_1(t) + \lambda z_1(t) = 0, \quad D_{a+}^{\beta,\psi}\delta_2(t) + \mu z_2(t) = 0, \quad t \in (a,b), \tag{22}$$

with the boundary conditions

$$\begin{aligned}
&\delta_1^{(i)}(a) = 0, \quad i = 0,1,\dots,n-2, \qquad \delta_2^{(i)}(a) = 0, \quad i = 0,1,\dots,m-2,\\
&D_{a+}^{\varsigma,\psi}\delta_1(b) = \sum_{i=1}^{c}\int_a^b D_{a+}^{\varrho_i,\psi}\delta_1(s)\,\mathrm{d}\mathcal{H}_i(s) + \sum_{i=1}^{d}\int_a^b D_{a+}^{\sigma_i,\psi}\delta_2(s)\,\mathrm{d}\mathcal{K}_i(s),\\
&D_{a+}^{\vartheta,\psi}\delta_2(b) = \sum_{i=1}^{p}\int_a^b D_{a+}^{\eta_i,\psi}\delta_1(s)\,\mathrm{d}\mathcal{P}_i(s) + \sum_{i=1}^{q}\int_a^b D_{a+}^{\theta_i,\psi}\delta_2(s)\,\mathrm{d}\mathcal{Q}_i(s).
\end{aligned} \tag{23}$$

Under assumptions (A1) and (A2), we obtain $\delta_1(t) \geqslant 0$, $\delta_2(t) \geqslant 0$ for all $t \in [a,b]$.

We will prove that there exists a solution (x,y) for the boundary value problem (20)–(21) with $x(t) \geqslant \delta_1(t)$ and $y(t) \geqslant \delta_2(t)$ on $[a,b]$, $x(t) > \delta_1(t)$ and $y(t) > \delta_2(t)$ on (a,b).

In this case, (u,v) with $u(t) = x(t) - \delta_1(t)$ and $v(t) = y(t) - \delta_2(t)$ for all $t \in [a,b]$ is a positive solution of problem (1)–(2). Indeed, by (20)–(23), for any $t \in (a,b)$, we derive

$$\begin{aligned}
D_{a+}^{\alpha,\psi}u(t) &= D_{a+}^{\alpha,\psi}x(t) - D_{a+}^{\alpha,\psi}\delta_1(t)\\
&= -\lambda f\bigl(t, [x(t)-\delta_1(t)]^*, [y(t)-\delta_2(t)]^*\bigr) - \lambda z_1(t) + \lambda z_1(t)\\
&= -\lambda f\bigl(t, u(t), v(t)\bigr),
\end{aligned}$$

$$\begin{aligned} D_{a+}^{\beta,\psi} v(t) &= D_{a+}^{\beta,\psi} y(t) - D_{a+}^{\beta,\psi} \delta_2(t) \\ &= -\mu g\big(t, \big[x(t) - \delta_1(t)\big]^*, \big[y(t) - \delta_2(t)\big]^*\big) - \mu z_2(t) + \mu z_2(t) \\ &= -\mu g\big(t, u(t), v(t)\big), \end{aligned}$$

and

$$\begin{aligned} u^{(i)}(a) &= x^{(i)}(a) - \delta_1^{(i)}(a) = 0, \quad i = 0, 1, \dots, n-2, \\ D_{a+}^{\varsigma,\psi} u(b) &= D_{a+}^{\varsigma,\psi} x(b) - D_{a+}^{\varsigma,\psi} \delta_1(b) \\ &= \sum_{i=1}^{c} \int_a^b D_{a+}^{\varrho_i,\psi}\big(x(s) - \delta_1(s)\big)\,\mathrm{d}\mathcal{H}_i(s) + \sum_{i=1}^{d} \int_a^b D_{a+}^{\sigma_i,\psi}\big(y(s) - \delta_2(s)\big)\,\mathrm{d}\mathcal{K}_i(s) \\ &= \sum_{i=1}^{c} \int_a^b D_{a+}^{\varrho_i,\psi} u(s)\,\mathrm{d}\mathcal{H}_i(s) + \sum_{i=1}^{d} \int_a^b D_{a+}^{\sigma_i,\psi} v(s)\,\mathrm{d}\mathcal{K}_i(s), \\ v^{(i)}(a) &= y^{(i)}(a) - \delta_2^{(i)}(a) = 0, \quad i = 0, 1, \dots, m-2, \\ D_{a+}^{\vartheta,\psi} v(b) &= D_{a+}^{\vartheta,\psi} y(b) - D_{a+}^{\vartheta,\psi} \delta_2(b) \\ &= \sum_{i=1}^{p} \int_a^b D_{a+}^{\eta_i,\psi}\big(x(s) - \delta_1(s)\big)\,\mathrm{d}\mathcal{P}_i(s) + \sum_{i=1}^{q} \int_a^b D_{a+}^{\theta_i,\psi}\big(y(s) - \delta_2(s)\big)\,\mathrm{d}\mathcal{Q}_i(s) \\ &= \sum_{i=1}^{p} \int_a^b D_{a+}^{\eta_i,\psi} u(s)\,\mathrm{d}\mathcal{P}_i(s) + \sum_{i=1}^{q} \int_a^b D_{a+}^{\theta_i,\psi} v(s)\,\mathrm{d}\mathcal{Q}_i(s). \end{aligned}$$

Therefore, we proceed to study the boundary value problem (20)–(21).

By using Lemma 2, problem (20)–(21) is equivalent to the system

$$\begin{aligned} x(t) &= \lambda \int_a^b G_1(t,s)\big[f\big(s, \big[x(s) - \delta_1(s)\big]^*, \big[y(s) - \delta_2(s)\big]^*\big) + z_1(s)\big]\,\mathrm{d}s \\ &\quad + \mu \int_a^b G_2(t,s)\big[g\big(s, \big[x(s) - \delta_1(s)\big]^*, \big[y(s) - \delta_2(s)\big]^*\big) + z_2(s)\big]\,\mathrm{d}s, \\ y(t) &= \lambda \int_a^b G_3(t,s)\big[f\big(s, \big[x(s) - \delta_1(s)\big]^*, \big[y(s) - \delta_2(s)\big]^*\big) + z_1(s)\big]\,\mathrm{d}s \\ &\quad + \mu \int_a^b G_4(t,s)\big[g\big(s, \big[x(s) - \delta_1(s)\big]^*, \big[y(s) - \delta_2(s)\big]^*\big) + z_2(s)\big]\,\mathrm{d}s, \end{aligned} \tag{24}$$

where $t \in [a,b]$.

We consider the Banach space $X = C[a,b]$ with the supremum norm $\|\cdot\|$ and the Banach space $Y = X \times X$ with the norm $\|(u,v)\|_Y = \|u\| + \|v\|$. We define the cone

$$P = \bigl\{(u,v) \in Y\colon\ u(t) \geqslant \gamma^{\alpha-1}(t)\|u\|,\ v(t) \geqslant \gamma^{\beta-1}(t)\|v\|,\ t \in [a,b]\bigr\}.$$

For $\lambda, \mu > 0$, we define the operators $T_1, T_2 : Y \to X$ and $T : Y \to Y$ by $T(x,y) = (T_1(x,y), T_2(x,y))$, $(x,y) \in Y$, with

$$\begin{aligned}
T_1(x,y)(t) = \lambda \int_a^b G_1(t,s)\bigl[f\bigl(s, \bigl[x(s)-\delta_1(s)\bigr]^*, \bigl[y(s)-\delta_2(s)\bigr]^*\bigr) + z_1(s)\bigr]\,\mathrm{d}s \\
+ \mu \int_a^b G_2(t,s)\bigl[g\bigl(s, \bigl[x(s)-\delta_1(s)\bigr]^*, \bigl[y(s)-\delta_2(s)\bigr]^*\bigr) + z_2(s)\bigr]\,\mathrm{d}s, \\
T_2(x,y)(t) = \lambda \int_a^b G_3(t,s)\bigl[f\bigl(s, \bigl[x(s)-\delta_1(s)\bigr]^*, \bigl[y(s)-\delta_2(s)\bigr]^*\bigr) + z_1(s)\bigr]\,\mathrm{d}s \\
+ \mu \int_a^b G_4(t,s)\bigl[g\bigl(s, \bigl[x(s)-\delta_1(s)\bigr]^*, \bigl[y(s)-\delta_2(s)\bigr]^*\bigr) + z_2(s)\bigr]\,\mathrm{d}s,
\end{aligned}$$

where $t \in [a,b]$.

One readily verifies that $(x,y) \in P$ solves problem (20)–(21) (or, equivalently, system (24)) if and only if (x,y) is a fixed point of operator T.

Lemma 5. *Assume that assumptions* (A1) *and* (A2) *hold. Then the operator* $T:\ P \to P$ *is a completely continuous operator.*

Proof. The operators T_1 and T_2 are well defined. To prove this, let $(x,y) \in P$ be fixed with $\|(x,y)\|_Y = M$, that is, $\|x\| + \|y\| = M$. Then we have

$$\begin{aligned}
\bigl[x(s)-\delta_1(s)\bigr]^* \leqslant \|x\| \leqslant \bigl\|(x,y)\bigr\|_Y = M, \quad s \in [a,b], \\
\bigl[y(s)-\delta_2(s)\bigr]^* \leqslant \|y\| \leqslant \bigl\|(x,y)\bigr\|_Y = M, \quad s \in [a,b].
\end{aligned}$$

If (A1) and (A2) hold, then we obtain

$$\begin{aligned}
T_1(x,y)(t) \leqslant \lambda \int_a^b J_1(s)\bigl[f\bigl(s, \bigl[x(s)-\delta_1(s)\bigr]^*, \bigl[y(s)-\delta_2(s)\bigr]^*\bigr) + z_1(s)\bigr]\,\mathrm{d}s \\
+ \mu \int_a^b J_2(s)\bigl[g\bigl(s, \bigl[x(s)-\delta_1(s)\bigr]^*, \bigl[y(s)-\delta_2(s)\bigr]^*\bigr) + z_2(s)\bigr]\,\mathrm{d}s \\
\leqslant 2\lambda M_1 \int_a^b J_1(s)\,\mathrm{d}s + 2\mu M_1 \int_a^b J_2(s)\,\mathrm{d}s < \infty, \quad t \in [a,b],
\end{aligned}$$

$$T_2(x,y)(t) \leqslant \lambda \int_a^b J_3(s)\big[f\big(s, \big[x(s)-\delta_1(s)\big]^*, \big[y(s)-\delta_2(s)\big]^*\big) + z_1(s)\big]\,\mathrm{d}s$$
$$+ \mu \int_a^b J_4(s)\big[g\big(s, \big[x(s)-\delta_1(s)\big]^*, \big[y(s)-\delta_2(s)\big]^*\big) + z_2(s)\big]\,\mathrm{d}s$$
$$\leqslant 2\lambda M_1 \int_a^b J_3(s)\,\mathrm{d}s + 2\mu M_1 \int_a^b J_4(s)\,\mathrm{d}s < \infty, \quad t \in [a,b],$$

where

$$M_1 = \max\Big\{ \max_{\substack{t\in[a,b]\\ x,y\in[0,M]}} \big|f(t,x,y)\big|, \max_{\substack{t\in[a,b]\\ x,y\in[0,M]}} \big|g(t,x,y)\big|, \max_{t\in[a,b]} \delta_1(t), \max_{t\in[a,b]} \delta_2(t)\Big\}.$$

Therefore $T_1(x,y)$ and $T_2(x,y)$ are well defined.

In addition, by Lemma 4, we infer that for all $(x,y) \in P$,

$$T_1(x,y)(t) \geqslant \gamma^{\alpha-1}(t) T_1(x,y)(\zeta), \qquad T_2(x,y)(t) \geqslant \gamma^{\beta-1}(t) T_2(x,y)(\zeta),$$

where $t, \zeta \in [a,b]$, and then

$$T_1(x,y)(t) \geqslant \gamma^{\alpha-1}(t)\big\|T_1(x,y)\big\|, \qquad T_2(x,y)(t) \geqslant \gamma^{\beta-1}(t)\big\|T_2(x,y)\big\|,$$

where $t \in [a,b]$. So we find that $(T_1(x,y), T_2(x,y)) \in P$ for all $(x,y) \in P$, and therefore $T(P) \subset P$.

An application of standard techniques shows that the operator $T: P \to P$ is completely continuous. □

Theorem 1. *We assume that* (A1)–(A3) *hold. Then there exist the constants* $\lambda_1 > 0$ *and* $\mu_1 > 0$ *such that for any* $\lambda \in (0, \lambda_1]$ *and* $\mu \in (0, \mu_1]$, *the boundary value problem* (1)–(2) *has at least one positive solution* $(u(t), v(t))$, $t \in [a,b]$.

Proof. Let $\varpi \in (0,1)$ be fixed. By (A2) and (A3), there exists $r_0 \in (0,1]$ such that

$$f(t,x,y) \geqslant \varpi f(t,0,0), \qquad g(t,x,y) \geqslant \varpi g(t,0,0), \tag{25}$$

where $t \in [a,b]$, $x, y \in [0, r_0]$.

We define

$$f_0 = \max_{\substack{t\in[a,b]\\ x,y\in[0,r_0]}} \big\{f(t,x,y) + z_1(t)\big\}, \qquad g_0 = \max_{\substack{t\in[a,b]\\ x,y\in[0,r_0]}} \big\{g(t,x,y) + z_2(t)\big\},$$

$$\varUpsilon_i = \int_a^b J_i(s)\,\mathrm{d}s, \quad i = 1, \dots, 4,$$

$$\lambda_1 = \begin{cases} \min\{\frac{r_0}{8f_0\varUpsilon_1}, \frac{r_0}{8f_0\varUpsilon_3}\} & \text{if } \varUpsilon_3 \neq 0, \\ \frac{r_0}{8f_0\varUpsilon_1} & \text{if } \varUpsilon_3 = 0, \end{cases} \qquad \mu_1 = \begin{cases} \min\{\frac{r_0}{8g_0\varUpsilon_2}, \frac{r_0}{8g_0\varUpsilon_4}\} & \text{if } \varUpsilon_2 \neq 0, \\ \frac{r_0}{8g_0\varUpsilon_4} & \text{if } \varUpsilon_2 = 0. \end{cases}$$

We see that

$$f_0 \geqslant \max_{t\in[a,b]}\bigl\{\varpi f(t,0,0) + z_1(t)\bigr\} > 0, \qquad g_0 \geqslant \max_{t\in[a,b]}\bigl\{\varpi g(t,0,0) + z_2(t)\bigr\} > 0.$$

We will prove that for any $\lambda \in (0, \lambda_1]$ and $\mu \in (0, \mu_1]$, there exists a positive solution for problem (1)–(2). Let $\lambda \in (0, \lambda_1]$ and $\mu \in (0, \mu_1]$ be arbitrary, but fixed for the moment. We introduce the set $\varXi = \{(x, y) \in P\colon \|(x, y)\|_Y < r_0\}$. We suppose that there exist $(x, y) \in \partial\varXi$, that is, $\|(x, y)\|_Y = r_0$ or $\|x\| + \|y\| = r_0$, and $\xi \in (0, 1)$ such that $(x, y) = \xi T(x, y)$; hence $x = \xi T_1(x, y)$ and $y = \xi T_2(x, y)$. Then we obtain

$$\bigl[x(t) - \delta_1(t)\bigr]^* = \begin{cases} x(t) - \delta_1(t) \leqslant x(t) \leqslant r_0 & \text{if } x(t) - \delta_1(t) \geqslant 0,\ t \in [a, b], \\ 0 & \text{if } x(t) - \delta_1(t) < 0,\ t \in [a, b], \end{cases}$$

and

$$\bigl[y(t) - \delta_2(t)\bigr]^* = \begin{cases} y(t) - \delta_2(t) \leqslant y(t) \leqslant r_0 & \text{if } y(t) - \delta_2(t) \geqslant 0,\ t \in [a, b], \\ 0 & \text{if } y(t) - \delta_2(t) < 0,\ t \in [a, b]. \end{cases}$$

Therefore, by Lemma 3, we find

$$\begin{aligned} x(t) &= \xi T_1(x, y)(t) \leqslant T_1(x, y)(t) \leqslant \lambda \int_a^b J_1(s) f_0\,\mathrm{d}s + \mu \int_a^b J_2(s) g_0\,\mathrm{d}s \\ &= \lambda_1 f_0 \varUpsilon_1 + \mu_1 g_0 \varUpsilon_2 \leqslant \frac{r_0}{8} + \frac{r_0}{8} = \frac{r_0}{4}, \quad t \in [a, b], \\ y(t) &= \xi T_2(x, y)(t) \leqslant T_2(x, y)(t) \leqslant \lambda \int_a^b J_3(s) f_0\,\mathrm{d}s + \mu \int_a^b J_4(s) g_0\,\mathrm{d}s \\ &= \lambda_1 f_0 \varUpsilon_3 + \mu_1 g_0 \varUpsilon_4 \leqslant \frac{r_0}{8} + \frac{r_0}{8} = \frac{r_0}{4}, \quad t \in [a, b]. \end{aligned}$$

Then $\|x\| \leqslant r_0/4$ and $\|y\| \leqslant r_0/4$. Hence $\|(x, y)\|_Y \leqslant r_0/2$, which is a contradiction.

Therefore, by using the nonlinear alternative of Leray–Schauder type, we infer that T has a fixed point $(x_1, y_1) \in \overline{\varXi}$. So $(x_1, y_1) = T(x_1, y_1)$, or, equivalently, $x_1 = T_1(x_1, y_1)$, $y_1 = T_2(x_1, y_1)$, and $\|(x_1, y_1)\|_Y = \|x_1\| + \|y_1\| \leqslant r_0$ with $x_1(t) \geqslant \gamma^{\alpha-1}(t)\|x_1\|$, $y_1(t) \geqslant \gamma^{\beta-1}(t)\|y_1\|$ for all $t \in [a, b]$.

Besides, by (25), we derive

$$
\begin{aligned}
x_1(t) = T_1(x_1, y_1)(t) &\geqslant \lambda \int_a^b G_1(t,s)\big(\varpi f(s,0,0) + z_1(s)\big)\,\mathrm{d}s \\
&\quad + \mu \int_a^b G_2(t,s)\big(\varpi g(s,0,0) + z_2(s)\big)\,\mathrm{d}s \\
&\geqslant \lambda \int_a^b G_1(t,s) z_1(s)\,\mathrm{d}s + \mu \int_a^b G_2(t,s) z_2(s)\,\mathrm{d}s = \delta_1(t), \quad t \in [a,b], \\
x_1(t) &> \lambda \int_a^b G_1(t,s) z_1(s)\,\mathrm{d}s + \mu \int_a^b G_2(t,s) z_2(s)\,\mathrm{d}s = \delta_1(t), \quad t \in (a,b], \\
y_1(t) = T_2(x_1, y_1)(t) &\geqslant \lambda \int_a^b G_3(t,s)\big(\varpi f(s,0,0) + z_1(s)\big)\,\mathrm{d}s \\
&\quad + \mu \int_a^b G_4(t,s)\big(\varpi g(s,0,0) + z_2(s)\big)\,\mathrm{d}s \\
&\geqslant \lambda \int_a^b G_3(t,s) z_1(s)\,\mathrm{d}s + \mu \int_a^b G_4(t,s) z_2(s)\,\mathrm{d}s = \delta_2(t), \quad t \in [a,b], \\
y_1(t) &> \lambda \int_a^b G_3(t,s) z_1(s)\,\mathrm{d}s + \mu \int_a^b G_4(t,s) z_2(s)\,\mathrm{d}s = \delta_2(t), \quad t \in (a,b].
\end{aligned}
$$

Hence $x_1(t) \geqslant \delta_1(t)$, $y_1(t) \geqslant \delta_2(t)$ for all $t \in [a,b]$, and $x_1(t) > \delta_1(t)$, $y_1(t) > \delta_2(t)$ for all $t \in (a,b]$. We consider $u_1(t) = x_1(t) - \delta_1(t)$ and $v_1(t) = y_1(t) - \delta_2(t)$ for all $t \in [a,b]$. Then $u_1(t) \geqslant 0$, $v_1(t) \geqslant 0$ for all $t \in [a,b]$, and $u_1(t) > 0$, $v_1(t) > 0$ for all $t \in (a,b]$. So (u_1, v_1) is a positive solution of problem (1)–(2). $\square$

4 An example

Let $\psi(t) = \ln t$, $t \in [a,b] = [1,\mathrm{e}]$, $\alpha = 5/2$, $n = 3$, $\beta = 10/3$, $m = 4$, $c = 1$, $d = 2$, $p = 1$, $q = 1$, $\varsigma = 5/4$, $\vartheta = 11/5$, $\varrho_1 = 2/7$, $\sigma_1 = 0$, $\sigma_2 = 5/3$, $\eta_1 = 14/15$, $\theta_1 = 17/12$. We also define

$$
\mathcal{H}_1(s) = \begin{cases} (s-1)^{7/8}, & s \in [1, \frac{5}{4}), \\ (\frac{1}{4})^{7/8}, & s \in [\frac{5}{4}, \mathrm{e}], \end{cases} \qquad \mathcal{K}_1(s) = \begin{cases} 1, & s \in [1, \frac{3}{2}), \\ \frac{3}{2}, & s \in [\frac{3}{2}, \mathrm{e}], \end{cases}
$$

$$\mathcal{K}_2(s) = \begin{cases} \frac{1}{3}, & s \in [1,2), \\ (s-2)^{1/4} + \frac{1}{3}, & s \in [2,\mathrm{e}], \end{cases} \qquad \mathcal{P}_1(s) = \begin{cases} 2, & s \in [1,\frac{7}{3}), \\ \frac{40}{19}, & s \in [\frac{7}{3},\mathrm{e}], \end{cases}$$

$$\mathcal{Q}_1(s) = \begin{cases} (s-\frac{1}{2})^{13/5}, & s \in [1,\frac{11}{6}), \\ (\frac{4}{3})^{13/5}, & s \in [\frac{11}{6},\mathrm{e}]. \end{cases}$$

In this way, we obtain the system of Hadamard fractional differential equations

$$\begin{aligned} D_{1+}^{5/2,\psi} u(t) + \lambda f\bigl(t,u(t),v(t)\bigr) &= 0, \quad t \in (1,\mathrm{e}), \\ D_{1+}^{10/3,\psi} v(t) + \mu g\bigl(t,u(t),v(t)\bigr) &= 0, \quad t \in (1,\mathrm{e}), \end{aligned} \tag{26}$$

with the boundary conditions

$$\begin{aligned} &u(1) = u'(1) = 0, \qquad v(1) = v'(1) = v''(1) = 0, \\ &D_{1+}^{5/4,\psi} u(\mathrm{e}) = \frac{7}{8} \int_1^{5/4} (s-1)^{-1/8} D_{1+}^{2/7,\psi} u(s)\,\mathrm{d}s + \frac{1}{2} v\left(\frac{3}{2}\right) \\ &\qquad + \frac{1}{4} \int_2^{\mathrm{e}} (s-2)^{-3/4} D_{1+}^{5/3,\psi} v(s)\,\mathrm{d}s, \\ &D_{1+}^{11/5,\psi} v(\mathrm{e}) = \frac{2}{19} D_{1+}^{14/15,\psi} u\left(\frac{7}{3}\right) + \frac{13}{5} \int_1^{11/6} \left(s-\frac{1}{2}\right)^{8/5} D_{1+}^{17/12,\psi} v(s)\,\mathrm{d}s. \end{aligned} \tag{27}$$

We have here $\gamma(t) = \ln t$, $t \in [1,\mathrm{e}]$, and after some computations, we obtain $\Lambda_1 \approx 1.44143276 \geqslant 0$, $\Lambda_2 \approx 2.41470243$, $\Lambda_3 \approx 0.14310234$, $\Lambda_4 \approx 0.54581281 \geqslant 0$, and $\Delta \approx 0.44120289 > 0$. So assumption (A1) is satisfied.

We also find

$$g_1(t,s) = \frac{1}{\Gamma(\frac{5}{2})} \begin{cases} \frac{1}{s}[\ln^{3/2} t\,(1-\ln s)^{1/4} - (\ln t - \ln s)^{3/2}], & 1 \leqslant s \leqslant t \leqslant \mathrm{e}, \\ \frac{1}{s} \ln^{3/2} t\,(1-\ln s)^{1/4}, & 1 \leqslant t \leqslant s \leqslant \mathrm{e}, \end{cases}$$

$$g_2(t,s) = \frac{1}{\Gamma(\frac{10}{3})} \begin{cases} \frac{1}{s}[\ln^{7/3} t\,(1-\ln s)^{2/15} - (\ln t - \ln s)^{7/3}], & 1 \leqslant s \leqslant t \leqslant \mathrm{e}, \\ \frac{1}{s} \ln^{7/3} t\,(1-\ln s)^{2/15}, & 1 \leqslant t \leqslant s \leqslant \mathrm{e}, \end{cases}$$

$$g_{11}(\tau,s) = \frac{1}{\Gamma(\frac{31}{14})} \begin{cases} \frac{1}{s}[\ln^{17/14} \tau\,(1-\ln s)^{1/4} - (\ln \tau - \ln s)^{17/14}], & 1 \leqslant s \leqslant \tau \leqslant \mathrm{e}, \\ \frac{1}{s} \ln^{17/14} \tau\,(1-\ln s)^{1/4}, & 1 \leqslant \tau \leqslant s \leqslant \mathrm{e}, \end{cases}$$

$$g_{21}(\tau,s) = \frac{1}{\Gamma(\frac{47}{30})} \begin{cases} \frac{1}{s}[\ln^{17/30} \tau\,(1-\ln s)^{1/4} - (\ln \tau - \ln s)^{17/30}], & 1 \leqslant s \leqslant \tau \leqslant \mathrm{e}, \\ \frac{1}{s} \ln^{17/30} \tau\,(1-\ln s)^{1/4}, & 1 \leqslant \tau \leqslant s \leqslant \mathrm{e}, \end{cases}$$

$$g_{31}(\tau,s) = \frac{1}{\Gamma(\frac{10}{3})} \begin{cases} \frac{1}{s}[\ln^{7/3} \tau\,(1-\ln s)^{2/15} - (\ln \tau - \ln s)^{7/3}], & 1 \leqslant s \leqslant \tau \leqslant \mathrm{e}, \\ \frac{1}{s} \ln^{7/3} \tau\,(1-\ln s)^{2/15}, & 1 \leqslant \tau \leqslant s \leqslant \mathrm{e}, \end{cases}$$

$$g_{32}(\tau,s)=\frac{1}{\Gamma(\frac{5}{3})}\begin{cases}\frac{1}{s}[\ln^{2/3}\tau\,(1-\ln s)^{2/15}-(\ln\tau-\ln s)^{2/3}], & 1\leqslant s\leqslant\tau\leqslant\mathrm{e},\\ \frac{1}{s}\ln^{2/3}\tau\,(1-\ln s)^{2/15}, & 1\leqslant\tau\leqslant s\leqslant\mathrm{e},\end{cases}$$

$$g_{41}(\tau,s)=\frac{1}{\Gamma(\frac{23}{12})}\begin{cases}\frac{1}{s}[\ln^{11/12}\tau\,(1-\ln s)^{2/15}-(\ln\tau-\ln s)^{11/12}], & 1\leqslant s\leqslant\tau\leqslant\mathrm{e},\\ \frac{1}{s}\ln^{11/12}\tau\,(1-\ln s)^{2/15}, & 1\leqslant\tau\leqslant s\leqslant\mathrm{e}.\end{cases}$$

Besides, we obtain

$$G_1(t,s)=g_1(t,s)+\frac{\ln^{3/2}t}{\Delta}\left[\Lambda_4\frac{7}{8}\int_1^{5/4}g_{11}(\tau,s)(\tau-1)^{-1/8}\,\mathrm{d}\tau+\Lambda_2\frac{2}{19}g_{21}\left(\frac{7}{3},s\right)\right],$$

$$G_2(t,s)=\frac{\ln^{3/2}t}{\Delta}\left\{\Lambda_4\left[\frac{1}{2}g_{31}\left(\frac{3}{2},s\right)+\frac{1}{4}\int_2^{\mathrm{e}}(\tau-2)^{-3/4}g_{32}(\tau,s)\,\mathrm{d}\tau\right]\right.$$
$$\left.+\Lambda_2\frac{13}{5}\int_1^{11/6}g_{41}(\tau,s)\left(\tau-\frac{1}{2}\right)^{8/5}\mathrm{d}\tau\right\},$$

$$G_3(t,s)=\frac{\ln^{7/3}t}{\Delta}\left[\Lambda_3\frac{7}{8}\int_1^{5/4}g_{11}(\tau,s)(\tau-1)^{-1/8}\,\mathrm{d}\tau+\Lambda_1\frac{2}{19}g_{21}\left(\frac{7}{3},s\right)\right],$$

$$G_4(t,s)=g_2(t,s)+\frac{\ln^{7/3}t}{\Delta}\left\{\Lambda_3\left[\frac{1}{2}g_{31}\left(\frac{3}{2},s\right)+\frac{1}{4}\int_2^{\mathrm{e}}g_{32}(\tau,s)(\tau-2)^{-3/4}\,\mathrm{d}\tau\right]\right.$$
$$\left.+\Lambda_1\frac{13}{5}\int_1^{11/6}g_{41}(\tau,s)\left(\tau-\frac{1}{2}\right)^{8/5}\mathrm{d}\tau\right\},\quad t,s\in[1,\mathrm{e}],$$

$$h_1(s)=\frac{1}{\Gamma(\frac{5}{2})s}\left[(1-\ln s)^{1/4}-(1-\ln s)^{3/2}\right],$$
$$h_2(s)=\frac{1}{\Gamma(\frac{10}{3})s}\left[(1-\ln s)^{2/15}-(1-\ln s)^{7/3}\right],\quad s\in[1,\mathrm{e}].$$

We consider now the functions

$$f(t,x,y)=(x+y)^2+10\cos y,\qquad g(t,x,y)=(x+y)^{1/3}+2\cos\bigl(x^2+y^7\bigr),\tag{28}$$

where $t\in[1,\mathrm{e}]$, $x,y\geqslant 0$. Here we have $z_1(t)=10$, $z_2(t)=2$ for all $t\in[1,\mathrm{e}]$ and $f(t,x,y)\geqslant 10\cos y\geqslant -10$, $g(t,x,y)\geqslant 2\cos(x^2+y^7)\geqslant -2$ for all $t\in[1,\mathrm{e}]$, $x,y\geqslant 0$. So assumption (A2) is satisfied.

Assumption (A3) is also satisfied because $f(t,0,0)=10>0$ and $g(t,0,0)=2>0$ for all $t\in[1,\mathrm{e}]$. We consider $\varpi=1/20<1$ and $r_0=1$. Then we obtain

$$f(t,x,y)\geqslant\varpi f(t,0,0)=\frac{1}{2},\qquad g(t,x,y)\geqslant\varpi g(t,0,0)=\frac{1}{10},$$

where $t \in [1, \mathrm{e}]$, $x, y \in [0, 1]$. Besides, we find

$$f_0 = \max_{\substack{t\in[1,\mathrm{e}]\\ x,y\in[0,1]}} \bigl\{f(t, x, y) + z_1(t)\bigr\} \approx 21.25166806,$$
$$g_0 = \max_{\substack{t\in[1,\mathrm{e}]\\ x,y\in[0,1]}} \bigl\{g(t, x, y) + z_2(t)\bigr\} \approx 4.9751295.$$

In addition, after complex computations with Mathematica program, we obtain $\varUpsilon_1 = \int_1^{\mathrm{e}} J_1(s)\,\mathrm{d}s \approx 0.47071797$, $\varUpsilon_2 = \int_1^{\mathrm{e}} J_2(s)\,\mathrm{d}s \approx 3.51870072$, $\varUpsilon_3 = \int_1^{\mathrm{e}} J_3(s)\,\mathrm{d}s \approx 0.09563862$, $\varUpsilon_4 = \int_1^{\mathrm{e}} J_4(s)\,\mathrm{d}s \approx 2.15462064$, and then

$$\lambda_1 = \min\left\{\frac{1}{8f_0\varUpsilon_1}, \frac{1}{8f_0\varUpsilon_3}\right\} \approx 0.0125,$$
$$\mu_1 = \min\left\{\frac{1}{8g_0\varUpsilon_2}, \frac{1}{8g_0\varUpsilon_4}\right\} \approx 0.0071.$$

By Theorem 1, for any $\lambda \in (0, \lambda_1]$ and $\mu \in (0, \mu_1]$, we deduce that problem (26)–(27) with the nonlinearities given by (28) has at least one positive solution $(u_1(t), v_1(t))$, $t \in [1, \mathrm{e}]$.

5 Conclusions

In this paper, we investigate the system of ψ-Riemann–Liouville fractional differential equations (1) with two positive parameters λ and μ and the sign-changing nonlinearities f and g. This system is subject to the general coupled boundary conditions (2), which involve Riemann–Stieltjes integrals and ψ-fractional derivatives of various orders. We begin by examining the linear version of problem (1)–(2) and derive its solution, the corresponding Green functions, and their main properties. Then we make a change of the unknown functions and transform our problem into an equivalent one, namely problem (20)–(21). Using the Green functions, we rewrite this last problem equivalently as the system of ψ-fractional integral equations (24). We introduce the operator T defined on the cone P in the Banach space Y, whose fixed points are the positive solutions of system (24). Under appropriate assumptions on the functions f and g, we identify intervals of the parameters λ and μ for which the operator T admits at least one fixed point. The proof of our main result relies on the Leray–Schauder nonlinear alternative. Finally, we provide an example that illustrates the applicability of the obtained result.

Conflicts of interest. The authors declare no conflicts of interest.

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