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On the d-Fenchel–Young distance function with fixed point results*

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Abstract. The aim of this work is to introduce a d-Fenchel–Young distance function associated with the 5-tuple $(\alpha, \beta, \gamma, \xi, f)$, where d is a metric on a nonempty set X, $\alpha : [0, +\infty) \to [0, +\infty)$ is a monotonically increasing and invertible function from $(0, +\infty)$ onto $(0, +\infty)$, $\xi : [0, +\infty) \to [0, +\infty)$ is an auxiliary function related to α through the structural condition $c_\alpha := \inf_{t>0} \xi(t)/\alpha^{-1}(t) > 0$, $\beta, \gamma : X \times X \to [0, +\infty)$ satisfy $\beta(x, y) \leqslant \gamma(x, y)$ for all distinct $x, y \in X$, and γ is related to a function $f : [0, +\infty) \to [0, +\infty)$. Within this framework, we establish several fundamental properties of the proposed distance and derive fixed point results for self-mappings on metric spaces satisfying μ-comparison contractive conditions with respect to this distance. For suitable choices of $(\alpha, \beta, \gamma, \xi, f)$, the proposed framework reduces to the classical metric and the d-Young distance. Consequently, the Banach fixed point theorem and related fixed point results in the d-Young setting are recovered as special cases. Finally, we provide an explicit example for which neither the Banach fixed point theorem nor the existing fixed point results based on the d-Young distance are applicable, whereas our generalized framework guarantees the existence and uniqueness of a fixed point.

Keywords: (c)-comparison functions, d-Fenchel–Young distance, d-Young distance, Fenchel–Young inequality, fixed points.

1 Introduction

Fixed point theory has long been recognized as one of the fundamental tools in modern nonlinear analysis. Its underlying principle is simple yet powerful: under suitable conditions, a mapping on a given space admits a point that remains invariant under the action of the mapping. This concept provides an important connection between abstract

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mathematical theory and a wide range of applications, including differential equations, dynamical systems, optimization, image processing, biology, and oceanology, where fixed points frequently arise in the study of stability, equilibrium, and long-term behavior. Owing to its broad applicability, fixed point theory continues to play a significant role in the analysis of nonlinear problems and complex systems.

The classical fixed point result, widely known as the Banach fixed point theorem in [1], is one of the fundamental results in fixed point theory and provides a simple and powerful criterion for the existence and uniqueness of fixed points. In a complete metric space, a Banach contraction admits a unique fixed point, and the sequence generated by successive iterations of the mapping converges to this fixed point from any initial point. Despite its fundamental importance, the Banach fixed point theorem requires a rather restrictive contractive condition, which is not satisfied by many nonlinear problems arising in theory and applications. This limitation has motivated the development of various generalized contractive conditions and broader fixed point frameworks.

To overcome the limitations of classical assumptions, researchers have pursued several strategies, including generalizing contractive conditions or relaxing strict contractive requirements (see, for instance, [3, 5–7, 10, 11, 13, 18]), and employing comparison functions, as illustrated by [4,8,12,17]. Another significant direction has been the development of new ways to measure distances between elements using distance-like quantities that extend beyond the traditional metric. These advances enhance the applicability of fixed point methods, allowing them to address problems where classical approaches are too restrictive. In particular, several generalized distance structures in abstract spaces have been introduced to provide more flexible and powerful frameworks for fixed point theory. For instance, the classical metric was elegantly extended through the introduction of the w-distance by Kada et al. [9], which was later further generalized to the τ-distance [16]. It is well established that every metric is a w-distance and every w-distance is a τ-distance, while the converses generally do not hold. Such generalized distances facilitate the study of mappings for which classical metric-based contractions are insufficient.

Most recently, Rakočević and Samet [14] proposed the d-Young distance function, which is a generalized distance function that provides additional analytical flexibility. This distance function is defined on a metric space (X, d) with respect to the 5-tuple $(p, q, \tau, \kappa, \xi)$, where $1 < p, q < +\infty$ with $1/p + 1/q = 1$, $\tau : X \times X \to [0, +\infty)$ is a given function, $\kappa > 0$, and $\xi : [0, +\infty) \to [0, +\infty)$ satisfies

$$c_\kappa := \inf_{t>0} \frac{\xi(t)}{t^\kappa} > 0.$$

This construction not only generalizes existing distance notions but also enables the analysis of contractive mappings that classical metrics cannot handle. Using this framework, they established existence and uniqueness results for fixed points of mappings satisfying contractive conditions defined via the d-Young distance. Remarkably, by selecting an appropriate 5-tuple $(p, q, \tau, \kappa, \xi)$, their results recover the Banach fixed point theorem as a special case, highlighting the versatility and power of the d-Young framework for addressing a broad class of nonlinear phenomena.

Motivated by the idea of the d-Young distance, we extend this concept to introduce a d-Fenchel–Young distance by incorporating additional auxiliary functions and pairwise functions. Using this framework, we establish several fixed point results that generalize classical theorems, including the Banach fixed point theorem, as well as results related to the d-Young distance in [14]. By making appropriate choices of the parameters and functions, our generalized distance reduces to the classical metric or the d-Young distance. In other words, every classical metric or d-Young distance can be seen as a special case of the d-Fenchel–Young distance, but not every d-Fenchel–Young distance reduces to one of these forms. To highlight the power of this approach, we provide an example where previous results are not applicable due to unmet assumptions or contractive conditions, while our framework satisfies the necessary conditions and guarantees both the existence and uniqueness of a fixed point for the corresponding self-mapping.

2 d-Fenchel–Young distance functions

In this section, we first introduce the relevant parameters and functions, then define the d-Fenchel–Young distance, and discuss some of its key properties. We also provide nontrivial examples to illustrate its behavior and show that this distance generalizes the classical d-Young distance.

Throughout the paper, we denote by Θ the set of all functions $\alpha : [0, +\infty) \to [0, +\infty)$ that are monotonically increasing and invertible from $(0, +\infty)$ onto $(0, +\infty)$. Using this, we define Ξ_α as the set of all functions $\xi : [0, +\infty) \to [0, +\infty)$ satisfying

$$c_\alpha := \inf_{t>0} \frac{\xi(t)}{\alpha^{-1}(t)} > 0. \tag{1}$$

For a given set X, let $\mathcal{L}_1(X)$ denote the family of all mappings $\beta : X \times X \to [0, +\infty)$ satisfying

$$\beta(x, y) = 0 \quad \text{if and only if} \quad x = y. \tag{2}$$

Furthermore, we define

$$X^{\#} := \big\{(x, y) \in X \times X \colon x \neq y\big\}.$$

Remark 1. It follows from (2) that if $\beta \in \mathcal{L}_1(X)$, then for all $(x, y) \in X^{\#}$,

$$\beta(x, y) > 0.$$

We denote by $\mathcal{F}$ the class of functions $f : [0, +\infty) \to [0, +\infty)$ satisfying $f(0) = 0$ and $f(t) > 0$ for all $t > 0$. For $f \in \mathcal{F}$, its Fenchel conjugate $f^* : [0, +\infty) \to [0, +\infty]$ is defined by

$$f^*(t) = \sup_{x \geqslant 0} \big\{xt - f(t)\big\}$$

for all $t \in [0, +\infty)$. By the definition of the Fenchel conjugate, we have the Fenchel–Young inequality

$$st \leqslant f(s) + f^*(t) \tag{3}$$

for all $s, t \in [0, +\infty)$.

Table 1. Principal classes of mappings used throughout the paper.

Notation	Generic element	Membership conditions
Θ	$\alpha : [0,+\infty) \to [0,+\infty)$	Monotonically increasing and invertible from $(0,+\infty)$ onto $(0,+\infty)$
Ξ_α	$\xi : [0,+\infty) \to [0,+\infty)$	$c_\alpha := \inf_{t>0} \xi(t)/\alpha^{-1}(t) > 0$
$\mathcal{L}_1(X)$	$\beta : X \times X \to [0,+\infty)$	$\beta(x,y) = 0 \iff x = y$
$\mathcal{F}$	$f : [0,+\infty) \to [0,+\infty)$	$f(0) = 0$ and $f(t) > 0$ for all $t > 0$
$\mathcal{L}_{(\beta,f)}(X)$	$\gamma : X \times X \to [0,+\infty)$	$\beta(x,y) \leqslant \gamma(x,y)$ and $f^*(\gamma(x,y)) < +\infty$ for all $(x,y) \in X^\#$

Given a function $\beta \in \mathcal{L}_1(X)$ and $f \in \mathcal{F}$, we denote by $\mathcal{L}_{(\beta,f)}(X)$ the collection of mappings $\gamma : X \times X \to [0,+\infty)$ such that

$$\beta(x,y) \leqslant \gamma(x,y) \quad \text{and} \quad f^*(\gamma(x,y)) < +\infty$$

for all $(x,y) \in X^\#$.

Throughout this paper, (X,d) is assumed to be a metric space unless otherwise stated. For the convenience of the reader and to improve clarity, we summarize the main notations used throughout the paper in Table 1.

Using the aforementioned classes of functions on a metric space (X,d), we introduce the d-Fenchel–Young distance function as follows:

Definition 1. Let (X,d) be a metric space, $\alpha \in \Theta$, $\xi \in \Xi_\alpha$, $\beta \in \mathcal{L}_1(X)$, $f \in \mathcal{F}$, and $\gamma \in \mathcal{L}_{(\beta,f)}(X)$. Then we define the mapping $D_{\xi_{[\alpha,\beta,\gamma,f]}} : X^\# \to [0,+\infty)$ by

$$D_{\xi_{[\alpha,\beta,\gamma,f]}}(x,y) := \xi\left[f\left(\frac{\alpha(d(x,y))}{\beta(x,y)}\right) + f^*\big(\gamma(x,y)\big)\right] \tag{4}$$

for all $(x,y) \in X^\#$. We say that the mapping $D_{\xi_{[\alpha,\beta,\gamma,f]}}$ is a d-Fenchel–Young distance function with respect to $(\alpha,\beta,\gamma,\xi,f)$. For simplicity, we write D_ξ instead of $D_{\xi_{[\alpha,\beta,\gamma,f]}}$ whenever there is no ambiguity.

The positivity of the d-Fenchel–Young distance $D_{\xi_{[\alpha,\beta,\gamma,f]}}$ on $X^\#$ is an immediate consequence of (1) and (4). For completeness, we state it as the following proposition.

Proposition 1. *Let $D_{\xi_{[\alpha,\beta,\gamma,f]}}$ be the d-Fenchel–Young distance on a metric space (X,d). For all $(x,y) \in X^\#$, we have*

$$D_{\xi_{[\alpha,\beta,\gamma,f]}}(x,y) > 0.$$

The d-Fenchel–Young distance function is, in general, not symmetric. Its symmetry properties depend on the behavior of the pairwise functions β and γ that define the underlying distance framework. The following proposition characterizes the conditions under which the d-Fenchel–Young distance becomes symmetric.

Proposition 2. *Let $D_{\xi_{[\alpha,\beta,\gamma,f]}}$ be the d-Fenchel–Young distance on a metric space (X,d). Suppose that the pairwise functions β and γ are symmetric on $X^\#$. Then $D_{\xi_{[\alpha,\beta,\gamma,f]}}$ is symmetric, that is, for all $(x,y) \in X^\#$,*

$$D_{\xi_{[\alpha,\beta,\gamma,f]}}(x,y) = D_{\xi_{[\alpha,\beta,\gamma,f]}}(y,x).$$

Proof. For each $(x, y) \in X^{\#}$, using the symmetry of d, β, and γ, we obtain

$$\begin{aligned} D_{\xi_{[\alpha,\beta,\gamma,f]}}(x, y) &= \xi\left[f\left(\frac{\alpha(d(x, y))}{\beta(x, y)}\right) + f^*\big(\gamma(x, y)\big)\right] \\ &= \xi\left[f\left(\frac{\alpha(d(y, x))}{\beta(y, x)}\right) + f^*\big(\gamma(y, x)\big)\right] \\ &= D_{\xi_{[\alpha,\beta,\gamma,f]}}(y, x). \end{aligned}$$

□

In the following, we present several examples to illustrate our definition of the d-Fenchel–Young distance. In particular, by choosing suitable values for the parameters and functions involved, the classical d-Young distance can be recovered as a special case. To highlight this relationship, we begin with the following example.

Example 1. Let (X, d) be a metric space, and let $d_{\xi_{[p,q,\tau,\kappa]}}$ be a d-Young distance function with respect to $(p, q, \tau, \kappa, \xi)$, where $p, q > 1$ satisfy $1/p+1/q = 1$, $\tau : X\times X \to [0, +\infty)$ is a mapping satisfying $\tau(x, y) = 0$ if and only if $x = y$, $\kappa > 0$, and $\xi : [0, +\infty) \to [0, +\infty)$ satisfies

$$c_\kappa := \inf_{t>0} \frac{\xi(t)}{t^\kappa} > 0. \tag{5}$$

Recall that $d_{\xi_{[p,q,\tau,\kappa]}}$ is defined by

$$d_{\xi_{[p,q,\tau,\kappa]}}(x, y) = \xi\left[\frac{1}{p}\frac{(d(x, y))^{p/\kappa}}{(\tau(x, y))^p} + \frac{(\tau(x, y))^q}{q}\right]$$

for all $(x, y) \in X^{\#}$.

We now show that the above d-Young distance can be recovered as a particular case of the d-Fenchel–Young distance. To this end, define $\alpha, f : [0, +\infty) \to [0, +\infty)$ by

$$\alpha(t) = t^{1/\kappa}, \quad f(t) = \frac{t^p}{p}$$

for all $t \geqslant 0$ and set $\beta := \tau$ and $\gamma := \tau$. Then

$$c_\alpha := \inf_{t>0} \frac{\xi(t)}{\alpha^{-1}(t)} = \inf_{t>0} \frac{\xi(t)}{t^\kappa} \overset{(5)}{=} c_\kappa > 0.$$

Moreover, the Fenchel conjugate of f is a function $f^* : [0, +\infty) \to [0, +\infty]$ defined by

$$f^*(t) = \frac{t^q}{q}$$

for all $t \geqslant 0$. Consequently, for each $(x, y) \in X^{\#}$, we obtain

$$\begin{aligned} D_{\xi_{[\alpha,\beta,\gamma,f]}}(x, y) &= \xi\left[f\left(\frac{\alpha(d(x, y))}{\beta(x, y)}\right) + f^*\big(\gamma(x, y)\big)\right] \\ &= \xi\left[\frac{1}{p}\left(\frac{d(x, y)^{1/\kappa}}{\tau(x, y)}\right)^p + \frac{(\tau(x, y))^q}{q}\right] \end{aligned}$$

$$= \xi\left[\frac{1}{p}\frac{(d(x,y))^{p/\kappa}}{(\tau(x,y))^{p}} + \frac{(\tau(x,y))^{q}}{q}\right]$$
$$= d_{\xi_{[p,q,\tau,\kappa]}}(x,y).$$

Hence, the d-Young distance introduced in [14] is recovered as a particular case of the d-Fenchel–Young distance.

The preceding example shows that the d-Fenchel–Young distance extends the class of d-Young distance functions. We next illustrate the flexibility of this framework by showing that the resulting d-Fenchel–Young distance may also coincide with the original metric d. The following two examples provide different constructions that exhibit this property.

Example 2. Let (X,d) be a metric space. Define $\beta,\gamma : X\times X \to [0,+\infty)$ by

$$\beta(x,y) = \gamma(x,y) = \big(d(x,y)\big)^{1/2}$$

for all $x,y\in X$. Consider also the functions $\alpha, f, \xi : [0,+\infty)\to[0,+\infty)$ defined by

$$\alpha(t) = t, \quad f(t) = \frac{t^2}{2}, \quad \xi(t) = t$$

for all $t\geqslant 0$. Clearly, $\alpha\in\Theta$, $f\in\mathcal{F}$, $\xi\in\Xi_\alpha$, and

$$c_\alpha = \inf_{t>0}\frac{\xi(t)}{\alpha^{-1}(t)} = \inf_{t>0}\frac{t}{t} = 1.$$

The Fenchel conjugate of f is a function $f^*:[0,+\infty)\to[0,+\infty]$ defined by

$$f^*(t) = \sup_{s\geqslant 0}\big\{st - f(s)\big\} = \sup_{s\geqslant 0}\left\{st - \frac{s^2}{2}\right\} = \frac{t^2}{2}$$

for all $t\in[0,+\infty)$. Then for each $(x,y)\in X^{\#}$, the d-Fenchel–Young distance reduces to

$$\begin{aligned} D_{\xi_{[\alpha,\beta,\gamma,f]}}(x,y) &= \xi\left[f\left(\frac{\alpha(d(x,y))}{\beta(x,y)}\right) + f^*\big(\gamma(x,y)\big)\right] \\ &= f\left(\frac{d(x,y)}{(d(x,y))^{1/2}}\right) + f^*\big((d(x,y))^{1/2}\big) \\ &= f\big(\big(d(x,y)\big)^{1/2}\big) + f^*\big(\big(d(x,y)\big)^{1/2}\big) \\ &= \frac{((d(x,y))^{1/2})^2}{2} + \frac{((d(x,y))^{1/2})^2}{2} \\ &= d(x,y). \end{aligned}$$

Hence, for this choice of α, β, γ, f, and ξ, the d-Fenchel–Young distance coincides with the original metric d.

The next example provides another nontrivial construction for which the resulting d-Fenchel–Young distance still coincides with the original metric d.

Example 3. Let (X, d) be a metric space. Define $\beta, \gamma : X \times X \to [0, +\infty)$ by

$$\beta(x, y) = \gamma(x, y) = \big(d(x, y)\big)^{1/6}$$

for all $x, y \in X$. Consider also the functions $\alpha, f, \xi : [0, +\infty) \to [0, +\infty)$ defined by

$$\alpha(t) = t^{1/3}, \qquad f(t) = \frac{t^2}{2}, \qquad \xi(t) = t^3$$

for all $t \geqslant 0$. Clearly, $\alpha \in \Theta$, $f \in \mathcal{F}$, and $\xi \in \Xi_\alpha$. Since $\alpha^{-1}(t) = t^3$, we have

$$c_\alpha = \inf_{t>0} \frac{\xi(t)}{\alpha^{-1}(t)} = \inf_{t>0} \frac{t^3}{t^3} = 1.$$

The Fenchel conjugate of f is a function $f^* : [0, +\infty) \to [0, +\infty]$ defined by

$$f^*(t) = \sup_{s \geqslant 0} \big\{ st - f(s) \big\} = \sup_{s \geqslant 0} \left\{ st - \frac{s^2}{2} \right\} = \frac{t^2}{2}$$

for all $t \in [0, +\infty)$. Then for each $(x, y) \in X^\#$, the d-Fenchel–Young distance reduces to

$$\begin{aligned}
D_{\xi_{[\alpha,\beta,\gamma,f]}}(x, y) &= \xi \left[f\left(\frac{\alpha(d(x, y))}{\beta(x, y)} \right) + f^*(\gamma(x, y)) \right] \\
&= \xi \left[f\left(\frac{(d(x, y))^{1/3}}{(d(x, y))^{1/6}} \right) + f^*\big((d(x, y))^{1/6}\big) \right] \\
&= \xi \left[f\big((d(x, y))^{1/6}\big) + f^*\big((d(x, y))^{1/6}\big) \right] \\
&= \xi \left[\frac{\big((d(x, y))^{1/6}\big)^2}{2} + \frac{\big((d(x, y))^{1/6}\big)^2}{2} \right] \\
&= \xi\big((d(x, y))^{1/3}\big) = \big((d(x, y))^{1/3}\big)^3 \\
&= d(x, y).
\end{aligned}$$

Hence, for this choice of α, β, γ, f, and ξ, the d-Fenchel–Young distance coincides with the original metric d.

Although the preceding examples show that the d-Fenchel–Young distance may coincide with the original metric d, it does not, in general, constitute a metric, since the triangle inequality may fail, as demonstrated in Example 4. Nevertheless, the d-Fenchel–Young distance possesses certain fundamental properties, including positivity and, under suitable conditions, symmetry. Importantly, the triangle inequality is not required for the fixed point results developed in this paper.

Example 4. Let $X = \mathbb{R}$ be equipped with the usual metric d. Define $\beta, \gamma : X \times X \to [0, +\infty)$ by

$$\beta(x, y) = \gamma(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Consider also the functions $\alpha, f, \xi : [0, +\infty) \to [0, +\infty)$ defined by

$$\alpha(t) = t, \qquad f(t) = \frac{t^3}{3}, \qquad \xi(t) = t$$

for all $t \geqslant 0$. Clearly, $\alpha \in \Theta$, $f \in \mathcal{F}$, and $\xi \in \Xi_\alpha$. Moreover,

$$c_\alpha = \inf_{t>0} \frac{\xi(t)}{\alpha^{-1}(t)} = \inf_{t>0} \frac{t}{t} = 1.$$

The Fenchel conjugate of f is a function $f^* : [0, +\infty) \to [0, +\infty]$ defined by

$$f^*(t) = \sup_{s \geqslant 0} \{st - f(s)\} = \sup_{s \geqslant 0} \left\{ st - \frac{s^3}{3} \right\} = \frac{2}{3} t^{3/2}$$

for all $t \in [0, +\infty)$. Then for each $(x, y) \in X^\#$, we have

$$\begin{aligned} D_{\xi_{[\alpha,\beta,\gamma,f]}}(x, y) &= \xi \left[f\left(\frac{\alpha(d(x, y))}{\beta(x, y)} \right) + f^*\big(\gamma(x, y)\big) \right] \\ &= f\big(|x - y|\big) + f^*(1) = \frac{|x - y|^3}{3} + \frac{2}{3}. \end{aligned}$$

Now, take $x = 0$, $y = 1$, and $z = 2$. Then

$$\begin{aligned} D_{\xi_{[\alpha,\beta,\gamma,f]}}(x, y) &= \frac{1^3}{3} + \frac{2}{3} = 1, \\ D_{\xi_{[\alpha,\beta,\gamma,f]}}(y, z) &= \frac{1^3}{3} + \frac{2}{3} = 1, \\ D_{\xi_{[\alpha,\beta,\gamma,f]}}(x, z) &= \frac{2^3}{3} + \frac{2}{3} = \frac{10}{3}. \end{aligned}$$

Consequently,

$$D_{\xi_{[\alpha,\beta,\gamma,f]}}(x, z) = \frac{10}{3} > 2 = D_{\xi_{[\alpha,\beta,\gamma,f]}}(x, y) + D_{\xi_{[\alpha,\beta,\gamma,f]}}(y, z).$$

Therefore, the triangle inequality fails. Therefore, in general, a d-Fenchel–Young distance need not be a metric.

The failure of the triangle inequality is not restricted to identity-type choices in the construction, as illustrated by the following example.

Example 5. Let $X = \mathbb{R}$ be equipped with the usual metric d. Define $\beta, \gamma : X \times X \to [0, +\infty)$ by

$$\beta(x, y) = \gamma(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Consider also the functions $\alpha, f, \xi : [0, +\infty) \to [0, +\infty)$ defined by

$$\alpha(t) = t^{1/3}, \qquad f(t) = \frac{t^4}{4}, \qquad \xi(t) = t^3$$

for all $t \geqslant 0$. Clearly, $\alpha \in \Theta$, $f \in \mathcal{F}$, and $\xi \in \Xi_\alpha$. Since $\alpha^{-1}(t) = t^3$, we have

$$c_\alpha = \inf_{t>0} \frac{\xi(t)}{\alpha^{-1}(t)} = \inf_{t>0} \frac{t^3}{t^3} = 1.$$

The Fenchel conjugate of f is a function $f^* : [0, +\infty) \to [0, +\infty]$ defined by

$$f^*(t) = \sup_{s \geqslant 0} \{st - f(s)\} = \sup_{s \geqslant 0} \left\{ st - \frac{s^4}{4} \right\} = \frac{3}{4} t^{4/3}$$

for all $t \in [0, +\infty)$. Then for each $(x, y) \in X^\#$, we have

$$\begin{aligned} D_{\xi_{[\alpha,\beta,\gamma,f]}}(x, y) &= \xi \left[f\left(\frac{\alpha(d(x, y))}{\beta(x, y)} \right) + f^*\big(\gamma(x, y)\big) \right] \\ &= \xi \left[f\big(\big(d(x, y)\big)^{1/3} \big) + \frac{3}{4} \right] \\ &= \xi \left[\frac{(d(x, y))^{4/3}}{4} + \frac{3}{4} \right] \\ &= \left(\frac{(d(x, y))^{4/3}}{4} + \frac{3}{4} \right)^3. \end{aligned}$$

Now, take $x = 0$, $y = 1$, and $z = 2$. Then

$$D_{\xi_{[\alpha,\beta,\gamma,f]}}(0, 1) = D_{\xi_{[\alpha,\beta,\gamma,f]}}(1, 2) = 1,$$

whereas

$$\begin{aligned} D_{\xi_{[\alpha,\beta,\gamma,f]}}(0, 2) &= \left(\frac{2^{4/3}}{4} + \frac{3}{4} \right)^3 > 2 \\ &= D_{\xi_{[\alpha,\beta,\gamma,f]}}(0, 1) + D_{\xi_{[\alpha,\beta,\gamma,f]}}(1, 2). \end{aligned}$$

Therefore, the triangle inequality fails. Therefore, in general, a d-Fenchel–Young distance need not be a metric.

3 Some generalized fixed point results

In this section, we study the existence and uniqueness of fixed points for a self-mapping T on a metric space (X, d), where T satisfies the contractive condition involving the d-Fenchel–Young distance function. In particular, this result covers the Banach fixed point theorem as well as many other results from the literature. We start with the necessary definitions that will be used throughout this section.

A function $\mu : [0, +\infty) \to [0, +\infty)$ is called a (c)-comparison function (see, e.g., [15]) if it is nondecreasing and satisfies $\sum_{n=0}^{+\infty} \mu^n(t) < +\infty$ for all $t > 0$.

Example 6. A classical and simple example of a (c)-comparison function is the mapping $\mu : [0, +\infty) \to [0, +\infty)$ defined by $\mu(t) = \ell t$ for all $t \geqslant 0$, where $\ell \in [0, 1)$. Clearly, μ is nondecreasing on $[0, \infty)$ since for any $s, t \in [0, +\infty)$ with $s \leqslant t$, we obtain

$$\mu(s) = \ell s \leqslant \ell t = \mu(t).$$

Moreover, for each $t > 0$, it follows by induction that $\mu^n(t) = \ell^n t$ for all $n \in \mathbb{N}$. Since $0 \leqslant \ell < 1$, we obtain

$$\sum_{n=0}^{+\infty} \mu^n(t) = \sum_{n=0}^{+\infty} \ell^n t = t \sum_{n=0}^{+\infty} \ell^n = \frac{t}{1-\ell} < +\infty$$

for every $t > 0$. Therefore, μ is a (c)-comparison function.

Example 7. Let $\mu : [0, +\infty) \to [0, +\infty)$ be the function defined by

$$\mu(t) = a \arctan t$$

for all $t \geqslant 0$, where $a \in [0, 1)$ is a constant. Then μ is a (c)-comparison function. Indeed, since

$$\mu'(t) = \frac{a}{1+t^2} > 0$$

for all $t \geqslant 0$, the function μ is nondecreasing on $[0, +\infty)$. Moreover, using the fact that $\arctan t \leqslant t$ for all $t > 0$, we obtain

$$\mu(t) = a \arctan t \leqslant at$$

for all $t \in [0, \infty)$. Hence, for each $t > 0$, it follows by induction that $\mu^n(t) \leqslant a^n t$ for all $n \in \mathbb{N}$. Since $0 \leqslant a < 1$, we obtain

$$\sum_{n=0}^{+\infty} \mu^n(t) \leqslant \sum_{n=0}^{+\infty} a^n t = t \sum_{n=0}^{+\infty} a^n = \frac{t}{1-a} < +\infty$$

for every $t > 0$. Therefore, μ is a (c)-comparison function.

Remark 2. If $\mu : [0, +\infty) \to [0, +\infty)$ is a (c)-comparison function, then the following assertions hold:

(i) $\lim_{n\to+\infty} \mu^n(t) = 0$ for all $t > 0$;
(ii) $\mu(t) < t$ for all $t > 0$;
(iii) $\mu(0) = 0$;
(iv) μ is continuous at $t = 0$.

We begin the main results in this section by introducing the notion of a d-Fenchel–Young-type μ-contraction as follows.

Definition 2. Let (X,d) be a metric space, $\alpha \in \Theta$, $\xi \in \Xi_\alpha$, $\beta \in \mathcal{L}_1(X)$, $f \in \mathcal{F}$, $\gamma \in \mathcal{L}_{(\beta,f)}(X)$, and let μ be a (c)-comparison function. A self-mapping $T : X \to X$ is called a d-Fenchel–Young-type μ-contraction if

$$D_{\xi_{[\alpha,\beta,\gamma,f]}}(Tx,Ty) \leqslant \mu\big[D_{\xi_{[\alpha,\beta,\gamma,f]}}(x,y)\big] \tag{6}$$

for all $x,y \in X$ with $(Tx,Ty) \in X^{\#}$.

Our first main result presents a fundamental fixed point theorem for d-Fenchel–Young-type μ-contractions, giving clear conditions under which a self-mapping on a metric space not only possesses a fixed point but also ensures its uniqueness. This theorem extends the classical Banach fixed point theorem and related results for the d-Young distance, all within the broader framework of d-Fenchel–Young distances.

Theorem 1. *Let (X,d) be a complete metric space, $\alpha \in \Theta$, $\xi \in \Xi_\alpha$, $\beta \in \mathcal{L}_1(X)$, $f \in \mathcal{F}$, $\gamma \in \mathcal{L}_{(\beta,f)}(X)$, and μ be a (c)-comparison function. Suppose that $T : X \to X$ is a self-mapping satisfying the following conditions*:

(i) *T is a d-Fenchel–Young type μ-contraction*;
(ii) *There exists $x_0 \in X$ such that T is Picard subsequentially continuous at x_0, that is, if $T^n x_0 \to y$ for some $y \in X$, then the sequence $\{T^n x_0\}$ admits a subsequence $\{T^{n_j} x_0\}$ such that $T(T^{n_j} x_0) \to Ty$.*

Then T has a unique fixed point. Moreover, the sequence $\{T^n x_0\}$ converges to this fixed point.

Proof. Let $x_0 \in X$ be the point given by assumption (ii), and define the Picard sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \in \mathbb{N} \cup \{0\}$. If there exists some $m \in \mathbb{N} \cup \{0\}$ such that $x_m = x_{m+1}$, then $x_m = Tx_m$ and hence x_m is a fixed point of T. Moreover, $x_n = x_m$ for all $n \geqslant m$ and so the sequence $\{x_n\}$ converges to x_m. To prove uniqueness, suppose that y is another fixed point of T with $y \neq x_m$. By Proposition 1, we obtain $D_\xi(x_m,y) > 0$. By the contractive condition and Remark 2(ii), we obtain

$$D_\xi(x_m,y) = D_\xi(Tx_m,Ty) \leqslant \mu\big(D_\xi(x_m,y)\big) < D_\xi(x_m,y),$$

which is a contradiction. Therefore, x_m is the unique fixed point of T, and the desired conclusion follows.

For the remainder of the proof, we may therefore assume that $x_n \neq x_{n+1}$. Applying the contractive condition (6) to the pair (x_0,x_1), we obtain

$$D_{\xi_{[\alpha,\beta,\gamma,f]}}(Tx_0,Tx_1) \leqslant \mu\big(D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_0,x_1)\big),$$

that is,

$$D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_1,x_2) \leqslant \mu\big(D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_0,x_1)\big).$$

Next, applying (6) to the pair (x_1,x_2) and using the monotonicity of μ, we obtain

$$\begin{aligned} D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_2,x_3) = D_{\xi_{[\alpha,\beta,\gamma,f]}}(Tx_1,Tx_2) &\leqslant \mu\big(D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_1,x_2)\big) \\ &\leqslant \mu^2\big(D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_0,x_1)\big). \end{aligned}$$

Proceeding inductively, we obtain

$$D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_n, x_{n+1}) \leqslant \mu^n \big(D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_0, x_1)\big) \tag{7}$$

for all $n \in \mathbb{N} \cup \{0\}$.

On the other hand, by (4), for each $n \in \mathbb{N} \cup \{0\}$, we have

$$D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_n, x_{n+1}) = \xi\left(f\left(\frac{\alpha(d(x_n, x_{n+1}))}{\beta(x_n, x_{n+1})}\right) + f^*\big(\gamma(x_n, x_{n+1})\big)\right).$$

Since $\xi \in \Xi_\alpha$, it follows from (1) that

$$D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_n, x_{n+1}) \geqslant c_\alpha \alpha^{-1}\left(f\left(\frac{\alpha(d(x_n, x_{n+1}))}{\beta(x_n, x_{n+1})}\right) + f^*\big(\gamma(x_n, x_{n+1})\big)\right).$$

Equivalently,

$$\alpha^{-1}\left(f\left(\frac{\alpha(d(x_n, x_{n+1}))}{\beta(x_n, x_{n+1})}\right) + f^*\big(\gamma(x_n, x_{n+1})\big)\right) \leqslant \frac{1}{c_\alpha} D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_n, x_{n+1}) \tag{8}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since $\beta \leqslant \gamma$, it follows from Remark 1 that

$$\alpha\big(d(x_n, x_{n+1})\big) \leqslant \frac{\alpha(d(x_n, x_{n+1}))}{\beta(x_n, x_{n+1})} \gamma(x_n, x_{n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$. For each in $n \in \mathbb{N} \cup \{0\}$, set

$$a_n := \frac{\alpha(d(x_n, x_{n+1}))}{\beta(x_n, x_{n+1})} \quad \text{and} \quad b_n := \gamma(x_n, x_{n+1}).$$

By the Fenchel–Young inequality (3), we obtain

$$\begin{aligned}\alpha\big(d(x_n, x_{n+1})\big) &\leqslant a_n b_n \leqslant f(a_n) + f^*(b_n) \\ &= f\left(\frac{\alpha(d(x_n, x_{n+1}))}{\beta(x_n, x_{n+1})}\right) + f^*\big(\gamma(x_n, x_{n+1})\big)\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since α^{-1} is increasing, we obtain

$$d(x_n, x_{n+1}) \leqslant \alpha^{-1}\left(f\left(\frac{\alpha(d(x_n, x_{n+1}))}{\beta(x_n, x_{n+1})}\right) + f^*\big(\gamma(x_n, x_{n+1})\big)\right) \tag{9}$$

for all $n \in \mathbb{N} \cup \{0\}$. Combining (8) and (9), we obtain

$$d(x_n, x_{n+1}) \leqslant \frac{1}{c_\alpha} D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_n, x_{n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$. Hence, by (7), we have

$$d(x_n, x_{n+1}) \leqslant \frac{1}{c_\alpha} \mu^n \big(D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_0, x_1)\big) \tag{10}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Next, using (10) together with the triangle inequality, for each $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$, we obtain

$$\begin{aligned} d(x_n, x_{n+m}) &\leqslant \sum_{i=n}^{n+m-1} d(x_i, x_{i+1}) \leqslant \frac{1}{c_\alpha} \sum_{i=n}^{n+m-1} \mu^i \big(D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_0, x_1)\big) \\ &\leqslant \frac{1}{c_\alpha} \sum_{i=n}^{+\infty} \mu^i \big(D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_0, x_1)\big). \end{aligned}$$

Hence,

$$d(x_n, x_{n+m}) \leqslant \frac{1}{c_\alpha} \sum_{i=n}^{+\infty} \mu^i \big(D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_0, x_1)\big) \tag{11}$$

for all $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$. Since μ is a (c)-comparison function, we obtain

$$\sum_{i=0}^{+\infty} \mu^i \big(D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_0, x_1)\big) < +\infty.$$

It implies that

$$\lim_{n \to +\infty} \sum_{i=n}^{+\infty} \mu^i \big(D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_0, x_1)\big) = 0.$$

Therefore, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geqslant N$,

$$\frac{1}{c_\alpha} \sum_{i=n}^{+\infty} \mu^i \big(D_{\xi_{[\alpha,\beta,\gamma,f]}}(x_0, x_1)\big) < \varepsilon.$$

It then follows from (11) that

$$d(x_n, x_{n+m}) < \varepsilon$$

for all $n \geqslant N$ and $m \in \mathbb{N}$. Hence, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $x^* \in X$ such that

$$\lim_{n \to +\infty} d(x_n, x^*) = 0.$$

By condition (ii), there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{j \to +\infty} d(Tx_{n_j}, Tx^*) = 0.$$

On the other hand, since

$$Tx_{n_j} = x_{n_j+1}$$

and $\{x_n\}$ converges to x^*, we have

$$\lim_{j \to +\infty} d(Tx_{n_j}, x^*) = 0.$$

By the uniqueness of the limit in the metric space (X, d), it follows that

$$Tx^* = x^*.$$

Thus, x^* is a fixed point of T.

It remains to prove uniqueness. Suppose that x^{**} is a fixed point of T with $x^{**} \neq x^*$. By Proposition 1, we have

$$D_{\xi_{[\alpha,\beta,\gamma,f]}}(x^*, x^{**}) > 0.$$

Applying the d-Fenchel–Young μ-contractive condition (6) to the pair (x^*, x^{**}), we obtain

$$\begin{aligned} D_{\xi_{[\alpha,\beta,\gamma,f]}}(x^*, x^{**}) = D_{\xi_{[\alpha,\beta,\gamma,f]}}(Tx^*, Tx^{**}) &\leqslant \mu\big(D_{\xi_{[\alpha,\beta,\gamma,f]}}(x^*, x^{**})\big) \\ &< D_{\xi_{[\alpha,\beta,\gamma,f]}}(x^*, x^{**}), \end{aligned}$$

where the last inequality follows from Remark 2(ii). This is a contradiction. Therefore, T has a unique fixed point. □

If T is Picard continuous or continuous, then condition (ii) of Theorem 1 is automatically satisfied. Consequently, we obtain the following corollary.

Corollary 1. *Let (X, d) be a complete metric space, $\alpha \in \Theta$, $\xi \in \Xi_\alpha$, $\beta \in \mathcal{L}_1(X)$, $f \in \mathcal{F}$, $\gamma \in \mathcal{L}_{(\beta,f)}(X)$, and let μ be a (c)-comparison function. Suppose that $T : X \to X$ satisfies the following conditions*:

(i) *T is a d-Fenchel–Young type μ-contraction*;
(ii) *T is Picard continuous or continuous with respect to d.*

Then T has a unique fixed point in X. Moreover, for any initial point $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.

We next demonstrate how Theorem 1 encompasses several known fixed point results as particular cases. In particular, it recovers the fixed point theorem for d-Young contractions established in [14], as well as the fixed point results for μ-contractions [2] and the Banach fixed point theorem [1]. Thus, Theorem 1 provides a unified framework that includes these results, among others, from the existing literature. We begin with the following corollary.

Corollary 2. (*See* [14].) *Let (X, d) be a complete metric space, $1 < p, q < +\infty$ with $1/p + 1/q = 1$, $\tau \in \mathcal{L}_1(X)$, $\kappa > 0$, $\xi \in \Xi_\kappa$, and let μ be a (c)-comparison function. Suppose that $T : X \to X$ satisfies the following conditions*:

(i) *T is a d-Young μ-contraction*;
(ii) *condition* (ii) *of Theorem* 1 *holds*; *alternatively, T is Picard continuous or continuous with respect to d.*

Then T has a unique fixed point in X. Moreover, for any initial point $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.

Proof. By choosing the functions as in Example 1, the d-Fenchel–Young distance reduces to the d-Young distance. Hence, the conclusion follows directly from Theorem 1. □

We further show that the classical fixed point results for μ-contractions and Banach contractions are also recovered as direct consequences of our main theorem.

Corollary 3. (*See* [2].) *Let (X,d) be a complete metric space, and let μ be a* (c)*-comparison function. Suppose that $T : X \to X$ is a mapping satisfying*

$$d(Tx,Ty) \leqslant \mu\big(d(x,y)\big) \tag{12}$$

for all $x, y \in X$. Then T has a unique fixed point, and for any initial point $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to this fixed point.

Proof. By adopting the setting of Example 2 (or Example 3), we have $D_{\xi_{[\alpha,\beta,\gamma,f]}}(x,y) = d(x,y)$ for all $x, y \in X$. Hence, (12) yields

$$D_{\xi_{[\alpha,\beta,\gamma,f]}}(Tx,Ty) \leqslant \mu\big(D_{\xi_{[\alpha,\beta,\gamma,f]}}(x,y)\big)$$

whenever $(Tx,Ty) \in X^{\#}$. Moreover, by Remark 2(ii), $\mu(t) < t$ for every $t > 0$. Thus, for $x \neq y$,

$$d(Tx,Ty) \leqslant \mu\big(d(x,y)\big) < d(x,y),$$

which shows that T is continuous with respect to d. Therefore, the conclusion follows from Corollary 1. □

Corollary 4. (*See* [1].) *Let (X,d) be a complete metric space, and let $T : X \to X$ be a mapping satisfying*

$$d(Tx,Ty) \leqslant \ell d(x,y)$$

for all $x, y \in X$, where $\ell \in [0,1)$ is a constant. Then T has a unique fixed point, and for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to this fixed point.

Proof. Define $\mu : [0,+\infty) \to [0,+\infty)$ by $\mu(t) = \ell t$ for all $t \geqslant 0$. By Example 6, μ is a (c)-comparison function. Hence, the conclusion follows directly from Corollary 3. □

We conclude this section with an example illustrating the applicability of Theorem 1. This example is particularly noteworthy because the self-mapping considered below is neither a Banach contraction with respect to the metric d nor a Banach-type contraction with respect to the d-Young distance $d_{\xi_{[p,q,\tau,\kappa]}}$. Nevertheless, by modifying only the functions β and γ while retaining the remaining choices associated with the d-Young distance, the mapping becomes a d-Fenchel–Young-type μ-contraction with respect to $D_{\xi_{[\alpha,\beta,\gamma,f]}}$.

Example 8. Let $X = \{1,2,3\}$ be a finite metric space with a metric d given by

$$d(1,2) = d(2,1) = 2, \qquad d(1,3) = d(3,1) = 3, \qquad d(2,3) = d(3,2) = 4,$$
$$d(i,i) = 0 \quad \text{for all } i = 1,2,3.$$

Define a self-mapping T on X by

$$T(1) = 1, \qquad T(2) = 3, \qquad T(3) = 1.$$

We begin by selecting the parameters $p = q = 2$, $\kappa = 2$, $\xi(t) = t^2$ for all $t \geqslant 0$, and

$$\tau(i,j) = \begin{cases} 1, & i \neq j, \\ 0, & i = j. \end{cases}$$

With these choices, we have $c_\kappa := 1 > 0$. In this setting, we observe that the mapping T does not satisfy the Banach-type contractive condition in the associated d-Young distance framework. Indeed, for all $(i,j) \in X^{\#}$, we have

$$d_{\xi_{[p,q,\tau,\kappa]}}(i,j) = \xi\left[\frac{1}{2}\frac{d(i,j)^{p/\kappa}}{\tau(i,j)^p} + \frac{1}{2}\tau(i,j)^q\right] = \left(\frac{1}{2}d(i,j) + \frac{1}{2}\right)^2,$$

which gives

$$d_{\xi_{[p,q,\tau,\kappa]}}(1,2) = 2.25, \qquad d_{\xi_{[p,q,\tau,\kappa]}}(1,3) = 4, \qquad d_{\xi_{[p,q,\tau,\kappa]}}(2,3) = 6.25.$$

Under this d-Young distance, we have

$$d_{\xi_{[p,q,\tau,\kappa]}}\big(T(1),T(2)\big) = d_{\xi_{[p,q,\tau,\kappa]}}(1,3) = 4 > 2.25 = d_{\xi_{[p,q,\tau,\kappa]}}(1,2). \tag{13}$$

Since $\mu(t) < t$ for every $t > 0$ whenever μ is a (c)-comparison function, (13) shows that T cannot satisfy the d-Young μ-contractive condition. Moreover,

$$d\big(T(1),T(2)\big) = d(1,3) = 3 > 2 = d(1,2).$$

Hence, T is not a Banach contraction with respect to the original metric d. For the same reason, T cannot be a μ-contraction with respect to d, since otherwise

$$3 = d\big(T(1),T(2)\big) \leqslant \mu\big(d(1,2)\big) = \mu(2) < 2,$$

which is impossible. Consequently, the fixed point results in [1, 2, 14] are not applicable to this mapping under the corresponding settings.

Next, define the symmetric functions $\beta, \gamma : X \times X \to [0, +\infty)$ by

$$\begin{aligned} &\beta(1,2) = 0.1, \qquad &&\beta(1,3) = \gamma(1,3) = 3^{1/4}, \\ &\gamma(1,2) = 10, \qquad &&\beta(2,3) = \gamma(2,3) = 4^{1/4} \end{aligned}$$

together with

$$\beta(i,j) = \beta(j,i), \qquad \gamma(i,j) = \gamma(j,i)$$

for all $i, j \in X$ and

$$\beta(i,i) = \gamma(i,i) = 0$$

for all $i \in X$. Also, $\alpha, \xi, f : [0, +\infty) \to [0, +\infty)$ are defined by

$$\alpha(t) = \sqrt{t}, \quad f(t) = \frac{t^2}{2}, \quad \text{and} \quad \xi(t) = t^2 \quad \text{for all } t \geqslant 0.$$

Clearly, $\alpha \in \Theta$. Moreover, $\alpha^{-1}(t) = t^2$ for all $t \geqslant 0$, and hence

$$c_\alpha = \inf_{t>0} \frac{\xi(t)}{\alpha^{-1}(t)} = \inf_{t>0} \frac{t^2}{t^2} = 1 > 0.$$

Thus, $\xi \in \Xi_\alpha$. Furthermore, $\beta \in \mathcal{L}_1(X)$ and $\gamma \in \mathcal{L}_{(\beta,f)}(X)$, since $\beta(x,y) = 0$ if and only if $x = y$, and the following conditions hold for each $(x,y) \in X^\#$:

$$\beta(x,y) \leqslant \gamma(x,y) \quad \text{and} \quad f^*(\gamma(x,y)) < +\infty.$$

Finally, $f \in \mathcal{F}$ follows from $f(0) = 0$ and $f(t) > 0$ for all $t > 0$.

From the definition of f its Fenchel conjugate f^* is given by $f^*(t) = t^2/2$ for all $t \geqslant 0$. Therefore, for each $(i,j) \in X^\#$, the corresponding d-Fenchel–Young distance is given by

$$\begin{aligned} D_{\xi_{[\alpha,\beta,\gamma,f]}}(i,j) &= \xi\left[f\left(\frac{\alpha(d(i,j))}{\beta(i,j)}\right) + f^*\big(\gamma(i,j)\big)\right] \\ &= \left[\frac{d(i,j)}{2\beta^2(i,j)} + \frac{\gamma^2(i,j)}{2}\right]^2. \end{aligned}$$

Consequently,

$$\begin{aligned} D_{\xi_{[\alpha,\beta,\gamma,f]}}(1,2) &= \left[\frac{2}{2(0.1)^2} + \frac{10^2}{2}\right]^2 = 22500, \\ D_{\xi_{[\alpha,\beta,\gamma,f]}}(1,3) &= \left[\frac{3}{2(3^{1/4})^2} + \frac{(3^{1/4})^2}{2}\right]^2 = 3, \\ D_{\xi_{[\alpha,\beta,\gamma,f]}}(2,3) &= \left[\frac{4}{2(4^{1/4})^2} + \frac{(4^{1/4})^2}{2}\right]^2 = 4. \end{aligned}$$

By symmetry, we have

$$D_{\xi_{[\alpha,\beta,\gamma,f]}}(2,1) = 22500, \qquad D_{\xi_{[\alpha,\beta,\gamma,f]}}(3,1) = 3, \qquad D_{\xi_{[\alpha,\beta,\gamma,f]}}(3,2) = 4.$$

Observe that the only pairs $(i,j) \in X \times X$ for which $(T(i),T(j)) \in X^\#$ are $(1,2)$, $(2,1)$, $(2,3)$, $(3,2)$. Therefore, it suffices to verify the contractive condition for these four pairs. For simplicity, throughout the remainder of this example, we write

$$D_\xi := D_{\xi_{[\alpha,\beta,\gamma,f]}}.$$

The corresponding values for the relevant pairs are summarized in Table 2.

Table 2. Values of $D_\xi(i,j)$, $D_\xi(T(i),T(j))$, and their ratios for all relevant pairs (i,j) satisfying $(T(i),T(j)) \in X^\#$.

(i,j)	$D_\xi(i,j)$	$D_\xi(T(i),T(j))$	Ratio: $\frac{D_\xi(T(i),T(j))}{D_\xi(i,j)}$
$(1,2)$	22500	3	$1.33 \cdot 10^{-4}$
$(2,1)$	22500	3	$1.33 \cdot 10^{-4}$
$(2,3)$	4	3	0.75
$(3,2)$	4	3	0.75

From Table 2 we observe that the largest ratio is $\ell = 0.75 < 1$. Define $\mu : [0, +\infty) \to [0, +\infty)$ by $\mu(t) = \ell t$ for all $t \geqslant 0$. By Example 6, μ is a (c)-comparison function. Moreover, Table 2 shows that

$$D_\xi\big(T(i), T(j)\big) \leqslant \ell D_\xi(i, j) = \mu\big(D_\xi(i, j)\big)$$

for all $(i, j) \in X \times X$ such that $(T(i), T(j)) \in X^{\#}$. Hence, T is a d-Fenchel–Young-type μ-contraction. Since every finite metric space is discrete, T is continuous with respect to d. Therefore, all the assumptions of Theorem 1 are satisfied, and hence T has a unique fixed point in X. In fact, the unique fixed point of T is 1.

4 Conclusion and future directions

In this paper, we introduced the d-Fenchel–Young distance generated by the five-tuple $(\alpha, \beta, \gamma, \xi, f)$ and investigated its fundamental properties together with its applications to fixed point theory. The proposed distance unifies both the classical metric and the d-Young distance within a single framework, while offering substantially greater flexibility for modeling contractive behavior through auxiliary pairwise functions. Based on μ-comparison contractive conditions, we established existence and uniqueness theorems that extend several classical fixed point results. In particular, appropriate choices of the five-tuple recover the Banach fixed point theorem, the fixed point theorem for μ-contractions, and the corresponding fixed point theory in the d-Young distance setting. Furthermore, we presented an example demonstrating that our framework remains applicable even in situations where neither the Banach contraction principle nor the classical d-Young approach can be applied.

The proposed framework also suggests several promising directions for future research. One natural direction is to investigate its role in the construction and analysis of fractal structures, especially within iterated function systems. Another is to extend the theory to multivalued mappings by combining the d-Fenchel–Young distance with suitable Hausdorff-type distances and comparison principles. It is also of interest to develop analogous theories in fuzzy, intuitionistic fuzzy, probabilistic, and other uncertainty-based metric environments. Moreover, incorporating admissibility conditions, simulation functions, or other nonlinear control mechanisms may lead to broader classes of contractive mappings and more general fixed point principles. We believe that the d-Fenchel–Young distance provides a versatile foundation for future developments in generalized metric analysis, nonlinear dynamics, fractal geometry, optimization, and mathematical models involving uncertainty.

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References

1. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundam. Math.*, **3**(1):133–181, 1922, https://doi.org/10.4064/fm-3-1-133-181.
2. V. Berinde, *Generalized Contractions and Applications*, Cub Press, Baia Mare, 1997 (in Romanian).
3. V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, *Nonlinear Anal. Forum*, **9**(1):43–53, 2004.
4. J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, *Trans. Am. Math. Soc.*, **215**:241–251, 1976, https://doi.org/10.1090/S0002-9947-1976-0394329-4.
5. S.K. Chatterjea, Fixed point theorems, *C. R. Acad. Bulg. Sci.*, **25**:727–730, 1972.
6. L.B. Ćirić, A generalization of Banach's contraction principle, *Proc. Am. Math. Soc.*, **45**(2): 267–273, 1974, https://doi.org/10.1090/S0002-9939-1974-0356011-2.
7. G.E. Hardy, T.D. Rogers, A generalization of a fixed point theorem of Reich, *Can. Math. Bull.*, **16**(2):201–206, 1973, https://doi.org/10.4153/CMB-1973-036-0.
8. M. Jleli, B. Samet, A generalized metric space and related fixed point theorems, *Fixed Point Theory Appl.*, **2015**:61, 2015, https://doi.org/10.1186/s13663-015-0312-7.
9. O. Kada, T. Suzuki, W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, *Math. Japon.*, **44**(2):381–391, 1996.
10. R. Kannan, Some results on fixed points. II, *Am. Math. Mon.*, **76**(4):405–408, 1969, https://doi.org/10.2307/2316437.
11. S. Kasahara, On some generalizations of the Banach contraction theorem, *Publ. Res. Inst. Math. Sci.*, **12**(2):427–437, 1976, https://doi.org/10.2977/PRIMS/1195190723.
12. F. Khojasteh, S. Shukla, S. Radenović, A new approach to the study of fixed point theory for simulation functions, *Filomat*, **29**(6):1189–1194, 2015, https://doi.org/10.2298/FIL1506189K.
13. M.G. Maia, Un'osservazione sulle contrazioni metriche, *Rend. Sem. Mat. Univ. Padova*, **40**: 139–143, 1968.
14. V. Rakočević, B. Samet, d-Young distance function and existence of fixed points in complete metric spaces, *Aequat. Math.*, **99**(4):1625–1639, 2025, https://doi.org/10.1007/s00010-024-01144-3.
15. I.A. Rus, *Generalized Contractions and Applications*, Cluj Univ. Press, Cluj-Napoca, 2001.
16. T. Suzuki, Generalized distance and existence theorems in complete metric spaces, *J. Math. Anal. Appl.*, **253**(2):440–458, 2001, https://doi.org/10.1006/jmaa.2000.7151.
17. D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012**:94, 2012, https://doi.org/10.1186/1687-1812-2012-94.
18. T. Zamfirescu, Fix point theorems in metric spaces, *Arch. Math.*, **23**(1):292–298, 1972, https://doi.org/10.1007/BF01304884.