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Linear map preserving modular frames

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Abstract. A frame can be thought of as a redundant coordinate system in the sense that it can contain more vectors that are necessary to represent each vector in a Hilbert space. In this paper, we aim to study the stability of $*$-$\mathcal{K}$-modular frames under some class of closed range operators.

Keywords: $*$-$\mathcal{K}$-frame, regular operator, minimal operator, pseudoinverse.

1 Introduction

In 1946, Gabor [13] proposed a decomposition of a signal in terms of elementary signals that displays simultaneously the local time and frequency content of the signal, in contrast to the classical Fourier transform, which provides only the global frequency content for the entire signal. Building on this development, Duffin and Schaeffer in 1952 [9] further contributed to the theory by addressing some problems related to the nonharmonic Fourier series. However, the frame theory had not attracted much attention until the landmark work by Daubechies, Crossman, and Meyer [8] because of their applications in wavelet analysis. Frames generalize orthonormal bases by allowing redundant, yet stable representations of vectors in a Hilbert space. For a dynamical system in Hilbert spaces, see [23]. Unlike bases, frames need not be linearly independent or complete, which makes them more flexible in various applications like signal processing [3], especially in connection with sampling theory [24], probability statistics [10], and much more.

To further apply this theory, many generalizations of frames have been proposed, for examples, frames of subspaces [4]. Also, frames and their generalization in Hilbert C^*-modules and locally compact abelian groups have been studied in [2]. It is well known that Hilbert C^*-modules are essentially objects like Hilbert spaces, except that

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the inner product, instead of being complex-valued, takes its values in a C^*-algebra. Since the geometry of these modules emerges from the C^*-valued inner product, some basic properties of Hilbert spaces like self-duality and decomposition into orthogonal complements do not hold.

A fundamental question is how to construct $*$-$\mathcal{K}$-modular frames from given $*$-$\mathcal{K}$-modular frames. Dastourian and Janfadaa [7] introduced $*$-$\mathcal{K}$-frames on Hilbert C^*-modules as a generalization of $\mathcal{K}$-frames and obtained some relations between $*$-$\mathcal{K}$-frames and $*$-atomic systems with respect to an adjointable operator and some characterizations of $*$-$\mathcal{K}$-frames. In this research, motivated by their results on $*$-$\mathcal{K}$-frames on Hilbert C^*-modules, we are seeking some closed range operators that allow the generation of $*$-$\mathcal{K}$-frames in Hilbert C^*-modules.

The outline of paper is organized as follows. In the next section, we establish the notation, and recall both some well known definitions and the main results from the literature we need. Section 3 focuses on constructing some $*$-$\mathcal{K}$-frames under various operators, such unitary operators and regular operators. Furthermore, we generate some $*$-$\mathcal{K}$-modular frames by means of minimal projections involving the Moore–Penrose inverse of a closed range operator.

2 Mathematical preliminaries

In the section, we present a brief account of basic definitions and some properties of Hilbert C^*-modules and their frames and operator theory that will be needed later. For further details and tools on frames, one can refer to [5, 19–21, 25, 27].

Let $\mathcal{A}$ be a unital C^*-algebra and $a \in \mathcal{A}$. The nonzero element a is called strictly nonzero if zero does not belong to the spectrum of a, and if $a \leqslant b$, then $cac^* \leqslant cbc^*$ for each $c \in \mathcal{A}$.

A linear space $\mathcal{H}$ equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ and endowed with the structure of an algebraic (left) $\mathcal{A}$-module is called a pre-Hilbert C^*-module if it satisfies the following properties (see [16]):

(i) $\langle x, x \rangle \geqslant 0$ with equality only when $x = 0$;
(ii) $\langle ax + y,\ z \rangle = a\langle x, z \rangle + \langle y, z \rangle$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$;
(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in \mathcal{H}$.

If $\mathcal{H}$ is a Banach space with respect to the norm induced by the $\mathcal{A}$-valued inner product, then $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is called a Hilbert C^*-module over $\mathcal{A}$.

The standard Hilbert $\mathcal{A}$-module $l^2(\mathcal{A})$ is defined by

$$l^2(\mathcal{A}) = \left\{ \{a_j\}_{j \in \mathbb{J}} \subseteq \mathcal{A} \colon \sum_{j \in \mathbb{J}} a_j a_j^* \text{ converges in } \mathcal{A} \right\}$$

with pointwise operations and the inner product

$$\left\langle \{a_j\}_{j \in \mathbb{J}}, \{b_j\}_{j \in \mathbb{J}} \right\rangle = \sum_{j \in \mathbb{J}} a_j b_j^*.$$

The notion of frames for Hilbert spaces was extended by Frank and Larson [12] to the notion of frames in Hilbert $\mathcal{A}$-modules. Then Alijani and Dehghan [2] defined $*$-frame for a Hilbert $\mathcal{A}$-module $\mathcal{H}$ as a family $\{h_j\}_{j\in\mathbb{J}}$ satisfying

$$\alpha\langle h,h\rangle\alpha^* \leqslant \sum_{j\in\mathbb{J}}\langle h,h_j\rangle\langle h_j,h\rangle \leqslant \beta\langle h,h\rangle\beta^*,$$

for all $h\in\mathcal{H}$ and some strictly nonzero elements α and β in $\mathcal{A}$.

Afterwards, Dastourian and Janfadaa [7] developed a new version of frames for Hilbert $\mathcal{A}$-modules, called $*$-$\mathcal{K}$-frames, defined as the family $\{h_j\}_{j\in\mathbb{J}}$ in a Hilbert $\mathcal{A}$-module $\mathcal{H}$, which satisfies

$$\alpha\langle \mathcal{K}^*h,\mathcal{K}^*x\rangle\alpha^* \leqslant \sum_{j\in\mathbb{J}}\langle h,h_j\rangle\langle h_j,h\rangle \leqslant \beta\langle h,h\rangle\beta^* \quad \forall h\in\mathcal{H}.$$

Throughout the paper, we denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded adjointable operators on $\mathcal{H}$ and by $\mathcal{CR}(\mathcal{H})$ the set of all closed-range operators on $\mathcal{H}$. For an operator $A\in\mathcal{L}(\mathcal{H})$, we denote by $\mathcal{R}(A)$ and $\mathcal{N}(A)$ the range and kernel of A, respectively, and by I is the identity operator. As usual, the orthogonal projection onto a submodule E is denoted by π_E. We denote the orthogonal complement of E in $\mathcal{H}$ by $E^\perp=\{y\in\mathcal{H}\colon \langle x,y\rangle=0\ \forall x\in\mathcal{H}\}$. We also write $\mathcal{R}^\infty(A)=\bigcap_{n\geqslant 0}\mathcal{R}(A^n)$ for the generalized range.

The following results are key tools for the proofs of our main results.

Lemma 1. (See [22].) *Let $A\in\mathcal{L}(\mathcal{H})$. Then*

$$\langle Ah,Ah\rangle \leqslant \|A\|^2\langle h,h\rangle \quad \forall h\in\mathcal{H}.$$

Lemma 2. (*See* [29].) *Let $A,B\in\mathcal{L}(\mathcal{H})$ such that $\mathcal{R}(B)$ is closed. Then the following statements are equivalent*:

(i) $\mathcal{R}(A)\subseteq\mathcal{R}(B)$;

(ii) $c\langle A^*h,A^*h\rangle \leqslant \langle B^*h,B^*h\rangle$ *for some $c>0$ and every $h\in\mathcal{H}$.*

Lemma 3. (*See* [26].) *Suppose that $A,B\in\mathcal{L}(\mathcal{H})$ are operators with closed ranges such that $AB=BA$. Then $\mathcal{R}(AB)$ is closed.*

The following quantity is closely connected with the closedness of the range [6]:

$$\gamma(A):=\inf\bigl\{\|Ah\|\colon\ h\in\mathcal{H},\ \operatorname{dist}\bigl(h,\mathcal{N}(A)\bigr)=1\bigr\}.$$

Formally, we set $\gamma(0):=\infty$. Clearly, $\gamma(A)>0$ if and only if $\mathcal{R}(A)$ is closed.

Example 1. Let $A\in\mathcal{L}(\mathbb{C}^3)$ be defined as follows:

$$A\colon\ \mathbb{C}^3\to\mathbb{C}^3,\ (h_1,h_2,h_3)\mapsto(h_1,h_1,h_1).$$

For $h=(h_1,h_2,h_3)\in\mathbb{C}^3$, we have

$$\|Ah\|=\sqrt{3}|h_1| \quad\text{and}\quad \operatorname{dist}\bigl(h,\mathcal{N}(A)\bigr)=|h_1|.$$

Then

$$\gamma(A)=\sqrt{3}.$$

Following Mbekhta [17], an operator $A \in \mathcal{L}(\mathcal{H})$ is said to be regular if $\mathcal{R}(A)$ is closed and $\mathcal{N}(A) \subset \mathcal{R}(A^n)$ for every $n \geqslant 1$.

Example 2. All surjective operators are regular. Some examples of regular operators may be found in [15].

Next, we collect below some useful properties related to regular operators.

Proposition 1. (*See* [1].) *Let $A \in \mathcal{L}(\mathcal{H})$ be regular. Then*

(i) $\mathcal{R}^\infty(A)$ *is closed*;
(ii) $\mathcal{R}^\infty(A - zI) = \mathcal{R}^\infty(A) = A(\mathcal{R}^\infty(A))$ *for all* $|z| < \gamma(A)$.

In [28], recall that the pseudoinverse of an operator $A \in \mathcal{L}(\mathcal{H})$ with closed range is defined as the unique operator $A^\dagger \in \mathcal{L}(\mathcal{H})$ satisfying

$$AA^\dagger h = h \quad \forall x \in \mathcal{R}(A).$$

So they are orthogonal projections. Recall that a projection $Q \in \mathcal{L}(\mathcal{H})$ is minimal if

$$\|Q\| \leqslant \|P\|$$

for every projection $P \in \mathcal{L}(\mathcal{H})$ (see [11]).

Remark 1. For $A \in \mathcal{CR}(\mathcal{H})$, $AA^\dagger = \pi_{\mathcal{R}(A)}$ is a minimal projection. Indeed, $\|AA^\dagger\| = 1$ and $\|P\| \geqslant 1$ for every projection $P \in \mathcal{L}(\mathcal{H})$.

To throw more light on the subject and understand the use of this concept, we exhibit below some examples.

Example 3. Let $A \in \mathcal{L}(\mathcal{H})$ such that $A^2 = A$. Then

$$A^\dagger = \pi_{\mathcal{R}(A^*)}\pi_{\mathcal{R}(A)}.$$

Now, we list below some useful properties related to Moore–Penrose inverses.

Proposition 2. (*See* [18].) *Let $A \in \mathcal{CR}(\mathcal{H})$. Then*

(i) $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*) = \mathcal{N}(A)^\perp$;
(ii) $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*) = \mathcal{R}(A)^\perp$.

Theorem 1. (*See* [14].) *Let $A \in \mathcal{CR}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H})$ be an arbitrary operator, which commutes with A. Then B commutes with $A^\dagger$.*

3 $*$-$\mathcal{K}$-modular frames via operators

In this section, we consider $A, \mathcal{K} \in \mathcal{CR}(\mathcal{H})$ and adopt the following notation:

$$m(A) = A^{\dagger *}.$$

Lemma 4. *Let $A \in \mathcal{CR}(\mathcal{H})$. Then*

$$c\langle m(\mathcal{K})^*h, m(\mathcal{K})^*h\rangle \leqslant \langle \mathcal{K}^*h, \mathcal{K}^*h\rangle \quad \forall h \in \mathcal{H}.$$

Proof. It follows from Proposition 2 that

$$\mathcal{R}\big(m(\mathcal{K})\big) = N(\mathcal{K}^\dagger)^\perp = \big(N(\mathcal{K}^*)\big)^\perp = \mathcal{R}(\mathcal{K}).$$

By Lemma 2, the result is obtained. □

Lemma 5. *Let V be unitary. Then*

$$m(V\mathcal{K}V^*) = Vm(\mathcal{K})V^*.$$

Proof. Clearly, for each $T \in \mathcal{L}(\mathcal{H})$, we have

$$\gamma(VT) = \gamma(T).$$

Thus

$$\gamma(V\mathcal{K}V^*) = \gamma(\mathcal{K}V^*) = \gamma(V\mathcal{K}^*) = \gamma(\mathcal{K}^*) = \gamma(\mathcal{K}) > 0.$$

Direct computations show that

$$(V\mathcal{K}^*V^*)^\dagger = V\mathcal{K}^{*\dagger}V^* = V\mathcal{K}^{\dagger *}V^*.$$

This implies

$$m(V\mathcal{K}V^*) = (V\mathcal{K}V^*)^{\dagger *} = (V\mathcal{K}V^*)^{*\dagger} = Vm(\mathcal{K})U^*.$$

This completes the proof. □

Theorem 2. *Let $\{h_j\}_{j\in\mathbb{J}}$ be a $*$-$\mathcal{K}$-frame for $\mathcal{H}$ and V be unitary. Then $\{Vh_j\}_{j\in\mathbb{J}}$ is a $*$-$Vm(\mathcal{K})V^*$-frame for $\mathcal{H}$.*

Proof. For $h \in \mathcal{H}$, by Lemma 1, we have

$$\begin{aligned}\big\langle (V\mathcal{K}V^*)^*h, (V\mathcal{K}V^*)^*h\big\rangle &\leqslant \|V^*\|^2\langle \mathcal{K}^*V^*h, \mathcal{K}^*V^*h\rangle \\ &\leqslant \langle \mathcal{K}^*V^*h, \mathcal{K}^*V^*h\rangle.\end{aligned}$$

By Lemma 4, there exists $c > 0$ such that

$$c\big\langle m(V\mathcal{K}V^*)^*h, m(V\mathcal{K}V^*)^*h\big\rangle \leqslant \big\langle (V\mathcal{K}V^*)^*h, (V\mathcal{K}V^*)^*h\big\rangle.$$

It follows from Lemma 5 that

$$c\big\langle \big(Vm(\mathcal{K})V^*\big)^*h, \big(Vm(\mathcal{K})V^*\big)^*h\big\rangle \leqslant \langle \mathcal{K}^*V^*h, \mathcal{K}^*V^*h\rangle.$$

By assumption, there exist strictly nonzero elements $\alpha, \beta \in \mathcal{A}$ such that

$$\begin{aligned}\alpha\langle \mathcal{K}^*V^*h, \mathcal{K}^*V^*h\rangle\alpha^* &\leqslant \sum_{j\in\mathbb{J}}\langle V^*h, h_j\rangle\langle h_j, V^*h\rangle \\ &\leqslant \beta\langle V^*h, V^*h\rangle\beta^*.\end{aligned}$$

This implies

$$\begin{aligned}\sqrt{c}\alpha\big\langle \big(Vm(\mathcal{K})V^*\big)^* h, \big(Vm(\mathcal{K})V^*\big)^* h\big\rangle(\sqrt{c}\alpha)^* \\ \leqslant \sum_{j\in\mathbb{J}}\langle h, Vh_j\rangle\langle Vh_j, h\rangle \leqslant \beta\langle V^*h, V^*h\rangle\beta^*.\end{aligned}$$

Therefore, $\{Vh_j\}_{j\in\mathbb{J}}$ is a $*$-$Vm(\mathcal{K})V^*$-frame for $\mathcal{H}$. □

The next result guarantees that the operator $\pi_{R(A)}$ preserves $*$-$\mathcal{K}$-modular frames.

Theorem 3. *Let $\{h_j\}_{j\in\mathbb{J}}$ be a $*$-$\mathcal{K}$-frame for $\mathcal{H}$, and let $A \in \mathcal{CR}(\mathcal{H})$ be such that $\mathcal{R}(\mathcal{K}^*) \subset \mathcal{R}(A)$. Then $\{(\pi_{\mathcal{R}(A)})h_j\}_{j\in\mathbb{J}}$ is a $*$-$\mathcal{K}$-frame for $\mathcal{H}$.*

Proof. By assumption, there exist strictly nonzero elements $\alpha, \beta \in \mathcal{A}$ such that

$$\begin{aligned}&\alpha\big\langle(\mathcal{K}\pi_{\mathcal{R}(A)})^*h, (\mathcal{K}\pi_{\mathcal{R}(A)})^*h\big\rangle\alpha^* \\ &\quad\leqslant \sum_{j\in\mathbb{J}}\big\langle h, \pi_{\mathcal{R}(A)}h_j\big\rangle\big\langle\pi_{\mathcal{R}(A)}h_j, h\big\rangle \leqslant \beta\big\langle\pi_{\mathcal{R}(A)}h, \pi_{\mathcal{R}(A)}h\big\rangle\beta^* \quad \forall h \in \mathcal{H}.\end{aligned}$$

If $h \in \mathcal{H}$, since $\mathcal{H} = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp$, there exist $r \in \mathcal{R}(A)$ and $w \in \mathcal{R}(A)^\perp$ such that

$$\mathcal{K}h = \mathcal{K}r + \mathcal{K}w.$$

From $\mathcal{R}(\mathcal{K}^*) \subseteq \mathcal{R}(A)$ we get $\mathcal{R}(A)^\perp \subset \mathcal{R}(\mathcal{K}^*)^\perp = \mathcal{N}(\mathcal{K})$ and

$$\mathcal{K}h = \mathcal{K}\big(AA^\dagger r\big).$$

It follows from Theorem 1 that $\mathcal{K}A^\dagger = A^\dagger\mathcal{K}$, and hence

$$\mathcal{K}h = (\pi_{\mathcal{R}(A)}\mathcal{K})(r + w).$$

By Lemma 2, there exists $c > 0$ such that

$$c\langle\mathcal{K}^*h, \mathcal{K}^*h\rangle \leqslant \big\langle(\pi_{\mathcal{R}(\mathcal{A})}\mathcal{K})^*h, (\pi_{\mathcal{R}(\mathcal{A})}\mathcal{K})^*h\big\rangle \quad \forall h \in \mathcal{H}.$$

Hence

$$\sqrt{c}\alpha\langle\mathcal{K}^*h, \mathcal{K}^*h\rangle(\sqrt{c}\alpha)^* \leqslant \alpha\big\langle(\pi_{\mathcal{R}(\mathcal{A})}\mathcal{K})^*h, (\pi_{\mathcal{R}(\mathcal{A})}\mathcal{K})^*h\big\rangle\alpha^*.$$

Therefore,

$$\begin{aligned}\sqrt{c}\alpha\langle\mathcal{K}^*h, \mathcal{K}^*h\rangle(\sqrt{c}\alpha)^* &\leqslant \sum_{j\in\mathbb{J}}\langle h, \pi_{\mathcal{R}(A)}h_j\rangle\langle\pi_{\mathcal{R}(A)}h_j, h\rangle \\ &\leqslant \beta\langle h, h\rangle\beta^* \quad \forall h \in \mathcal{H}.\end{aligned}$$

□

Example 4. Let $A \in \mathcal{L}(l^2(\mathbb{R}))$ be defined as follows:

$$A(h_1, h_2, \dots) = (h_3 - h_1, 0, h_3, 0, 0, \dots).$$

Then $Ae_1 = -e_1$, $Ae_2 = 0$, $Ae_3 = e_1 + e_3$, and $Ae_n = 0$ for $n \geqslant 4$, where e_n denotes the nth vector of the canonical basis of $l^2(\mathbb{C})$. Moreover,

$$\mathcal{R}(A) = \{e_2\}^{\perp} \cap \big(\operatorname{span}\{e_n\colon\, n \geqslant 4\}\big)^{\perp},$$

and so it is closed. Now, let $\mathcal{K} \in \mathcal{L}(\mathcal{H})$ be such that

$$\mathcal{K}(h_1, h_2, \dots) = (2h_1 - h_3, 0, 0, \dots),$$

and

$$\mathcal{K}e_1 = 2e_1, \quad \mathcal{K}e_2 = 0, \quad \mathcal{K}e_3 = -e_1, \quad \text{and} \quad \mathcal{K}_n = 0, \quad n \geqslant 4.$$

Then the adjoint operator $\mathcal{K}^*$ satisfies $\mathcal{K}^*e_1 = 2e_1 - e_3$ and $\mathcal{K}^*e_n = 0$, $n \geqslant 2$.

By straightforward computations, we obtain $\mathcal{R}(\mathcal{K}^*) \subset \mathcal{R}(A)$ and $\mathcal{K}A = A\mathcal{K}$.

Set

$$h_1 = \sqrt{2}e_1, \qquad \text{and} \quad h_n = \frac{1}{n}e_n, \quad n \geqslant 2.$$

For $h = (h_n)_{n\geqslant 1} \in l^2(\mathbb{C})$, we obtain

$$\sum_{n\geqslant 1} \big|\langle h, h_n\rangle\big|^2 = 2|h_1|^2 + \sum_{n\geqslant 2} \frac{1}{n^2}|h_n|^2$$

and

$$\|\mathcal{K}^*h\|^2 = 5|h_1|^2.$$

Therefore,

$$\frac{1}{5}\|\mathcal{K}^*h\|^2 \leqslant \sum_{n\geqslant 1} \big|\langle h, h_n\rangle\big|^2 \leqslant 2\|h\|^2.$$

Also, we have

$$\frac{1}{5}\|\mathcal{K}^*h\|^2 \leqslant \sum_{n\geqslant 1} \big|\langle h, \pi_{\mathcal{R}(A)}h_n\rangle\big|^2 \leqslant 2\|h\|^2.$$

Theorem 4. *Let $\{h_j\}_{j\in\mathbb{J}}$ be a $*$-$\mathcal{K}$-frame for $\mathcal{H}$, and let $L \in \mathcal{L}(\mathcal{H})$ be such that $\mathcal{K}L\mathcal{K} = \mathcal{K}$. Then $\{(\mathcal{K}L)h_j\}_{j\in\mathbb{J}}$ is a $*$-$\mathcal{K}$-frame for $\mathcal{H}$.*

Proof. By assumption, there exist strictly nonzero elements $\alpha, \beta \in \mathcal{A}$ such that

$$\alpha\langle \mathcal{K}^*h, \mathcal{K}^*h\rangle\alpha^* \leqslant \sum_{j\in\mathbb{J}} \langle h, h_j\rangle\langle h_j, x\rangle \leqslant \beta\langle h, h\rangle\beta^* \quad \forall h \in \mathcal{H}.$$

Thus

$$\begin{aligned}\alpha\langle \mathcal{K}^*L^*\mathcal{K}^*h, \mathcal{K}^*L^*\mathcal{K}^*h\rangle\alpha^* &\leqslant \sum_{j\in\mathbb{J}} \big\langle h, (\mathcal{K}L)h_j\big\rangle\big\langle (\mathcal{K}L)h_j, h\big\rangle \\ &\leqslant \beta\langle L^*\mathcal{K}^*h, L^*\mathcal{K}^*h\rangle\beta^*.\end{aligned}$$

By Lemma 1, we obtain

$$\langle L^*\mathcal{K}^*h, L^*\mathcal{K}^*h\rangle \leqslant \|\mathcal{K}L\|^2\langle h,h\rangle.$$

This implies

$$\begin{aligned}\alpha\langle\mathcal{K}^*h,\mathcal{K}^*h\rangle\alpha^* &\leqslant \sum_{j\in\mathbb{J}}\big\langle h,(\mathcal{K}L)h_j\big\rangle\big\langle(\mathcal{K}L)h_j,h\big\rangle\\ &\leqslant \beta\|\mathcal{K}L\|\langle h,h\rangle\big(\beta\|\mathcal{K}L\|\big)^*.\end{aligned}$$

Therefore, $\{(\mathcal{K}L)h_j\}_{j\in\mathbb{J}}$ is a $*$-$\mathcal{K}$-frame for $\mathcal{H}$. □

Example 5. Let $a\in\mathbb{R}$ and $\mathcal{K}, L\in\mathcal{L}(\mathbb{C}^)$ be defined as follows:

$$\mathcal{K}=\begin{pmatrix}0 & 0\\ \mathrm{e}^{\mathrm{i}a} & 0\end{pmatrix},\qquad L=\begin{pmatrix}2 & \mathrm{e}^{-\mathrm{i}a}\\ 0 & 0\end{pmatrix},$$

where $\mathrm{i}^2=-1$.

It is easy to verify that $\mathcal{K}L\mathcal{K}=\mathcal{K}$. Then $\mathcal{K}L$ is not a minimal projection.

Now let $h=(h_1,h_2)\in\mathbb{C}^2$. Then

$$\|\mathcal{K}^*h\|^2=|h_2|^2.$$

Set

$$h_1=\frac{\mathrm{i}}{\sqrt{2}}e_1\quad\text{and}\quad h_2=2e_2.$$

Then we obtain

$$\sum_{j=1}^{2}\langle h,h_j\rangle\langle h_j,h\rangle=\frac{1}{2}|h_1|^2+4|h_2|^2.$$

Thus

$$\|\mathcal{K}^*h\|^2\leqslant\sum_{j=1}^{2}\langle h,h_j\rangle\langle h_j,h\rangle\leqslant 4\|h\|^2.$$

So $\{h_1,h_2\}$ is a $*$-$\mathcal{K}$-frame for $\mathbb{C}^2$.

On the other hand, we get

$$\sum_{j=1}^{2}\big\langle h,(\mathcal{K}L)h_j\big\rangle\big\langle(\mathcal{K}L)h_j,h\big\rangle=6|h_2|^2.$$

So

$$\|\mathcal{K}^*h\|^2\leqslant\sum_{j=1}^{2}\big\langle h,(\mathcal{K}L)h_j\big\rangle\big\langle(\mathcal{K}L)h_j,h\big\rangle\leqslant 6\|h\|^2,\quad h\in\mathbb{C}^2.$$

We deduce that $\{(\mathcal{K}L)h_j\}_{j\in\mathbb{J}}$ is a $*$-$\mathcal{K}$-frame for $\mathcal{L}(\mathbb{C}^2)$.

Theorem 5. *Suppose that $\{h_j\}_{j\in\mathbb{J}}$ is a $*$-$\mathcal{K}$-frame for $\mathcal{H}$ and A is regular such that $\mathcal{K}A=A\mathcal{K}$ and $\mathcal{R}(\mathcal{K}^*)\subset\mathcal{R}^\infty(A)$. Then $\{Ah_j\}_{j\in\mathbb{J}}$ is a $*$-$\mathcal{K}$-frame for $\mathcal{H}$. Moreover, $\{(A-zI)h_j\}_{j\in\mathbb{J}}$ is also a $*$-$\mathcal{K}$-frame for $\mathcal{H}$ for all $|z|<\gamma(A)$).*

Proof. By assumption, there exist strictly nonzero elements $\alpha, \beta \in \mathcal{A}$ such that

$$\begin{aligned}\alpha\langle \mathcal{K}^* A^* h, \mathcal{K}^* A^* h\rangle \alpha^* &\leqslant \sum_{j\in\mathbb{J}} \langle h, Ah_j\rangle\langle Ah_j, h\rangle \\ &\leqslant \beta\langle A^* h, A^* h\rangle\beta^* \quad \forall h \in \mathcal{H}.\end{aligned}$$

If $h \in \mathcal{H}$, since $\mathcal{H} = \mathcal{R}^\infty(A) \oplus R^\infty(A)^\perp$, there exist $r, r' \in \mathcal{R}^\infty(A)$ and $t \in \mathcal{R}^\infty(A)^\perp$ such that

$$\mathcal{K}h = \mathcal{K}r + \mathcal{K}t = \mathcal{K}Ar' + \mathcal{K}t.$$

From $\mathcal{R}(\mathcal{K}^*) \subseteq \mathcal{R}^\infty(A)$ we get $\mathcal{R}^\infty(A)^\perp \subset \mathcal{R}(\mathcal{K}^*)^\perp = \mathcal{N}(\mathcal{K})$ and

$$\mathcal{K}h = (A\mathcal{K})(r' + t).$$

By Lemma 2, there exists $c > 0$ such that

$$c\langle \mathcal{K}^* h, \mathcal{K}^* h\rangle \leqslant \big\langle (A\mathcal{K})^* h, (A\mathcal{K})^* h\big\rangle \quad \forall h \in \mathcal{H}.$$

Hence

$$\sqrt{c}\alpha\langle \mathcal{K}^* h, \mathcal{K}^* h\rangle(\sqrt{c}\alpha)^* \leqslant \alpha\big\langle (A\mathcal{K})^* h, (A\mathcal{K})^* h\big\rangle\alpha^*.$$

Therefore,

$$\begin{aligned}\sqrt{c}\alpha\langle \mathcal{K}^* h, \mathcal{K}^* h\rangle(\sqrt{c}\alpha)^* &\leqslant \sum_{j\in\mathbb{J}} \langle h, Ah_j\rangle\langle Ah_j, h\rangle \\ &\leqslant \beta\|A\|\langle h, h\rangle\big(\beta\|A\|\big)^* \quad \forall h \in \mathcal{H}.\end{aligned}$$

On the other hand, we have

$$\mathcal{R}^\infty(A - zI) = \mathcal{R}^\infty(A).$$

The proof is completed. □

4 Conclusion

In this paper, we have studied the effect of operators on $*$-$\mathcal{K}$-modular frames. The paper suggests a deeper analysis, and, in fact, one of our future projects is to use some concepts of algebraic theory to construct $*$-$\mathcal{K}$-modular frames. Finally, given a $*$-modular frame $\{h_j\}_{j\in\mathbb{J}}$, the characterization of a bounded operator $A \in \mathcal{L}(\mathcal{H})$ for which $\{Ah_j\}_{j\in\mathbb{J}}$ is also a $*$-module frame is still an open problem.

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