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Unified approach to uniqueness of solutions for a coupled system of fractional differential equations with three types of boundary value conditions*

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Abstract. This paper presents a unified approach to investigate a coupled system of Caputo fractional differential equations with three types of boundary conditions: multi-point, infinite-point, and Riemann–Stieltjes integral boundary conditions. Under a generalized Lipschitz condition, we first prove the uniqueness of solutions for the system with multi-point boundary conditions via the Banach contraction mapping principle. By taking limits, we obtain the existence of solutions for the remaining two boundary conditions directly without requiring any additional proof. The feasibility of this limiting argument lies in overcoming the complexity of the associated space and operator. Finally, numerical examples verify the theoretical findings.

Keywords: Caputo fractional differential system, multi-point boundary condition, infinite-point boundary condition, Riemann–Stieltjes integral condition, uniqueness of solutions.

1 Introduction

Differential equation systems are an enduring research topic in the field of mathematics and have significant theoretical guiding value in many disciplines such as biology, physics, and chemistry. A prominent example is the predator–prey model [16, 17], known as the Lotka–Volterra equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \beta xy, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \delta xy - \gamma y,$$

where α is the prey's growth rate without predators, β is the rate at which predators kill prey, δ is how efficiently predators turn eaten prey into new predators, and γ is the

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predator's death rate without prey. This model characterizes the competitive dynamic between two species in a shared ecological niche through a system of equations. Its biological significance is profound, providing a foundational theoretical framework for deciphering how interspecific interactions affect population dynamics. The model is a key component in the early development of differential equation systems, and it has spurred extensive subsequent research in the field [3, 9, 10].

On the other hand, the global dependence of fractional calculus, which is pivotal for describing memory and genetic properties, has brought fractional calculus to the forefront of research. For example, by introducing fractional derivatives to describe the relationship between strain and stress, [5,6] incorporated fractional models to characterize behaviors of viscoelastic materials such as creep, stress relaxation, and hysteresis. The theoretical results showed close agreement with experimental data, thereby providing a crucial theoretical basis for the study of viscoelastic material properties. Naturally, fractional calculus, with its rapid theoretical advances and modeling advantages, has been applied to ecological systems. For example, in [4] the authors explored the following Caputo-type fractional-order predator–prey system:

$$\begin{aligned} {}^{c}D^{\alpha}_{0^+}x_1(t) &= x_1(t)\big(r - ax_1(t) - bx_2(t)\big), \quad t \in (0,T], \\ {}^{c}D^{\alpha}_{0^+}x_2(t) &= x_2(t)\big(-d + cx_1(t)\big), \quad t \in (0,T], \end{aligned}$$

with the boundary conditions

$$x_1(0) = \gamma, \qquad x_2(0) = \eta,$$

where $0 < \alpha \leqslant 1$, r, a, b, c, d, γ, and η are positive constants. The authors proved the existence and uniqueness of solutions and studied the stability of equilibrium points. Subsequently, fractional differential systems have gained significant research attention, leading to many excellent works [2, 14, 18, 19] and numerous influential papers. For instance, in [15] the authors investigated a class of Hadamard fractional differential systems. The authors proved the existence of solutions by applying the Leray–Schauder nonlinear alternative and obtained the uniqueness result through the Banach contraction mapping principle. Using the same methods, the work in [13] investigated a class of Caputo-type fractional boundary value problems, while the authors in [22] studied a class of coupled fractional conformable Langevin differential systems on a circular graph. Later, the authors of [7] employed a coupled iterative technique to investigate a class of coupled p-Laplacian systems, establishing the uniqueness and iterative properties of the positive solution. Similar to the problem in [15], the work in [21] dealt with a Hadamard fractional differential system. By applying several fixed-point theorems, the existence and uniqueness of solutions were obtained. Following the same methods as in [13, 15, 22], the authors in [1] considered the following Caputo-type coupled fractional differential system:

$$\begin{aligned} {}^{c}D^{q_1}_{0^+}\varphi(t) &= \sigma_1\big(t,\varphi(t),\psi(t)\big), \quad t \in [0,T], \\ {}^{c}D^{q_2}_{0^+}\psi(t) &= \sigma_2\big(t,\varphi(t),\psi(t)\big), \quad t \in [0,T], \end{aligned}$$

with the boundary conditions

$$\varphi(T)=\sum_{i=1}^{m}\big(\alpha_i\psi(\xi_i)+\beta_iT\psi'(\xi_i)\big),\qquad T\varphi'(T)=\sum_{i=1}^{m}\big(\gamma_i\psi(\xi_i)+\delta_iT\psi'(\xi_i)\big),$$
$$\psi(T)=\sum_{i=1}^{m}\big(\mu_i\varphi(\xi_i)+\nu_iT\varphi'(\xi_i)\big),\qquad T\psi'(T)=\sum_{i=1}^{m}\big(\rho_i\varphi(\xi_i)+\omega_iT\varphi'(\xi_i)\big),$$

where $q_j\in(1,2]$, $\sigma_j\in C(T\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ $(j=1,2)$, $\alpha_i,\beta_i,\gamma_i,\delta_i,\mu_i,\nu_i,\rho_i,\omega_i\in\mathbb{R}$, and $\xi_i\in(0,T)$ $(i=1,2,3,\dots,m)$.

It is noted that existing research on such systems is predominantly conducted under a single boundary condition, while unified methods capable of handling multiple boundary conditions remain scarce. For instance, in [11] the authors investigated the following coupled system of integro-differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=g_1\left(t,\,y(t),\int_0^t f_1\big(s,y(s)\big)\,\mathrm{d}s\right),\quad\text{a.e. } t\in(0,T],$$
$$\frac{\mathrm{d}y}{\mathrm{d}t}=g_2\left(t,\,x(t),\int_0^t f_2\big(s,x(s)\big)\,\mathrm{d}s\right),\quad\text{a.e. } t\in(0,T].$$

This system was considered with three types of boundary conditions: multi-point, infinite-point, and Riemann–Stieltjes integral boundary conditions. By applying Schauder's fixed point theorem, the existence of solutions was proved under the assumption that the nonlinear terms satisfy the Carathéodory condition. Using a similar approach, the authors in [12] investigated a class of differential equation problems involving three types of boundary conditions. In our previous work [8], we studied a class of coupled system of Caputo fractional differential equations

$$\begin{aligned}{}^cD^{\alpha}_{0^+}x(t)=g_1\Bigg(&t,\,x(t),\,y(t),\int_0^t f_1\big(s,x(s),y(s)\big)\,\mathrm{d}s,\\&\int_0^t h_1\big(s,x(s),y(s)\big)\,\mathrm{d}s\Bigg),\quad\text{a.e. } t\in(0,1),\end{aligned}$$
$$\begin{aligned}{}^cD^{\beta}_{0^+}y(t)=g_2\Bigg(&t,\,x(t),\,y(t),\int_0^t f_2\big(s,x(s),y(s)\big)\,\mathrm{d}s,\\&\int_0^t h_2\big(s,x(s),y(s)\big)\,\mathrm{d}s\Bigg),\quad\text{a.e. } t\in(0,1).\end{aligned}$$

By using Schauder's fixed point theorem, we obtained the existence of solutions for the system with three types of boundary conditions under the assumption that the nonlinear terms satisfy the Carathéodory condition.

With the aim of further investigating the case where the nonlinear term involves different fractional derivatives, analyzing the uniqueness of such systems, and developing a unified approach for treating three types of boundary conditions, we consider the following class of systems:

$$\begin{aligned}
{}^{c}D^{\alpha}_{0^+}x(t) &= f_1\Bigg(t,\, x(t),\, y(t),\, \int_0^t g_1\big(s,x(s),y(s)\big)\,\mathrm{d}s,\, \int_0^1 h_1\big(s,x(s),y(s)\big)\,\mathrm{d}s,\\
&\qquad {}^{c}D^{\gamma_1}_{0^+}x(t),\, {}^{c}D^{\nu_1}_{0^+}y(t)\Bigg), \quad \text{a.e. } t\in(0,1),\\
{}^{c}D^{\beta}_{0^+}y(t) &= f_2\Bigg(t,\, x(t),\, y(t),\, \int_0^t g_2\big(s,x(s),y(s)\big)\,\mathrm{d}s,\, \int_0^1 h_2\big(s,x(s),y(s)\big)\,\mathrm{d}s,\\
&\qquad {}^{c}D^{\gamma_2}_{0^+}x(t),\, {}^{c}D^{\nu_2}_{0^+}y(t)\Bigg), \quad \text{a.e. } t\in(0,1),
\end{aligned} \tag{1}$$

with the three types of boundary conditions:

$$\begin{aligned}
x(0)+\sum_{j=1}^{m} a_j x(\tau_j) &= x_0, \quad a_j>0,\ \tau_j\in(0,1),\\
y(0)+\sum_{j=1}^{m} b_j y(\varsigma_j) &= y_0, \quad b_j>0,\ \varsigma_j\in(0,1);
\end{aligned} \tag{2}$$

$$\begin{aligned}
x(0)+\sum_{j=1}^{\infty} a_j x(\tau_j) &= x_0, \quad a_j>0,\ \tau_j\in(0,1),\ 0\leqslant\sum_{j=1}^{\infty} a_j<+\infty,\\
y(0)+\sum_{j=1}^{\infty} b_j y(\varsigma_j) &= y_0, \quad b_j>0,\ \varsigma_j\in(0,1),\ 0\leqslant\sum_{j=1}^{\infty} b_j<+\infty;
\end{aligned} \tag{3}$$

$$x(0)+\int_0^1 x(s)\,\mathrm{d}l_1(s)=x_0, \qquad y(0)+\int_0^1 y(s)\,\mathrm{d}l_2(s)=y_0, \tag{4}$$

where ${}^{c}D^{\alpha}_{0^+}$, ${}^{c}D^{\beta}_{0^+}$, ${}^{c}D^{\gamma_i}_{0^+}$, ${}^{c}D^{\nu_i}_{0^+}$ are the Caputo fractional derivatives, $0<\alpha,\beta,\gamma_i,\nu_i\leqslant 1$, $\alpha-\gamma_i>1/p_1$, $\beta-\nu_i>1/p_2$ $(i=1,2)$, and $p_1,p_2>1$ are fixed constants; $x_0,y_0\in\mathbb{R}$ are fixed constants; $f_i:[0,1]\times\mathbb{R}^6\to\mathbb{R}$, $g_i,h_i:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ $(i=1,2)$; $l_i:[0,1]\to\mathbb{R}$ $(i=1,2)$ are increasing functions of bounded variation. For convenience, throughout this paper, let q_1 and q_2 denote the conjugate exponent of p_1 and p_2, respectively, i.e., $1/p_1+1/q_1=1$ and $1/p_2+1/q_2=1$.

In this paper, we investigate the coupled system (1), in which the nonlinear terms involve Volterra integrals, Fredholm integrals, and fractional derivatives of different orders. This makes our system more general than those in previous works, e.g., [8, 11, 12]. We assume that the nonlinear terms satisfy a Lipschitz condition with L^p-integrable functional coefficients, thereby allowing them to be more general. Under this assumption, we first establish the uniqueness of the solution for system (1) with multi-point boundary conditions by the Banach contraction mapping principle. Building on this result, we extend our study to two more general cases: infinite-point boundary conditions and Riemann–Stieltjes integral conditions. In both settings, the existence and uniqueness of solutions are established via a limit argument, with no additional proof required. In this way, we develop a unified approach to investigate the existence of solutions for three types of boundary conditions simultaneously. It is worth noting that the existence of the required limits is proved, despite the difficulties caused by the complexity of the function space. To the best of our knowledge, this result has not been reported in the literature. Finally, numerical examples are provided to verify the validity of the theoretical results.

The outline of this paper is organized as follows. In Section 2, we introduce the preliminary settings. In Section 3, we establish the uniqueness result for the system with m-point boundary conditions. Building on this, we further investigate the existence and uniqueness of solutions with infinite-point boundary conditions in Section 4 and Riemann–Stieltjes integral boundary conditions in Section 5. In Section 6, we present numerical examples to validate the theoretical results.

2 Preliminaries and lemmas

In this section, we will provide some definitions and lemmas for later use.

Definition 1. (See [14].) Let $\gamma > 0$. The Riemann–Liouville fractional integral of order γ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s)\,\mathrm{d}s,$$

where the right-hand side exists a.e. on $(0, \infty)$.

Definition 2. (See [14].) Let $0 < \gamma \leqslant 1$. The Caputo fractional derivative of order γ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$${}^{c}D_{0^+}^{\gamma} f(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{f'(s)}{(t-s)^{\gamma}}\,\mathrm{d}s,$$

where the right-hand side exists a.e. on $(0, \infty)$. In particular, if $\gamma = 1$, then ${}^{c}D_{0^+}^{\gamma} f(t) = f'(t)$.

Lemma 1. (*See* [14].) *Let* $0 < \gamma \leqslant 1$, $f \in AC^1[0,1]$. *Then*

$$I_{0+}^{\gamma}\, {}^{c}D_{0+}^{\gamma} f(t) = f(t) - C, \quad C \in \mathbb{R}.$$

Lemma 2. *Let* $u \in C(0,1) \cap L^{p_1}(0,1)$, $v \in C(0,1) \cap L^{p_2}(0,1)$. *Then the system*

$${}^{c}D_{0+}^{\alpha} x(t) = u(t), \quad {}^{c}D_{0+}^{\beta} y(t) = v(t), \quad t \in (0,1),$$

with the boundary conditions

$$x(0) + \sum_{j=1}^{m} a_j x(\tau_j) = x_0, \qquad y(0) + \sum_{j=1}^{m} b_j y(\varsigma_j) = y_0$$

has a unique solution (x, y) *on* $[0,1]$, *which can be expressed by*

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,\mathrm{d}s \\ &\quad + \frac{1}{1+\sum_{j=1}^{m} a_j} \left(x_0 - \frac{\sum_{j=1}^{m} a_j}{\Gamma(\alpha)} \int_0^{\tau_j} (\tau_j - s)^{\alpha-1} u(s)\,\mathrm{d}s \right), \quad t \in [0,1], \\ y(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s)\,\mathrm{d}s \\ &\quad + \frac{1}{1+\sum_{j=1}^{m} b_j} \left(y_0 - \frac{\sum_{j=1}^{m} b_j}{\Gamma(\beta)} \int_0^{\varsigma_j} (\varsigma_j - s)^{\beta-1} v(s)\,\mathrm{d}s \right), \quad t \in [0,1]. \end{aligned}$$

Proof. Since the proof is similar to that of Lemma 4 in [8], we omit it here. □

For later use, we state the following assumptions.

(A1) For any $R > 0$, there exist functions $\mathfrak{a}_i \in L^{p_i}((0,1), \mathbb{R}^+)$ such that for any $|x_j| \leqslant R$, $|y_j| \leqslant R$ $(j = 1, 2, \ldots, 6)$, $t \in (0,1)$,

$$\left| f_i(t, x_1, x_2, x_3, x_4, x_5, x_6) - f_i(t, y_1, y_2, y_3, y_4, y_5, y_6) \right| \leqslant \mathfrak{a}_i(t) \sum_{j=1}^{6} |x_j - y_j|,$$

where $f_i(\cdot, 0, 0, 0, 0, 0, 0) \in L^{p_i}(0,1)$ and $\mathbf{M}_{f_i} = \|f_i(;0,0,0,0,0,0)\|_{L^{p_i}}$, $i = 1, 2$.

(A2) For any $R > 0$, there exist functions $\mathfrak{b}_i \in L^{p_i}((0,1), \mathbb{R}^+)$ such that for any $|x_j| \leqslant R$, $|y_j| \leqslant R$ $(j = 1, 2)$, $t \in (0,1)$,

$$\left| g_i(t, x_1, x_2) - g_i(t, y_1, y_2) \right| \leqslant \mathfrak{b}_i(t) \big(|x_1 - y_1| + |x_2 - y_2| \big),$$

where $\mathbf{M}_{g_i} = \int_0^1 |g_i(s, 0, 0)|\,\mathrm{d}s < +\infty$, $i = 1, 2$.

(A3) For any $R > 0$, there exist functions $\mathfrak{c}_i \in L^{p_i}((0,1),\mathbb{R}^+)$ such that for any $|x_j| \leqslant R$, $|y_j| \leqslant R$ $(j = 1,2)$, $t \in (0,1)$,

$$\big|h_i(t,x_1,x_2) - h_i(t,y_1,y_2)\big| \leqslant \mathfrak{c}_i(t)\big(|x_1 - y_1| + |x_2 - y_2|\big),$$

where $\mathbf{M}_{h_i} = \int_0^1 |h_i(s,0,0)|\,\mathrm{d}s < +\infty$, $i = 1,2$.

Let

$$X = \big\{x \bigm| x \in C[0,1]\colon {}^cD_{0^+}^{\gamma_i} x \in C[0,1],\ i = 1,2\big\},$$
$$Y = \big\{y \bigm| y \in C[0,1]\colon {}^cD_{0^+}^{\nu_i} y \in C[0,1],\ i = 1,2\big\}.$$

Evidently, X is a Banach space with norm $\|x\|_X = \max\{\|x\|, \|{}^cD_{0^+}^{\gamma_1} x\|, \|{}^cD_{0^+}^{\gamma_2} x\|\}$, Y is a Banach space with norm $\|y\|_Y = \max\{\|y\|, \|{}^cD_{0^+}^{\nu_1} y\|, \|{}^cD_{0^+}^{\nu_2} y\|\}$, $E = X \times Y$ is a Banach space with norm $\|(x,y)\|_E = \|x\|_X + \|y\|_Y$, where $\|x\| = \max_{t\in[0,1]} |x(t)|$. The proof of completeness is essentially the same as in [20]; we omit the details. Here we define three operators $T_1 : E \to X$, $T_2 : E \to Y$, and $T : E \to E$ as follows:

$$\begin{aligned} &T_1(x,y)(t) \\ &\quad = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U_{1,(x,y)}(s)\,\mathrm{d}s \\ &\qquad + \frac{1}{1+\sum_{j=1}^m a_j} \left(x_0 - \frac{\sum_{j=1}^m a_j}{\Gamma(\alpha)} \int_0^{\tau_j} (\tau_j - s)^{\alpha-1} U_{1,(x,y)}(s)\,\mathrm{d}s \right), \quad t \in [0,1], \end{aligned}$$

$$\begin{aligned} &T_2(x,y)(t) \\ &\quad = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} U_{2,(x,y)}(s)\,\mathrm{d}s \\ &\qquad + \frac{1}{1+\sum_{j=1}^m b_j} \left(y_0 - \frac{\sum_{j=1}^m b_j}{\Gamma(\beta)} \int_0^{\varsigma_j} (\varsigma_j - s)^{\beta-1} U_{2,(x,y)}(s)\,\mathrm{d}s \right), \quad t \in [0,1], \end{aligned}$$

$$T(x,y)(t) = \big(T_1(x,y)(t), T_2(x,y)(t)\big), \quad t \in [0,1],$$

where

$$\begin{aligned} U_{i,(x,y)}(t) = f_i \Bigg(&t,\, x(t),\, y(t),\, \int_0^t g_i\big(\theta, x(\theta), y(\theta)\big)\,\mathrm{d}\theta,\, \int_0^1 h_i\big(\theta, x(\theta), y(\theta)\big)\,\mathrm{d}\theta, \\ &{}^cD_{0^+}^{\gamma_i} x(t),\, {}^cD_{0^+}^{\nu_i} y(t) \Bigg), \quad t \in [0,1],\ i = 1,2. \end{aligned}$$

Obviously, the fixed point of the operator T in E is a solution of system (1), (2).

3 Multi-point boundary conditions

In this section, we use the following notation:

$$\Lambda_1 = \max\Bigg\{ \frac{2(1+2\sum_{j=1}^{m} a_j)\|\mathfrak{a}_1\|_{L^{p_1}}(\|\mathfrak{b}_1\|_{L^{p_1}} + \|\mathfrak{c}_1\|_{L^{p_1}} + 2)}{(1+\sum_{j=1}^{m} a_j)\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}}, \\ \frac{2\|\mathfrak{a}_1\|_{L^{p_1}}(\|\mathfrak{b}_1\|_{L^{p_1}} + \|\mathfrak{c}_1\|_{L^{p_1}} + 2)}{\Gamma(\alpha-\gamma_i)((\alpha-\gamma_i-1)q_1+1)^{1/q_1}},\ i=1,2\Bigg\}, \tag{5}$$

$$\Lambda_2 = \max\Bigg\{ \frac{2(1+2\sum_{j=1}^{m} b_j)\|\mathfrak{a}_2\|_{L^{p_2}}(\|\mathfrak{b}_2\|_{L^{p_2}} + \|\mathfrak{c}_2\|_{L^{p_2}} + 2)}{(1+\sum_{j=1}^{m} b_j)\Gamma(\beta)((\beta-1)q_2+1)^{1/q_2}}, \\ \frac{2\|\mathfrak{a}_2\|_{L^{p_2}}(\|\mathfrak{b}_2\|_{L^{p_2}} + \|\mathfrak{c}_2\|_{L^{p_2}} + 2)}{\Gamma(\beta-\nu_i)((\beta-\nu_i-1)q_2+1)^{1/q_2}},\ i=1,2\Bigg\}, \tag{6}$$

$$\Xi_1 = \max\Bigg\{ \frac{1+2\sum_{j=1}^{m} a_j}{1+\sum_{j=1}^{m} a_j}\left(\frac{\|\mathfrak{a}_1\|_{L^{p_1}}(\mathbf{M}_{g_1}+\mathbf{M}_{h_1}) + \mathbf{M}_{f_1}}{\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}} \right) + \frac{|x_0|}{1+\sum_{j=1}^{m} a_j}, \\ \frac{\|\mathfrak{a}_1\|_{L^{p_1}}(\mathbf{M}_{g_1}+\mathbf{M}_{h_1}) + \mathbf{M}_{f_1}}{\Gamma(\alpha-\gamma_i)((\alpha-\gamma_i-1)q_1+1)^{1/q_1}},\ i=1,2\Bigg\}, \tag{7}$$

$$\Xi_2 = \max\Bigg\{ \frac{1+2\sum_{j=1}^{m} b_j}{1+\sum_{j=1}^{m} b_j}\left(\frac{\|\mathfrak{a}_2\|_{L^{p_2}}(\mathbf{M}_{g_2}+\mathbf{M}_{h_2}) + \mathbf{M}_{f_2}}{\Gamma(\beta)((\beta-1)q_2+1)^{1/q_2}} \right) + \frac{|y_0|}{1+\sum_{j=1}^{m} b_j}, \\ \frac{\|\mathfrak{a}_2\|_{L^{p_2}}(\mathbf{M}_{g_2}+\mathbf{M}_{h_2}) + \mathbf{M}_{f_2}}{\Gamma(\beta-\nu_i)((\beta-\nu_i-1)q_2+1)^{1/q_2}},\ i=1,2\Bigg\}, \tag{8}$$

$$\Lambda = \Lambda_1 + \Lambda_2, \qquad \Xi = \Xi_1 + \Xi_2. \tag{9}$$

Lemma 3. *Suppose that* (A1)–(A3) *hold. Then for any $R > 0$, the operator $T : B_R \to E$ is well defined, where $B_R = \{(x,y) \in E\colon \|(x,y)\|_E \leqslant R\}$.*

Proof. By (A1)–(A3), we know that for any $(x,y) \in B_R$,

$$\left| \int_0^s g_1\big(\theta, x(\theta), y(\theta)\big)\,\mathrm{d}\theta \right| \leqslant 2\|\mathfrak{b}_1\|_{L^{p_1}} R + \mathbf{M}_{g_1}, \quad s \in [0,1], \tag{10}$$

$$\left| \int_0^1 h_1\big(\theta, x(\theta), y(\theta)\big)\,\mathrm{d}\theta \right| \leqslant 2\|\mathfrak{c}_1\|_{L^{p_1}} R + \mathbf{M}_{h_1}, \quad s \in [0,1], \tag{11}$$

and

$$\big|U_{1,(x,y)}(s)\big| \leqslant \mathfrak{a}_1(s)\big(2\big(\|\mathfrak{b}_1\|_{L^{p_1}} + \|\mathfrak{c}_1\|_{L^{p_1}} + 2\big)R + \mathbf{M}_{g_1} + \mathbf{M}_{h_1}\big) \\ + \big|f_1(s,0,0,0,0,0,0,0)\big|, \quad s \in [0,1]. \tag{12}$$

From (10)–(12) we obtain

$$\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U_{1,(x,y)}(s)\,\mathrm{d}s\right| \leqslant \frac{2\|\mathfrak{a}_1\|_{L^{p_1}}(\|\mathfrak{b}_1\|_{L^{p_1}}+\|\mathfrak{c}_1\|_{L^{p_1}}+2)}{\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}}R + \frac{\|\mathfrak{a}_1\|_{L^{p_1}}(\mathbf{M}_{g_1}+\mathbf{M}_{h_1})+\mathbf{M}_{f_1}}{\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}}, \quad t\in[0,1]. \tag{13}$$

It then follows that

$$\begin{aligned}\big|T_1(x,y)(t)\big| &\leqslant \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U_{1,(x,y)}(s)\,\mathrm{d}s\right| + \frac{|x_0|}{1+\sum_{j=1}^m a_j} \\ &\quad + \left|\frac{\sum_{j=1}^m a_j}{(1+\sum_{j=1}^m a_j)\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-1}U_{1,(x,y)}(s)\,\mathrm{d}s\right| \\ &\leqslant \frac{1+2\sum_{j=1}^m a_j}{1+\sum_{j=1}^m a_j}\left(\frac{2\|\mathfrak{a}_1\|_{L^{p_1}}(\|\mathfrak{b}_1\|_{L^{p_1}}+\|\mathfrak{c}_1\|_{L^{p_1}}+2)}{\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}}R\right. \\ &\quad \left. + \frac{\|\mathfrak{a}_1\|_{L^{p_1}}(\mathbf{M}_{g_1}+\mathbf{M}_{h_1})+\mathbf{M}_{f_1}}{\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}}\right) + \frac{|x_0|}{1+\sum_{j=1}^m a_j} \\ &\leqslant \Lambda_1 R+\Xi_1<+\infty, \quad t\in[0,1].\end{aligned} \tag{14}$$

Similar to (13), we can prove that

$$\begin{aligned}\big|{}^cD_{0^+}^{\gamma_i}T_1(x,y)(t)\big| &\leqslant \frac{2\|\mathfrak{a}_1\|_{L^{p_1}}(\|\mathfrak{b}_1\|_{L^{p_1}}+\|\mathfrak{c}_1\|_{L^{p_1}}+2)}{\Gamma(\alpha-\gamma_i)((\alpha-\gamma_i-1)q_1+1)^{1/q_1}}R \\ &\quad + \frac{\|\mathfrak{a}_1\|_{L^{p_1}}(\mathbf{M}_{g_1}+\mathbf{M}_{h_1})+\mathbf{M}_{f_1}}{\Gamma(\alpha-\gamma_i)((\alpha-\gamma_i-1)q_1+1)^{1/q_1}} \\ &\leqslant \Lambda_1 R+\Xi_1<+\infty, \quad t\in[0,1],\ i=1,2.\end{aligned} \tag{15}$$

Based on (14) and (15), we deduce that

$$\begin{aligned}\big\|T_1(x,y)\big\|_X &= \max\big\{\big\|T_1(x,y)\big\|,\ \big\|{}^cD_{0^+}^{\gamma_1}T_1(x,y)\big\|,\ \big\|{}^cD_{0^+}^{\gamma_2}T_1(x,y)\big\|\big\} \\ &\leqslant \Lambda_1 R+\Xi_1<+\infty.\end{aligned} \tag{16}$$

Furthermore, by a similar proof, we derive that

$$\begin{aligned}\big\|T_2(x,y)\big\|_Y &= \max\big\{\big\|T_2(x,y)\big\|,\ \big\|{}^cD_{0^+}^{\nu_1}T_2(x,y)\big\|,\ \big\|{}^cD_{0^+}^{\nu_2}T_2(x,y)\big\|\big\} \\ &\leqslant \Lambda_2 R+\Xi_2<+\infty.\end{aligned} \tag{17}$$

It follows from (16) and (17) that

$$\big\|T(x,y)\big\|_E = \big\|T_1(x,y)\big\|_X + \big\|T_2(x,y)\big\|_Y \leqslant \Lambda R + \Xi < +\infty. \tag{18}$$

In conclusion, the operator $T : B_R \to E$ is well defined. □

Theorem 1. *Suppose that* $\Lambda < 1$ *and* (A1)–(A3) *hold. Then system* (1), (2) *has a unique solution.*

Proof. First, we show that $T : B_R \to B_R$. Since $\Lambda < 1$, we can choose $R > 0$ such that $R > \Xi/(1-\Lambda)$. It follows from (18) that

$$\big\|T(x,y)\big\|_E \leqslant \Lambda R + \Xi < R,$$

which means $T : B_R \to B_R$.

In the following, we prove that $T : B_R \to B_R$ is a contraction mapping. For any $(x_1, y_1), (x_2, y_2) \in B_R$, using (A1)–(A3), we have

$$\begin{aligned}&\big|U_{1,(x_1,y_1)}(s) - U_{1,(x_2,y_2)}(s)\big| \\ &\quad\leqslant \mathfrak{a}_1(s)\big(\|\mathfrak{b}_1\|_{L^{p_1}} + \|\mathfrak{c}_1\|_{L^{p_1}} + 2\big)\big(\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y\big), \quad s \in [0,1].\end{aligned}$$

It then follows that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|U_{1,(x_1,y_1)}(s) - U_{1,(x_2,y_2)}(s)\big|\,\mathrm{d}s \\ &\quad\leqslant \frac{\|\mathfrak{a}_1\|_{L^{p_1}}(\|\mathfrak{b}_1\|_{L^{p_1}} + \|\mathfrak{c}_1\|_{L^{p_1}} + 2)}{\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}}\big(\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y\big), \quad t \in [0,1].\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}&\big|T_1(x_1,y_1)(t) - T_1(x_2,y_2)(t)\big| \\ &\quad\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|U_{1,(x_1,y_1)}(s) - U_{1,(x_2,y_2)}(s)\big|\,\mathrm{d}s \\ &\qquad+ \frac{\sum_{j=1}^m a_j}{(1+\sum_{j=1}^m a_j)\Gamma(\alpha)}\int_0^{\tau_j} (\tau_j - s)^{\alpha-1}\big|U_{1,(x_1,y_1)}(s) - U_{1,(x_2,y_2)}(s)\big|\,\mathrm{d}s \\ &\quad\leqslant \frac{(1+2\sum_{j=1}^m a_j)\|\mathfrak{a}_1\|_{L^{p_1}}(\|\mathfrak{b}_1\|_{L^{p_1}} + \|\mathfrak{c}_1\|_{L^{p_1}} + 2)}{(1+\sum_{j=1}^m a_j)\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}}\big(\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y\big) \\ &\quad\leqslant \frac{\Lambda_1}{2}\big(\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y\big)\end{aligned} \tag{19}$$

and

$$\begin{aligned}
&\left|{}^{c}D_{0^+}^{\gamma_i}T_1(x_1,y_1)(t)-{}^{c}D_{0^+}^{\gamma_i}T_1(x_2,y_2)(t)\right|\\
&\quad=\left|\frac{1}{\Gamma(\alpha-\gamma_i)}\int_0^t(t-s)^{\alpha-\gamma_i-1}\big(U_{1,(x_1,y_1)}(s)-U_{1,(x_2,y_2)}(s)\big)\,\mathrm{d}s\right|\\
&\quad\leqslant\frac{\|\mathfrak{a}_1\|_{L^{p_1}}\big(\|\mathfrak{b}_1\|_{L^{p_1}}+\|\mathfrak{c}_1\|_{L^{p_1}}+2\big)}{\Gamma(\alpha-\gamma_i)\big((\alpha-\gamma_i-1)q_1+1\big)^{1/q_1}}\big(\|x_1-x_2\|_X+\|y_1-y_2\|_Y\big)\\
&\quad\leqslant\frac{\Lambda_1}{2}\big(\|x_1-x_2\|_X+\|y_1-y_2\|_Y\big),\quad i=1,2.
\end{aligned}\tag{20}$$

It follows from (19) and (20) that

$$\big\|T_1(x_1,y_1)-T_1(x_2,y_2)\big\|_X\leqslant\frac{\Lambda_1}{2}\|(x_1-x_2,y_1-y_2)\|_E.$$

Similarly, we can prove the following result:

$$\big\|T_2(x_1,y_1)-T_1(x_2,y_2)\big\|_X\leqslant\frac{\Lambda_2}{2}\|(x_1-x_2,y_1-y_2)\|_E.$$

Hence, we infer that

$$\begin{aligned}
\big\|T(x_1,y_1)-T(x_2,y_2)\big\|_E&\leqslant\left(\frac{\Lambda_1}{2}+\frac{\Lambda_2}{2}\right)\big\|(x_1-x_2,y_1-y_2)\big\|_E\\
&=\frac{\Lambda}{2}\|(x_1-x_2,y_1-y_2)\|_E.
\end{aligned}$$

Noting that $\Lambda/2<1/2<1$, we conclude that $T:B_R\to B_R$ is a contraction. By the Banach contraction principle, we deduce that the operator T has a unique fixed point in B_R. That is, system (1), (2) has a unique solution (x,y), which can be expressed by

$$\begin{aligned}
x(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}U_{1,(x,y)}(s)\,\mathrm{d}s\\
&\quad+\frac{1}{1+\sum_{j=1}^m a_j}\left(x_0-\frac{\sum_{j=1}^m a_j}{\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-1}U_{1,(x,y)}(s)\,\mathrm{d}s\right),\\
y(t)&=\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}U_{2,(x,y)}(s)\,\mathrm{d}s\\
&\quad+\frac{1}{1+\sum_{j=1}^m b_j}\left(y_0-\frac{\sum_{j=1}^m b_j}{\Gamma(\beta)}\int_0^{\varsigma_j}(\varsigma_j-s)^{\beta-1}U_{2,(x,y)}(s)\,\mathrm{d}s\right).
\end{aligned}$$

□

4 Infinite-point boundary conditions

In this section, we use the following notation:

$$\begin{aligned}
\Lambda_1' &= \max\Bigg\{\frac{2(1+2\sum_{j=1}^{\infty}a_j)\|\mathfrak{a}_1\|_{L^{p_1}}(\|\mathfrak{b}_1\|_{L^{p_1}}+\|\mathfrak{c}_1\|_{L^{p_1}}+2)}{(1+\sum_{j=1}^{\infty}a_j)\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}},\\
&\qquad \frac{2\|\mathfrak{a}_1\|_{L^{p_1}}(\|\mathfrak{b}_1\|_{L^{p_1}}+\|\mathfrak{c}_1\|_{L^{p_1}}+2)}{\Gamma(\alpha-\gamma_i)((\alpha-\gamma_i-1)q_1+1)^{1/q_1}},\quad i=1,2\Bigg\},\\
\Lambda_2' &= \max\Bigg\{\frac{2(1+2\sum_{j=1}^{\infty}b_j)\|\mathfrak{a}_2\|_{L^{p_2}}(\|\mathfrak{b}_2\|_{L^{p_2}}+\|\mathfrak{c}_2\|_{L^{p_2}}+2)}{(1+\sum_{j=1}^{\infty}b_j)\Gamma(\beta)((\beta-1)q_2+1)^{1/q_2}},\\
&\qquad \frac{2\|\mathfrak{a}_2\|_{L^{p_2}}(\|\mathfrak{b}_2\|_{L^{p_2}}+\|\mathfrak{c}_2\|_{L^{p_2}}+2)}{\Gamma(\beta-\nu_i)((\beta-\nu_i-1)q_2+1)^{1/q_2}},\quad i=1,2\Bigg\},\\
\Xi_1' &= \max\Bigg\{\frac{1+2\sum_{j=1}^{\infty}a_j}{1+\sum_{j=1}^{\infty}a_j}\left(\frac{\|\mathfrak{a}_1\|_{L^{p_1}}(\mathbf{M}_{g_1}+\mathbf{M}_{h_1})+\mathbf{M}_{f_1}}{\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}}\right)+\frac{|x_0|}{1+\sum_{j=1}^{\infty}a_j},\\
&\qquad \frac{\|\mathfrak{a}_1\|_{L^{p_1}}(\mathbf{M}_{g_1}+\mathbf{M}_{h_1})+\mathbf{M}_{f_1}}{\Gamma(\alpha-\gamma_i)((\alpha-\gamma_i-1)q_1+1)^{1/q_1}},\quad i=1,2\Bigg\},\\
\Xi_2' &= \max\Bigg\{\frac{1+2\sum_{j=1}^{\infty}b_j}{1+\sum_{j=1}^{\infty}b_j}\left(\frac{\|\mathfrak{a}_2\|_{L^{p_2}}(\mathbf{M}_{g_2}+\mathbf{M}_{h_2})+\mathbf{M}_{f_2}}{\Gamma(\beta)((\beta-1)q_2+1)^{1/q_2}}\right)+\frac{|y_0|}{1+\sum_{j=1}^{\infty}b_j},\\
&\qquad \frac{\|\mathfrak{a}_2\|_{L^{p_2}}(\mathbf{M}_{g_2}+\mathbf{M}_{h_2})+\mathbf{M}_{f_2}}{\Gamma(\beta-\nu_i)((\beta-\nu_i-1)q_2+1)^{1/q_2}},\quad i=1,2\Bigg\},
\end{aligned}$$

$$\Lambda' = \Lambda_1' + \Lambda_2', \quad \Xi' = \Xi_1' + \Xi_2'.$$

Remark 1. In this section, our analysis is based on system (1), (2). Observe that for each $m \in \mathbb{N}_+$, the quantities Λ_1, Ξ_1, Λ_2, Ξ_2, Λ, and Ξ (defined in (5)–(9)), as well as the boundary conditions (2), depend on m. Therefore, throughout this section, we will write

$$\Lambda_{1,m},\ \Xi_{1,m},\ \Lambda_{2,m},\ \Xi_{2,m},\ \Lambda_m,\ \Xi_m, \text{ and } (2)_m,$$

respectively.

Lemma 4. *Suppose* $\Lambda' < 1$ *and* (A1)–(A3) *hold. Then for any* $m \in \mathbb{N}_+$, *system* (1), $(2)_m$ *has a unique solution* (x_m, y_m), *which can be expressed as*

$$\begin{aligned}
x_m(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U_{1,(x_m,y_m)}(s)\,\mathrm{d}s\\
&\quad + \frac{1}{1+\sum_{j=1}^{m}a_j}\left(x_0 - \frac{\sum_{j=1}^{m}a_j}{\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-1}U_{1,(x_m,y_m)}(s)\,\mathrm{d}s\right), \qquad (21_1)
\end{aligned}$$

$$y_m(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} U_{2,(x_m,y_m)}(s)\,\mathrm{d}s + \frac{1}{1+\sum_{j=1}^m b_j} \left(y_0 - \frac{\sum_{j=1}^m b_j}{\Gamma(\beta)} \int_0^{\varsigma_j} (\varsigma_j - s)^{\beta-1} U_{2,(x_m,y_m)}(s)\,\mathrm{d}s \right). \quad (21_2)$$

Moreover, the sequence $\{(x_m, y_m)\}|_{m\in\mathbb{N}_+}$ *contains a uniformly convergent subsequence.*

Proof. It is easy to verify that $\Lambda_{i,m} \leqslant \Lambda_i'$ and $\Xi_{i,m} \leqslant \Xi_i'$ $(i = 1, 2)$. Consequently, we obtain $\Lambda_m \leqslant \Lambda'$ and $\Xi_m \leqslant \Xi'$. It follows from $\Lambda' < 1$ that $\Lambda_m < 1$. Hence, by Theorem 1, we know that system (1), $(2)_m$ has a unique solution (x_m, y_m) given by (21).

In what follows, we prove the existence of a convergent subsequence by the Arzelà–Ascoli theorem. First, we prove that $\{x_m\}|_{m\in\mathbb{N}_+}$ and $\{y_m\}|_{m\in\mathbb{N}_+}$ are uniformly bounded. Since $\Lambda' < 1$, we can choose $R > 0$ such that $R > \Xi'/(1-\Lambda')$. By (16) and (17), we obtain

$$\begin{aligned} \|x_m\|_X &\leqslant \Lambda_{1,m}R + \Xi_{1,m} \leqslant \Lambda_1'R + \Xi_1' \\ &\leqslant \Lambda'R + \Xi' < R, \quad m \in \mathbb{N}_+, \\ \|y_m\|_Y &\leqslant \Lambda_{2,m}R + \Xi_{2,m} \leqslant \Lambda_2'R + \Xi_2' \\ &\leqslant \Lambda'R + \Xi' < R, \quad m \in \mathbb{N}_+. \end{aligned}$$

This proves the uniform boundedness of $\{x_m\}|_{m\in\mathbb{N}_+}$ and $\{y_m\}|_{m\in\mathbb{N}_+}$, respectively.

Next, we show that $\{x_m\}|_{m\in\mathbb{N}_+}$ and $\{y_m\}|_{m\in\mathbb{N}_+}$ are equicontinuous. From (12) we derive

$$\big|U_{1,(x_m,y_m)}(s)\big| \leqslant C_{1,R}\mathfrak{a}_1(s) + \big|f_1(s,0,0,0,0,0,0)\big|, \quad s \in [0,1],\ m \in \mathbb{N}_+, \quad (22)$$

where

$$C_{1,R} = 2\big(\|\mathfrak{b}_1\|_{L^{p_1}} + \|\mathfrak{c}_1\|_{L^{p_1}} + 2\big)R + \mathbf{M}_{g_1} + \mathbf{M}_{h_1}.$$

Thus, for any $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, let $\Delta t = t_2 - t_1$. We have

$$\begin{aligned} &\big|x_m(t_1+\Delta t) - x_m(t_1)\big| \\ &\quad\leqslant \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \big|(t_1+\Delta t-s)^{\alpha-1} - (t-s)^{\alpha-1}\big|\big|U_{1,(x_m,y_m)}(s)\big|\,\mathrm{d}s \\ &\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_1+\Delta t} |t_1+\Delta t-s|^{\alpha-1}\big|U_{1,(x_m,y_m)}(s)\big|\,\mathrm{d}s \\ &\quad=: I_1 + I_2. \end{aligned} \quad (23)$$

To estimate (23), we split it into two parts. First term:

$$\begin{aligned} I_1 &\leqslant \frac{(1-\alpha)}{\Gamma(\alpha)} \int_0^{t_1} \left(\int_{t_1}^{t_1+\Delta t} (\tau - s)^{\alpha-2} \,\mathrm{d}\tau \right) \big(C_{1,R}\mathfrak{a}_1(s) + \big| f_1(s,0,0,0,0,0,0) \big| \big) \,\mathrm{d}s \\ &\leqslant \frac{(1-\alpha)(C_{1,R}\|\mathfrak{a}_1\|_{L^{p_1}} + \mathbf{M}_{f_1})}{\Gamma(\alpha)} \int_{t_1}^{t_1+\Delta t} \left(\int_0^{t_1} (\tau - s)^{(\alpha-2)q_1} \mathrm{d}s \right)^{1/q_1} \mathrm{d}\tau \\ &= \frac{(1-\alpha)(C_{1,R}\|\mathfrak{a}_1\|_{L^{p_1}} + \mathbf{M}_{f_1})}{\Gamma(\alpha)} (\Delta t)^{\alpha-1/p_1} \\ &\quad \times \int_0^1 \left(\frac{(\frac{t_1}{\Delta t} + v)^{(\alpha-2)q_1+1} - v^{(\alpha-2)q_1+1}}{(\alpha-2)q_1+1} \right)^{1/q_1} \mathrm{d}v, \end{aligned} \tag{24}$$

where we have used the fact that

$$\begin{aligned} &\int_{t_1}^{t_1+\Delta t} \left(\int_0^{t_1} (\tau - s)^{(\alpha-2)q_1} \,\mathrm{d}s \right)^{1/q_1} \mathrm{d}\tau \\ &\quad \overset{\tau = t_1 + v\Delta t}{=} (\Delta t)^{\alpha-1/p_1} \int_0^1 \left(\frac{(\frac{t_1}{\Delta t} + v)^{(\alpha-2)q_1+1} - v^{(\alpha-2)q_1+1}}{(\alpha-2)q_1+1} \right)^{1/q_1} \mathrm{d}v. \end{aligned}$$

Second term:

$$\begin{aligned} I_2 &\leqslant \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_1+\Delta t} (t_1 + \Delta t - s)^{\alpha-1} \big(C_{1,R}\mathfrak{a}_1(s) + \big| f_1(s,0,0,0,0,0,0) \big| \big) \,\mathrm{d}s \\ &\leqslant \frac{C_{1,R}\|\mathfrak{a}_1\|_{L^{p_1}} + \mathbf{M}_{f_1}}{\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}} (\Delta t)^{\alpha-1/p_1}. \end{aligned} \tag{25}$$

Substituting (24) and (25) into (23) yields

$$\begin{aligned} &\big| x_m(t_1 + \Delta t) - x_m(t_1) \big| \\ &\leqslant \left\{ \frac{(1-\alpha)(C_{1,R}\|\mathfrak{a}_1\|_{L^{p_1}} + \mathbf{M}_{f_1})}{\Gamma(\alpha)} \int_0^1 \left(\frac{(\frac{t_1}{\Delta t} + v)^{(\alpha-2)q_1+1} - v^{(\alpha-2)q_1+1}}{(\alpha-2)q_1+1} \right)^{1/q_1} \mathrm{d}v \right. \\ &\quad \left. + \frac{C_{1,R}\|\mathfrak{a}_1\|_{L^{p_1}} + \mathbf{M}_{f_1}}{\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}} \right\} (\Delta t)^{\alpha-1/p_1}. \end{aligned} \tag{26}$$

Similarly, we have

$$\begin{aligned}
&\left|{}^{c}D_{0+}^{\gamma_i}x_m(t_1+\Delta t)-{}^{c}D_{0+}^{\gamma_i}x_m(t_1)\right| \\
&\quad\leqslant\Bigg\{\frac{(1-\alpha+\gamma_i)(C_{1,R}\|\mathfrak{a}_1\|_{L^{p_1}}+\mathbf{M}_{f_1})}{\Gamma(\alpha-\gamma_i)} \\
&\qquad\times\int_0^1\left(\frac{(\frac{t_1}{\Delta t}+v)^{(\alpha-\gamma_i-2)q_1+1}-v^{(\alpha-\gamma_i-2)q_1+1}}{(\alpha-\gamma_i-2)q_1+1}\right)^{1/q_1}\mathrm{d}v \\
&\qquad+\frac{C_{1,R}\|\mathfrak{a}_1\|_{L^{p_1}}+\mathbf{M}_{f_1}}{\Gamma(\alpha-\gamma_i)((\alpha-\gamma_i-1)q_1+1)^{1/q_1}}\Bigg\}(\Delta t)^{\alpha-\gamma_i-1/p_1},\quad i=1,2. \qquad (27)
\end{aligned}$$

By the Lebesgue dominated convergence theorem, the integral terms in (26) and (27) have finite limits as $\Delta t\to 0$. Together with the continuity of $t^{\alpha-1/p_1}$ and $t^{\alpha-\gamma_i-1/p_1}$ $(i=1,2)$ on $[0,1]$, we conclude that for any $\varepsilon>0$, there exists $\delta>0$ such that for all $m\in\mathbb{N}_+$ and $t_1,t_2\in[0,1]$, if $|t_2-t_1|<\delta$, then

$$\left|x_m(t_2)-x_m(t_1)\right|<\varepsilon,\quad\text{and}\quad\left|{}^{c}D_{0+}^{\gamma_i}x_m(t_2)-{}^{c}D_{0+}^{\gamma_i}x_m(t_1)\right|<\varepsilon,\quad i=1,2.$$

This shows that $\{x_m\}_{m\in\mathbb{N}_+}$ is equicontinuous. The equicontinuity of $\{y_m\}_{m\in\mathbb{N}_+}$ can be established in a similar manner.

The Arzelà–Ascoli theorem implies that $\{x_m\}$ and $\{y_m\}$ admit uniformly convergent subsequences. A standard diagonal argument then yields a uniformly convergent subsequence $\{(x_{m_{k_j}},y_{m_{k_j}})\}$ of $\{(x_m,y_m)\}$. □

Theorem 2. *Suppose that* $\Lambda'<1$ *and* (A1)–(A3) *hold. Then system* (1), (3) *has a solution given by*

$$\begin{aligned}
x(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}U_{1,(x,y)}(s)\,\mathrm{d}s \\
&\quad+\frac{1}{1+\sum_{j=1}^{\infty}a_j}\left(x_0-\frac{\sum_{j=1}^{\infty}a_j}{\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j-s)^{\alpha-1}U_{1,(x,y)}(s)\,\mathrm{d}s\right), \\
y(t)&=\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}U_{2,(x,y)}(s)\,\mathrm{d}s \\
&\quad+\frac{1}{1+\sum_{j=1}^{\infty}b_j}\left(y_0-\frac{\sum_{j=1}^{\infty}b_j}{\Gamma(\beta)}\int_0^{\varsigma_j}(\varsigma_j-s)^{\beta-1}U_{2,(x,y)}(s)\,\mathrm{d}s\right).
\end{aligned}\qquad (28)$$

Proof. By Lemma 4, we know that for any $m\in\mathbb{N}_+$, system (1), $(2)_m$ has a unique solution (x_m,y_m) given by (21). Moreover, the sequence $\{(x_m,y_m)\}|_{m\in\mathbb{N}_+}$ contains a uniformly convergent subsequence. Without loss of generality, we still denote the convergent subsequence by $\{(x_m,y_m)\}|_{m\in\mathbb{N}_+}$, and its limit by (x,y).

Let $m \to \infty$. Then the boundary conditions $(2)_m$ become (3). Noting that (21) is a solution of system (1), $(2)_m$, we will justify passing to the limit in (21) and thereby derive the explicit expression for the limit functions x and y. It then follows that (x, y) is a solution to system (1), (3).

In view of the form of (21), the key to establishing the existence of this limit lies in proving the convergence of the corresponding series and justifying the interchange of limit and integral. From (22) we obtain

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j - s)^{\alpha-1}U_{1,(x,y)}(s)\,\mathrm{d}s\right| \\
&\quad\leqslant \frac{1}{\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j - s)^{\alpha-1}\big(C_{1,R}\mathfrak{a}_1(s) + \big|f_1(s,0,0,0,0,0,0)\big|\big)\,\mathrm{d}s \\
&\quad\leqslant \frac{C_{1,R}\|\mathfrak{a}_1\|_{L^{p_1}} + \mathbf{M}_{f_1}}{\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}}. \qquad (29)
\end{aligned}$$

Combining (29) with the convergence of $\sum_{j=1}^{\infty} a_j$, we conclude that

$$\frac{\sum_{j=1}^{m} a_j}{\Gamma(\alpha)}\int_0^{\tau_j}(\tau_j - s)^{\alpha-1}U_{1,(x,y)}(s)\,\mathrm{d}s$$

is convergent. On the other hand, we prove that

$$\lim_{m\to+\infty}\int_0^t (t-s)^{\alpha-1}U_{1,(x_m,y_m)}(s)\,\mathrm{d}s = \int_0^t (t-s)^{\alpha-1}U_{1,(x,y)}(s)\,\mathrm{d}s, \quad t\in[0,1]. \qquad (30)$$

The uniform convergence of the sequence $\{(x_m, y_m)\}$ to (x, y) on $[0, 1]$ implies that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}_+$ such that for all $m > N$ and $s \in [0, 1]$,

$$\begin{aligned}
&\big|x_m(s) - x(s)\big| < \varepsilon \quad \text{and} \quad \big|{}^cD_{0^+}^{\gamma_1}x_m(s) - {}^cD_{0^+}^{\gamma_1}x(s)\big| < \varepsilon, \\
&\big|y_m(s) - y(s)\big| < \varepsilon \quad \text{and} \quad \big|{}^cD_{0^+}^{\nu_1}y_m(s) - {}^cD_{0^+}^{\nu_1}y(s)\big| < \varepsilon.
\end{aligned}$$

Combining the above fact with hypotheses (A1)–(A3), we conclude that

$$\big|U_{1,(x_m,y_m)}(s) - U_{1,(x,y)}(s)\big| \leqslant 2\mathfrak{a}_1(s)\big(\|\mathfrak{b}_1\|_{L^{p_1}} + \|\mathfrak{c}_1\|_{L^{p_1}} + 2\big)\varepsilon, \quad s\in[0,1].$$

It follows that

$$\begin{aligned}
&\left|\int_0^t (t-s)^{\alpha-1}U_{1,(x_m,y_m)}(s)\,\mathrm{d}s - \int_0^t (t-s)^{\alpha-1}U_{1,(x,y)}(s)\,\mathrm{d}s\right| \\
&\quad\leqslant \frac{2\|\mathfrak{a}_1\|_{L^{p_1}}(\|\mathfrak{b}_1\|_{L^{p_1}} + \|\mathfrak{c}_1\|_{L^{p_1}} + 2)}{((\alpha-1)q_1+1)^{1/q_1}}\varepsilon, \quad t\in[0,1].
\end{aligned}$$

This establishes the limit in (30). Thus, we obtain

$$\begin{aligned} x(t) &= \lim_{m\to\infty} x_m(t) \\ &= \lim_{m\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U_{1,(x_m,y_m)}(s)\,\mathrm{d}s \\ &\quad + \lim_{m\to\infty} \frac{1}{1+\sum_{j=1}^m a_j} \left(x_0 - \frac{\sum_{j=1}^m a_j}{\Gamma(\alpha)} \int_0^{\tau_j} (\tau_j - s)^{\alpha-1} U_{1,(x_m,y_m)}(s)\,\mathrm{d}s \right) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U_{1,(x,y)}(s)\,\mathrm{d}s \\ &\quad + \frac{1}{1+\sum_{j=1}^\infty a_j} \left(x_0 - \frac{\sum_{j=1}^\infty a_j}{\Gamma(\alpha)} \int_0^{\tau_j} (\tau_j - s)^{\alpha-1} U_{1,(x,y)}(s)\,\mathrm{d}s \right). \end{aligned}$$

The same argument applies to $y(t)$. Therefore, we deduce that system (1), (3) has a solution (x, y) given by (28). □

Remark 2. Given $\Lambda' < 1$, by a proof similar to that of Theorem 1, we know that system (1), (3) has a unique solution. Therefore, the solution (x, y) obtained in Theorem 2 is the unique solution to system (1), (3).

5 Riemann–Stieltjes integral conditions

In this section, we use the following notation:

$$\begin{aligned} \Lambda_1'' &= \max\left\{ \frac{2(1+2(l_1(1)-l_1(0)))\|\mathfrak{a}_1\|_{L^{p_1}}(\|\mathfrak{b}_1\|_{L^{p_1}} + \|\mathfrak{c}_1\|_{L^{p_1}} + 2)}{(1+l_1(1)-l_1(0))\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}}, \right. \\ &\qquad \left. \frac{2\|\mathfrak{a}_1\|_{L^{p_1}}(\|\mathfrak{b}_1\|_{L^{p_1}} + \|\mathfrak{c}_1\|_{L^{p_1}} + 2)}{\Gamma(\alpha-\gamma_i)((\alpha-\gamma_i-1)q_1+1)^{1/q_1}},\ i=1,2 \right\}, \\ \Lambda_2'' &= \max\left\{ \frac{2(1+2(l_2(1)-l_2(0)))\|\mathfrak{a}_2\|_{L^{p_2}}(\|\mathfrak{b}_2\|_{L^{p_2}} + \|\mathfrak{c}_2\|_{L^{p_2}} + 2)}{(1+l_2(1)-l_2(0))\Gamma(\beta)((\beta-1)q_2+1)^{1/q_2}}, \right. \\ &\qquad \left. \frac{2\|\mathfrak{a}_2\|_{L^{p_2}}(\|\mathfrak{b}_2\|_{L^{p_2}} + \|\mathfrak{c}_2\|_{L^{p_2}} + 2)}{\Gamma(\beta-\nu_i)((\beta-\nu_i-1)q_2+1)^{1/q_2}},\ i=1,2 \right\}, \\ \Xi_1'' &= \max\left\{ \frac{1+2(l_1(1)-l_1(0))}{1+l_1(1)-l_1(0)} \left(\frac{\|\mathfrak{a}_1\|_{L^{p_1}}(\mathbf{M}_{g_1}+\mathbf{M}_{h_1})+\mathbf{M}_{f_1}}{\Gamma(\alpha)((\alpha-1)q_1+1)^{1/q_1}} \right) + \frac{|x_0|}{1+l_1(1)-l_1(0)}, \right. \\ &\qquad \left. \frac{\|\mathfrak{a}_1\|_{L^{p_1}}(\mathbf{M}_{g_1}+\mathbf{M}_{h_1})+\mathbf{M}_{f_1}}{\Gamma(\alpha-\gamma_i)((\alpha-\gamma_i-1)q_1+1)^{1/q_1}},\ i=1,2 \right\}, \end{aligned}$$

$$\Xi_2'' = \max\left\{ \frac{1+2(l_2(1)-l_2(0))}{1+l_2(1)-l_2(0)} \left(\frac{\|\mathfrak{a}_2\|_{L^{p_2}}(\mathbf{M}_{g_2}+\mathbf{M}_{h_2})+\mathbf{M}_{f_2}}{\Gamma(\beta)((\beta-1)q_2+1)^{1/q_2}} \right) + \frac{|y_0|}{1+l_2(1)-l_2(0)}, \right.$$
$$\left. \frac{\|\mathfrak{a}_2\|_{L^{p_2}}(\mathbf{M}_{g_2}+\mathbf{M}_{h_2})+\mathbf{M}_{f_2}}{\Gamma(\beta-\nu_i)((\beta-\nu_i-1)q_2+1)^{1/q_2}},\ i=1,2 \right\},$$
$$\Lambda'' = \Lambda_1'' + \Lambda_2'', \qquad \Xi'' = \Xi_1'' + \Xi_2''.$$

Theorem 3. *Suppose* $\Lambda'' < 1$ *and* (A1)–(A3) *hold. Then system* (1), (4) *has a solution given by*

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U_{1,(x,y)}(s)\,\mathrm{d}s \\ &\quad + \frac{1}{1+(l_1(1)-l_1(0))} \left(x_0 - \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\int_0^t (t-s)^{\alpha-1} U_{1,(x,y)}(s)\,\mathrm{d}s \right) \mathrm{d}l_1(t) \right), \\ y(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} U_{2,(x,y)}(s)\,\mathrm{d}s \\ &\quad + \frac{1}{1+(l_2(1)-l_2(0))} \left(y_0 - \frac{1}{\Gamma(\beta)} \int_0^1 \left(\int_0^t (t-s)^{\beta-1} U_{2,(x,y)}(s)\,\mathrm{d}s \right) \mathrm{d}l_2(t) \right). \end{aligned} \tag{31}$$

Proof. Our analysis of system (1), (4) is based system (1), (2). First, we take a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $\tau_j, \varsigma_j \in (t_{j-1}, t_j)$ for $j = 1, 2, \ldots, m$ and $\max_{1 \leqslant j \leqslant m}(t_j - t_{j-1}) < 1/m$. By taking $a_j = l_1(t_j) - l_1(t_{j-1})$, $b_j = l_2(t_j) - l_2(t_{j-1})$ $(j = 1, 2, \ldots, m)$ in the boundary conditions (2), we obtain the following boundary conditions:

$$\begin{aligned} x(0) + \sum_{j=1}^m a_j x(\tau_j) &= x(0) + \sum_{j=1}^m \big(l_1(t_j) - l_1(t_{j-1})\big) x(\tau_j) = x_0, \\ y(0) + \sum_{j=1}^m b_j y(\varsigma_j) &= y(0) + \sum_{j=1}^m \big(l_2(t_j) - l_2(t_{j-1})\big) y(\varsigma_j) = y_0. \end{aligned} \tag{32}$$

Since Λ (defined in (9)) and the boundary conditions (32) depend on m, we will henceforth denote them as Λ_m and $(32)_m$, respectively. Since l_1 and l_2 are increasing, we obtain that $\Lambda_m < \Lambda'' < 1$. By Theorem 1, we know that system (1), $(32)_m$ has a unique solution (x_m, y_m), which can be expressed by

$$\begin{aligned} x_m(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U_{1,(x_m,y_m)}(s)\,\mathrm{d}s \\ &\quad + \frac{1}{1+\sum_{j=1}^m a_j} \left(x_0 - \frac{\sum_{j=1}^m a_j}{\Gamma(\alpha)} \int_0^{\tau_j} (\tau_j - s)^{\alpha-1} U_{1,(x_m,y_m)}(s)\,\mathrm{d}s \right), \end{aligned}$$

$$y_m(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} U_{2,(x_m,y_m)}(s)\,\mathrm{d}s + \frac{1}{1+\sum_{j=1}^m b_j} \left(y_0 - \frac{\sum_{j=1}^m b_j}{\Gamma(\beta)} \int_0^{\varsigma_j} (\varsigma_j - s)^{\beta-1} U_{2,(x_m,y_m)}(s)\,\mathrm{d}s \right).$$

Let $m \to \infty$. Then the boundary conditions $(32)_m$ become the boundary conditions (4):

$$x(0) + \lim_{m\to\infty} \sum_{j=1}^m \big(l_1(t_j) - l_1(t_{j-1})\big) x(\tau_j) = x(0) + \int_0^1 x(s)\,\mathrm{d}l_1(s) = x_0,$$

$$y(0) + \lim_{m\to\infty} \sum_{j=1}^m \big(l_2(t_j) - l_2(t_{j-1})\big) y(\varsigma_j) = y(0) + \int_0^1 y(s)\,\mathrm{d}l_2(s) = y_0.$$

Moreover, following the same argument as in the proof of Lemma 4, we conclude that $\{(x_m, y_m)\}|_{m\in\mathbb{N}_+}$ admits a uniformly convergent subsequence. Without loss of generality, we still denote this subsequence by $\{(x_m, y_m)\}|_{m\in\mathbb{N}_+}$ and its limit by (x, y). It follows that (x, y) is a solution to system (1), (4). We now proceed to further analyze the expression of (x, y). In view of the form of x_m, the crucial point lies in the existence of the following limit:

$$\lim_{m\to\infty} \sum_{j=1}^m a_j \int_0^{\tau_j} (\tau_j - s)^{\alpha-1} U_{1,(x_m,y_m)}(s)\,\mathrm{d}s = \int_0^1 \left(\int_0^t (t-s)^{\alpha-1} U_{1,(x,y)}(s)\,\mathrm{d}s \right) \mathrm{d}l_1(t).$$

In fact, by (30) and the definition of the Riemann–Stieltjes integral, we obtain

$$\begin{aligned} &\left| \sum_{j=1}^m a_j \int_0^{\tau_j} (\tau_j - s)^{\alpha-1} U_{1,(x_m,y_m)}(s)\,\mathrm{d}s - \int_0^1 \left(\int_0^t (t-s)^{\alpha-1} U_{1,(x,y)}(s)\,\mathrm{d}s \right) \mathrm{d}l_1(t) \right| \\ &\quad \leqslant \left| \sum_{j=1}^m a_j \int_0^{\tau_j} (\tau_j - s)^{\alpha-1} U_{1,(x_m,y_m)}(s)\,\mathrm{d}s - \sum_{j=1}^m a_j \int_0^{\tau_j} (\tau_j - s)^{\alpha-1} U_{1,(x,y)}(s)\,\mathrm{d}s \right| \\ &\qquad + \left| \sum_{j=1}^m a_j \int_0^{\tau_j} (\tau_j - s)^{\alpha-1} U_{1,(x,y)}(s)\,\mathrm{d}s - \int_0^1 \left(\int_0^t (t-s)^{\alpha-1} U_{1,(x,y)}(s)\,\mathrm{d}s \right) \mathrm{d}l_1(t) \right| \\ &\quad \to 0, \quad m \to \infty. \end{aligned}$$

It follows that

$$\begin{aligned} x(t) &= \lim_{m\to\infty} x_m(t) \\ &= \lim_{m\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U_{1,(x_m,y_m)}(s)\,\mathrm{d}s \\ &\quad + \lim_{m\to\infty} \frac{1}{1+\sum_{j=1}^m a_j} \left(x_0 - \frac{\sum_{j=1}^m a_j}{\Gamma(\alpha)} \int_0^{\tau_j} (\tau_j - s)^{\alpha-1} U_{1,(x_m,y_m)}(s)\,\mathrm{d}s \right) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U_{1,(x,y)}(s)\,\mathrm{d}s \\ &\quad + \frac{1}{1+(l_1(1)-l_1(0))} \left(x_0 - \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\int_0^t (t-s)^{\alpha-1} U_{1,(x,y)}(s)\,\mathrm{d}s \right) \mathrm{d}l_1(t) \right). \end{aligned}$$

Using the same argument for $y(t)$, system (1), (4) has a solution given by (31). □

Remark 3. Given $\Lambda'' < 1$, by a proof similar to that of Theorem 1, we know that system (1), (4) has a unique solution. Therefore, the solution (x, y) obtained in Theorem 3 is the unique solution to system (1), (4).

6 Examples

In this section, we consider the system

$$\begin{aligned} {}^cD_{0^+}^{1/2} x(t) &= f_1\left(t,\, x(t),\, y(t),\, \int_0^t g_1\big(s, x(s), y(s)\big)\,\mathrm{d}s,\, \int_0^1 h_1\big(s, x(s), y(s)\big)\,\mathrm{d}s,\right. \\ &\qquad \left. {}^cD_{0^+}^{1/4} x(t),\, {}^cD_{0^+}^{1/8} y(t) \right), \\ {}^cD_{0^+}^{1/4} y(t) &= f_2\left(t,\, x(t),\, y(t),\, \int_0^t g_2\big(s, x(s), y(s)\big)\,\mathrm{d}s,\, \int_0^1 h_2\big(s, x(s), y(s)\big)\,\mathrm{d}s,\right. \\ &\qquad \left. {}^cD_{0^+}^{3/8} x(t),\, {}^cD_{0^+}^{1/5} y(t) \right) \end{aligned} \tag{33}$$

with the following three boundary conditions:

$$x(0) + \frac{32}{35} x\left(\frac{1}{4}\right) + \frac{17}{10} x\left(\frac{1}{2}\right) = 4, \qquad y(0) + 128 y\left(\frac{1}{16}\right) + \frac{181}{16} y\left(\frac{1}{4}\right) = 2, \tag{34}$$

$$x(0) + \sum_{j=1}^{\infty} \frac{1}{80 \cdot 2^j} x\left(\frac{1}{j}\right) = 4, \qquad y(0) + \sum_{j=1}^{\infty} \frac{1}{200 \cdot 2^j} y\left(\frac{1}{j^2}\right) = 2, \tag{35}$$

$$x(0)+\int_0^1 x(s)\,\mathrm{d}l_1(s)=4,\qquad y(0)+\int_0^1 y(s)\,\mathrm{d}l_2(s)=2, \tag{36}$$

where

$$\begin{aligned}
&f_1(t,x_1,x_2,x_3,x_4,x_5,x_6)\\
&\quad=10^{-8}\big(t^{-1/10}x_1x_2+t^{-1/40}x_5x_6-t^{63/20}+t^{83/20}-t^{33/20}\big)+10^{-4}(x_3-x_4),\\
&f_2(t,x_1,x_2,x_3,x_4,x_5,x_6)\\
&\quad=10^{-8}\bigg(t^{-1/40}x_1x_2+\frac{\Gamma(\frac{25}{8})}{\Gamma(\frac{7}{2})}t^{11/10}x_5-\frac{\Gamma(\frac{51}{20})}{\Gamma(\frac{11}{4})}t^{7/40}x_6+t^{169/40}-\frac{17}{20}t^{89/40}\bigg)\\
&\qquad-10^{-4}(x_3-x_4),\\
&g_1(t,x,y)=10^{-4}\bigg(t^{1/10}x+\frac{4312\,\Gamma(\frac12)\Gamma(\frac34)}{325\,\Gamma(\frac14)\Gamma(\frac58)}y^{57/35}-t^{8/5}+t^{13/5}-t^{1/10}\bigg)\\
&\qquad\qquad+10^{4}\bigg(\Gamma\Big(\frac52\Big)-\Gamma\Big(\frac72\Big)t\bigg),\\
&g_2(t,x,y)=10^{-4}\big(x+t^{-1/40}y-t^{3/2}+t^{5/2}-t^{69/40}\big),\\
&h_1(t,x,y)=10^{-4}\bigg(x+\frac{14364\,\Gamma(\frac12)\Gamma(\frac34)}{1625\,\Gamma(\frac14)\Gamma(\frac58)}y^{37/35}-t^{3/2}+t^{5/2}-1\bigg),\\
&h_2(t,x,y)=10^{-4}\big(x+t^{-1/50}y^4-t^{3/2}+t^{5/2}-t^{349/50}\big)+10^4\frac{3\Gamma(\frac{11}{4})}{2\Gamma(\frac52)}t^{1/2},
\end{aligned}$$

$$l_1(s)=\begin{cases}0, & s=0,\\ 1, & 0<s<\frac12,\\ 1.003, & \frac12\leqslant s\leqslant 1,\end{cases}\qquad\text{and}\quad l_2(s)=0.$$

Let

$$\begin{aligned}
&\mathfrak{a}_1(t)=10^{-4}+10^{-8}R\big(t^{-0.1}+t^{-0.025}\big),\\
&\mathfrak{a}_2(t)=10^{-4}\big(1+t^{11/10}+t^{7/40}\big)+10^{-8}R\,t^{-1/40},
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{b}_1(t)=10^{-4}t^{0.1}+10^{-3}\cdot R^{22/35}, && \mathfrak{c}_1(t)=10^{-4}\big(1+4R^{2/35}\big),\\
&\mathfrak{b}_2(t)=10^{-4}\big(1+t^{-1/40}\big), && \mathfrak{c}_2(t)=10^{-4}\big(1+4R^3t^{-1/50}\big).
\end{aligned}$$

It can be verified by computation that assumptions (A1)–(A3) are satisfied with $R=100$, $p_1=17/2$, $p_2=41/2$. A direct computation yields $\Lambda=\Lambda_1+\Lambda_2\approx 0.032<1$, which implies that the hypotheses of Theorem 1 are satisfied. Hence, we conclude that system (33), (34) has a unique solution. Building on these computational results, an approximate solution (x,y) is obtained and shown in Fig. 1. The corresponding numerical

Table 1. The approximate solution (x, y) for system (33), (34).

t	0	0.2	0.4	0.6	0.8	1
$x(t)$	1.0000	1.0716	1.1518	1.1859	1.1431	1.0000
$y(t)$	0.0000	0.0598	0.2012	0.4090	0.6767	1.0000

Table 2. The approximate solution (x, y) for system (33), (35).

t	0	0.2	0.4	0.6	0.8	1.0
$x(t)/10^2$	0.0000	1.8993	2.6300	3.1280	3.5129	3.8290
$y(t)/10^2$	0.0000	1.1265	2.9286	4.4554	5.7950	7.0000

Table 3. The approximate solution (x, y) for system (33), (36).

t	0	0.2	0.4	0.6	0.8	1.0
$x(t)/10^3$	0.0010	0.1961	0.4763	0.8416	1.2426	1.6610
$y(t)/10^3$	0.0020	−0.0353	0.6627	1.4245	2.1756	2.9020

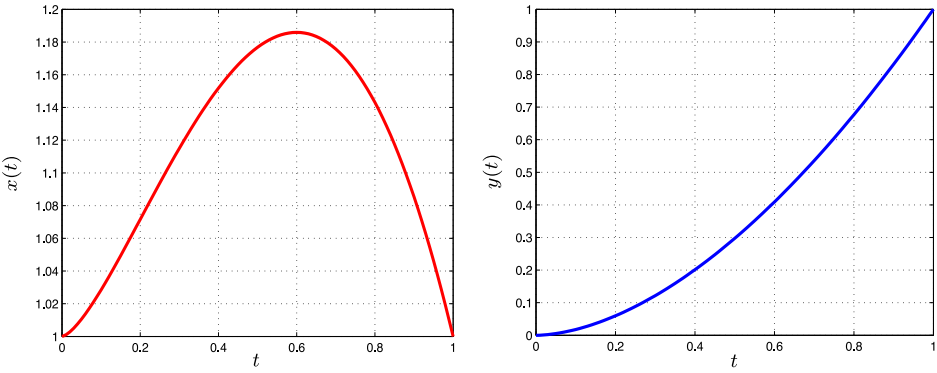


Figure 1. Trajectory of the solution for system (33), (34).

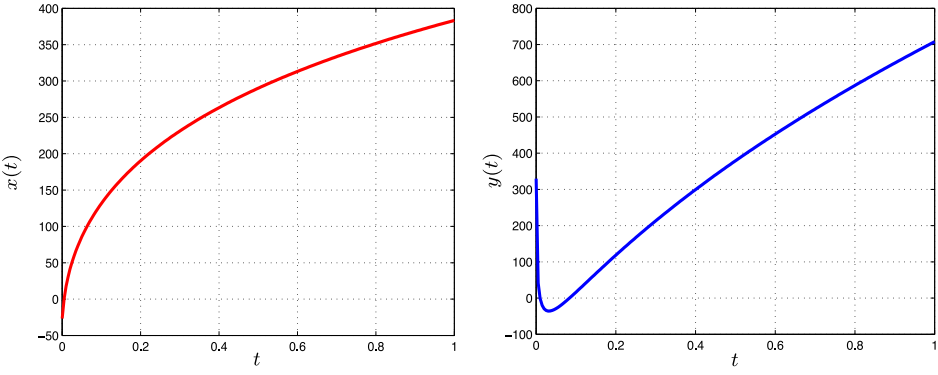


Figure 2. Trajectory of the solution for system (33), (35).

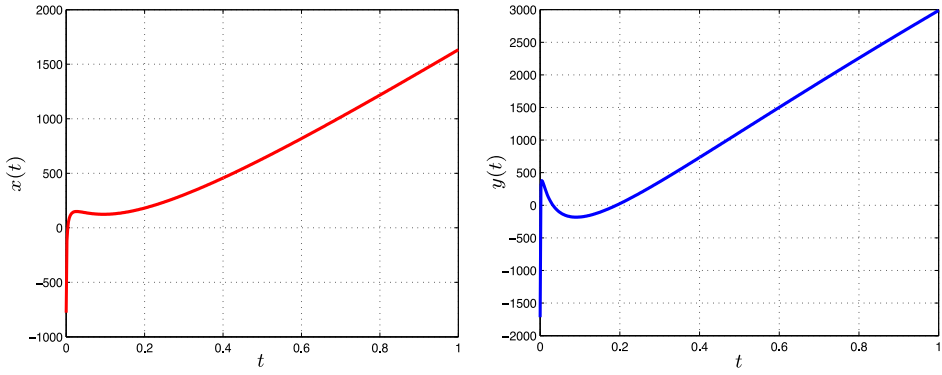


Figure 3. Trajectory of the solution for system (33), (36).

values are presented in Table 1. Similarly, the condition $\Lambda' = \Lambda'_1 + \Lambda'_2 \approx 0.031 < 1$ is verified for Theorem 2. The approximate solution for system (33), (35) is illustrated in Fig. 2, with the corresponding numerical values listed in Table 2. Finally, we confirm that $\Lambda'' = \Lambda''_1 + \Lambda''_2 \approx 0.031 < 1$ for Theorem 3. The resulting approximate solution for system (33), (36) is illustrated in Fig. 3. Table 3 provides a summary of the corresponding numerical values. These results collectively validate our theoretical findings.

7 Conclusions

In this paper, a unified approach is proposed to study the coupled system (1) with three types of boundary conditions. The complexity of the nonlinear terms makes the analysis challenging. We generalize the classical Lipschitz condition by replacing the Lipschitz constant with an L^p function, which makes the condition more general. By using the Banach contraction principle, we establish the uniqueness of the solution for the coupled system (1) with multi-point boundary conditions. Based on the existence of solutions under multi-point boundary conditions, we extend the results to the cases of infinite-point and Riemann–Stieltjes integral boundary conditions by taking limits. The existence of the corresponding limits is a key point in these generalizations. Thus, we have established a unified approach to investigate the existence of solutions for three types of boundary conditions, with no further proof required. Finally, numerical examples are provided along with graphical illustrations to demonstrate the feasibility of our results.

Author contributions. All authors (F. Wang, Y. Chen, L. Liu, H. Gu, and X. Yang) have contributed as follows: methodology, F. Wang, Y. Chen, and L. Liu; formal analysis, F. Wang, Y. Chen, and L. Liu; software, F. Wang; validation, L. Liu, H. Gu, and X. Yang; writing – original draft preparation, F. Wang and Y. Chen; writing – review & editing, F. Wang and L. Liu. All authors have read and approved the published version of the manuscript.

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